EXISTENCE OF PROPER CLASSES

Suppose that the Class Construction Axiom holds. We define the property

$$P(x) : x \notin x.$$

By the Class Construction Axiom, there exists a class $\mathscr{R}$ such that

$$x \in \mathscr{R} \Leftrightarrow x \text{ is an element and } P(x) \tag{1}$$

We refer to such a class as "Russell Class".

Proposition. The Russell Class is a proper class.

Proof. We argue by contradiction. Suppose that $\mathscr{R}$ is an element. Then

$$\mathscr{R} \in \mathscr{R} \Rightarrow P(\mathscr{R}) \Rightarrow \mathscr{R} \notin \mathscr{R} \tag{2}$$

which is a contradiction.

If $\mathscr{R} \notin \mathscr{R}$, then $P(\mathscr{R})$ is true. Since $\mathscr{R}$ is an element, by the implication $\Leftarrow$ in (1), it follows that $\mathscr{R} \in \mathscr{R}$. Then

$$\mathscr{R} \notin \mathscr{R} \Rightarrow \mathscr{R} \in \mathscr{R}. \tag{3}$$

Then, from (2) and (3), the statement $\mathscr{R} \in \mathscr{R}$ is neither true or false. Then $\mathscr{R}$ is not an element. □