SOLUTIONS OF THE EXERCISES OF WEEK SIX

Exercise 1. Let $g\colon [c,d] \to [a,b]$ be a continuous function such that

$$\{g(c), g(d)\} = \{a,b\}. \tag{1}$$

Then g is surjective.

Solution. Clearly, a and b belong to the image of g. Now, let

$$a < t < b \tag{2}$$

be a point of $[a,b]$.

$$\phi\colon [c,d] \to \mathbb{R}, \quad \phi(s) := g(s) - t.$$

From (1) and (2), ϕ changes sign at the endpoints. Since ϕ is continuous, there exists $s \in [c,d]$ such that

$$\phi(s) = 0$$

which implies $g(s) = t$. □

Exercise 2. Let g be the function defined below

$$g(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

State whether g is continuous at O. Do all the directional derivatives exist at O?

Solution. Firstly, we notice that all the directional derivatives exist at the point $O(0,0)$: in fact, if v is such that $v_1 = v_2$, we have

$$\frac{g(tv_1, tv_1)}{t} = 0 \Rightarrow \partial_v g(O) = 0.$$

If $v_1 \neq v_2$

$$\partial_v g(O) = \lim_{t\to 0} \frac{g(tv_1, tv_2)}{t} = \frac{t^2 v_1 v_2}{t(v_1 - v_2)} \cdot \frac{1}{t} = \frac{v_1 v_2}{v_1 - v_2}.$$

The function g is not continuous at O. Firstly, we notice that, since g has directional derivatives at O, g is continuous if evaluated on lines containing the if we evaluate g on lines containing the origin. However, g is not continuous on $\mathbb{R}^2$. If we consider the sequence of points

$$P_n := \left(\frac{1}{n}, \frac{1}{n} - \frac{1}{n^3}\right)$$

we have

$$g(P_n) = \frac{1/n^2}{1/n^3} = n$$

then the limit $\lim_{n\to\infty} g(P_n)$ does not exists. □

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