Standing-waves with small energy/charge ratio

Garrisi Daniele

College of Mathematics Education, Inha University
daniele.garrisi@inha.ac.kr

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Given $N \geq 1$ and $k \geq 1$, a system of non-linear Klein-Gordon equations is

$$(k\text{-NLKG}) \qquad v^i_{tt} - \Delta v_i + m_i^2 v_i + \partial_{z_i} G(v) = 0$$

where

$$v \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}^k$$

and $m_i > 0$, and

$$G \colon \mathbb{C}^k \to \mathbb{R}$$

is continuously differentiable.

Standing-wave solutions

A standing-wave is a solution to (k-NLKG)

$$v_i(t,x) = u_i(x)e^{-i\omega_i t}$$

where $u_i \in H^1(\mathbb{R}^N;\mathbb{R})$ and $\omega_i \in \mathbb{R}$. If

(G0)
$$G(z) = G(|z_1|, \ldots, |z_k|)$$

then v solves (k-NLKG) if and only if

$$-\Delta u_i + (m_i^2 - \omega_i^2)u_i + \partial_{z_i} G(u) = 0 \quad 1 \leq i \leq k$$

Our goal is to prove the existence of standing-wave solutions which are radially symmetric

$$|x| = |y| \Rightarrow u_i(x) = u_i(y)$$

and positive

$$u_i > 0.$$

Conserved quantities

If v is a solution to (k-NLKG), then we have conserved quantities associated to it: the *energy*, the *charges* and the *hylenic charge*.

$$\mathbf{E}(t) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\partial_t v(t,x)|^2 + |Dv(t,x)|^2\right)dx + \int_{\mathbb{R}^N} F(v(t,x))dx$$

$$\mathbf{C}_i(t) = -\mathrm{Im}\int_{\mathbb{R}^N}\partial_t v_i(t,x)\overline{v_i(t,x)}dx$$

$$\mathbf{\Lambda}(t) := \frac{\mathbf{E}(t)}{|\sum_{i=1}^k \mathbf{C}_i(t)|}.$$

When $\mathbf{\Lambda} < \min\{m_i \mid 1 \leq i \leq k\}$, solutions to ($k$-NLKG) do not disperd

$$\liminf_{t\to+\infty}\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} > 0.$$

Energy and charges of standing-waves

If we define the conserved quantities on standing-wave solutions, we obtain

$$\mathbf{E}(t) = E(u,\omega) := \frac{1}{2}\int_{\mathbb{R}^N} |Du(x)|^2 dx + \frac{1}{2}\sum_{i=1}^{k} \omega_i^2 \int_{\mathbb{R}^N} u_i(x)^2 dx + \int_{\mathbb{R}^N} F(u(x)) dx$$

$$\mathbf{C}_i(t) = C_i(u,\omega) := \omega_i \int_{\mathbb{R}^N} u_i(x)^2 dx$$

$$\mathbf{\Lambda}(t) = \Lambda(u,\omega) := \frac{E(u,\omega)}{\left|\sum_{i=1}^{k} C_i(u,\omega)\right|}.$$

Λ enters in the variational method.

Variational approach

We define

$$H_1^r = \Big\{ f \in H^1 \mid |x| = |y| \Rightarrow f_i(x) = f_i(y) \Big\}.$$

E and C_i are defined between the spaces

$$E, C_i \colon H_r^1 \times \mathbb{R}^k \to \mathbb{R} \quad 1 \leq i \leq k.$$

We seek solutions of the elliptic system among the minima of the functional E over the constraint

$$M_\sigma^r = \{(u, \omega) \in H_r^1 \times \mathbb{R}^k \mid C_i(u, \omega) = \sigma_i\}$$

$$C_i(u, \omega) = \omega_i \int_{\mathbb{R}^N} u_i^2.$$

The sub-critical growth conditions

We require

(G1) $$|DG(u)| \leq C(|u|^{p-1} + |u|^{q-1})$$

where

$$2 < p \leq q < \frac{2N}{N-2}$$

if $N \geq 3$ and

$$2 < p \leq q$$

if $N \geq 2$. Then E is well-defined on M^r_σ.

(G2) $$F(z) := \frac{1}{2} \sum_{i=1}^{k} m_i^2 z_i^2 + G(z) \geq 0.$$

Properties of E

Properties of E

1. Minima of E over M_σ^r are solutions to the elliptic system
2. E is coercive
3. if $(u_n, \omega_n) \in H_r^1(\mathbb{R}^N; \mathbb{R}^k) \times \mathbb{R}^k$ is a Palais-Smale sequence of E over M_σ^r such that

$$\omega_n^i \to \omega_i < m := \min\{m_i \mid 1 \leq i \leq k\}$$

then $(u_n)_{n\geq 1}$ has a converging subsequence.

3. Follows from the Radial Lemma (W. Strauss, Comm. Pure and App. Math., 1977).

The role of Λ is providing estimates from above of ω_i.

Properties of Λ

Properties of Λ

1. $\Lambda > 0$
2. $\inf \Lambda = \sqrt{2\alpha}$ where
$$\alpha := \inf \frac{F(z)}{|z|^2}.$$
3. $$\Lambda(u,\omega) = \frac{1}{2}\left(\frac{\xi^2(u) + \sum_{i=1}^k \omega_i^2 \|u_i\|_{L^2}^2}{\sum_{i=1}^k \omega_i \|u_i\|_{L^2}^2}\right)$$
where
$$\xi(u) = \left(\frac{\int_{\mathbb{R}^N} |Du|^2 + 2\int_{\mathbb{R}^N} F(u)}{\int_{\mathbb{R}^N} |u|^2}\right)^{1/2}$$
4. $\Lambda \geq \xi$
5. $\inf \xi = \sqrt{2\alpha}$.

$k = 1$ (V. Benci and D. Fortunato, Dyn. PDE, 2009)

Λ provides bounds for ω.

$$4\Lambda(\Lambda - \xi) \geq (\omega - \xi)^2.$$

So they assumed that

(G3)
$$\inf \frac{F(z)}{|z|^2} < \frac{m^2}{2}.$$

So, $\omega < m$ if $(\Lambda - \inf(\Lambda))$ is small enough.

$k \geq 2$

We define

(G4) $$\alpha_i := \inf \frac{F(z)}{\sum_{j \neq i} z_j^2}.$$

For systems we need the following assumption

$$\alpha < \alpha_i \text{ for every } 1 \leq i \leq k.$$

Lemma (arXiv:1110.6495)

If (G3) and (G4) hold, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Lambda(u, \omega) < \sqrt{2\alpha} + \delta \Rightarrow |\omega_i - \sqrt{2\alpha}| < (m - \sqrt{2\alpha})/2.$$

Theorem (arXiv:1110.6495)

If G satisfies assumptions (G0-G4), then there exists an open set $\Omega \subset \mathbb{R}^k_+$ such that $\inf_{M^r_\sigma} E$ is achieved for every $\sigma \in \Omega$.

Choose δ_0 such that

$$\Lambda(u,\omega) < \sqrt{2\alpha} + \delta_0 \Rightarrow |\omega_i - \sqrt{2\alpha}| < (m - \sqrt{2\alpha})/2$$

and (u', ω') such that

$$\Lambda(u',\omega') < \sqrt{2\alpha} + \delta_0 \quad \sigma_i' := \omega_i' \int_{\mathbb{R}^N} (u_i')^2.$$

Given a minimising Palais-Smale sequence

$$E(u_n, \omega_n) \to \inf_{M^r_{\sigma'}} E$$

then

$$\Lambda(u_n,\omega_n) = \frac{E(u_n,\omega_n)}{\sum_i \sigma_i'} \leq \frac{E(u',\omega')}{\sum_i \sigma_i'} = \Lambda(u',\omega') < \sqrt{2\alpha} + \delta_0.$$

Thus $\omega_n^i \to \omega_i < m \leq m_i$ and the property 3 of E applies.

We compare solutions to

(1) $$E(u,\omega) = \inf_{M_\sigma^r} E$$

with solutions to

(2) $$E(u,\omega) = \inf_{M_\sigma} E$$

where

$$M_\sigma := \{(u,\omega) \in H^1 \times \mathbb{R}^k \mid C_i(u,\omega) = \sigma_i\}.$$

(3) $$\inf_{M_\sigma} E \leq \inf_{M_\sigma^r} E.$$

$\inf_{M_\sigma} E = \inf_{M_\sigma^r} E$

For the minimization problem in higher dimension (E, M_σ) we account two references:

V. Benci, C. Bonanno *et al.*, Adv. Nonlinear Stud., 2010 $(k = 1)$

G., Adv. Nonlinear Stud., 2012 $(k = 2)$

In both references, it is required that

$$\text{(S)} \qquad \int_{\mathbb{R}^N} G(u_1^*, u_2^*, \dots, u_k^*) \leq \int_{\mathbb{R}^N} G(u_1, u_2, \dots, u_k)$$

for every u_i in $L_+^2(\mathbb{R}^N)$ with compact support.

By u_i^* we denote the symmetric decreasing rearrangement of u_i.

G is not sensitive to the symmetric rearrangement

Our assumptions (G0-G4) does not include (S). This follows from

Proposition, Arxiv:1110.6495

If $k = 2$, G is well behaved with respect to the symmetric rearrangement if and only if the coupling term

$$G_0(u, v) := G(u, v) - G(u, 0) - G(0, v)$$

is monotonically decreasing on u and v.

$$(N = 3, k = 2) \qquad G_1(u, v) = -u^2 v^2 + u^4 + v^4$$

$$(N = 3, k = 2) \qquad G_2(u, v) = -u^2 v^2 + u^3 v^3 + u^4 + v^4.$$

The first non-linearity satisfies (S). The second does not.

Weaker assumptions than (S)

Despite of examples G_2 and G_1 we might still have symmetric solutions. A weaker version of (S) is:

Weaker symmetric rearrangement property

For every $u, v \in L^2_+$ with compact support there exists $y \in \mathbb{R}^N$ such that

$$\text{(Sw)} \qquad \int_{\mathbb{R}^N} G(u^*, v^*(\cdot - y)) \leq \int_{\mathbb{R}^N} G(u, v).$$

We are interested on a complete characterisation of nonlinearity satisfying (Sw).

So far, we do not have an example of G where

$$\inf_{M_\sigma} E < \inf_{M_\sigma^r} E.$$