INVERTIBLE FUNCTIONS

Theorem 1 (Exercise 4, §2.2, [1]). *Let $f, g\colon A \to B$ be two functions such that $f \subseteq g$. Then $f = g$.*

Proof. We show that $g \subseteq f$. Suppose that $(x, y) \in g$. Let me use the formal language.

(1) $(x, y) \in g$
(2) $(x, y) \in g \Rightarrow x \in \mathrm{dom}(g) = A = \mathrm{dom}(f) \Rightarrow x \in \mathrm{dom}(f)$
(3) $x \in \mathrm{dom}(f) \Rightarrow \exists z \cdot \ni \cdot (x, z) \in f$
(4) $(x, z) \in f$
(5) $f \subseteq g$ (hypothesis) $\Rightarrow (x, z) \in g$
(6) since g satisfies F2, $(1) \wedge (5) \Rightarrow y = z$
(7) $y = z$
(8) $(4) \wedge (7) \Rightarrow (x, y) \in f$.

∎

Proposition 1. *Let G be a graph. Then*

$$id_A \subseteq G^{-1} \circ G \tag{1}$$

$$id_B \subseteq G \circ G^{-1} \tag{2}$$

where $A = \mathrm{dom}(G)$ and $B = \mathrm{ran}(G)$.

Proof. Let $(x, x) \in id_A$

(1) $(x, x) \in id_A \Rightarrow x \in A = \mathrm{dom}(G)$
(2) $x \in \mathrm{dom}(G) \Rightarrow \exists y \cdot \ni \cdot (x, y) \in G$
(3) $(x, y) \in G$
(4) $(x, y) \in G \Rightarrow (y, x) \in G^{-1}$
(5) $(y, x) \in G^{-1}$
(6) $(3) \wedge (5) \Rightarrow (x, x) \in G^{-1} \circ G$
(7) $(x, x) \in G^{-1} \circ G$.

We proved (1). We want to prove (2) without going through all the implications (1-7) above. Then we set

$$H := G^{-1}.$$

We apply (1) to H. Then

$$id_{\mathrm{dom}(H)} \subseteq H^{-1} \circ H.$$

Since $\mathrm{dom}(H) = \mathrm{dom}(G^{-1}) = \mathrm{ran}(G) = B$. Since $H^{-1} = G$, we obtain (2). ∎

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Definition 1. A function $f\colon A \to B$ is said *invertible* if and only if $f^{-1}\colon B \to A$ is a function. On this case, f^{-1} is called *inverse function*.

Given a function $f\colon A \to B$, the following are equivalent definitions of invertible function:

$$f \text{ bijective} \tag{1}$$

$$f \text{ is invertible} \tag{2}$$

$$(f^{-1} \circ f = id_A) \wedge (f \circ f^{-1} = id_B), \tag{3}$$

$$\exists g\colon B \to A \cdot \ni \cdot (f \circ g = id_B) \wedge (g \circ f = id_A). \tag{4}$$

In the next theorem we prove that the above facts are equivalent.

Theorem 2. *The facts listed in (1-4) are all equivalent.*

Proof. (1) $\Rightarrow$ (2). Suppose that f is bijective. Then

$$\mathrm{dom}(f) = A, \quad \mathrm{ran}(f) = B$$

whence

$$\mathrm{dom}(f^{-1}) = B, \quad \mathrm{ran}(f^{-1}) = A.$$

We prove F2:

$$(y_1, x), (y_2, x) \in f^{-1} \Rightarrow (x, y_1), (x, y_2) \in f \Rightarrow y_1 = y_2$$

Then f^{-1} is a function.
(2) $\Rightarrow$ (3). Since f^{-1} is a function, both compositions are functions. By Proposition 1,

$$id_A \subseteq f^{-1} \circ f.$$

By Theorem 1, $f^{-1} \circ f = id_A$.
Similarly, By Proposition 1,

$$id_B \subseteq f \circ f^{-1}.$$

So, By Theorem 1, $f \circ f^{-1} = id_B$.
(3) $\Rightarrow$ (4). We set $g := f^{-1}$. We only need to prove that $f^{-1}\colon B \to A$ is a function.
$\mathrm{ran}(f^{-1}) \subseteq A$. $\mathrm{ran}(f^{-1}) = \mathrm{dom}(f) = A$. Then, in particular, $\mathrm{ran}(f^{-1}) \subseteq A$.
$\mathrm{dom}(f^{-1}) = B$. Since

$$f \circ f^{-1} = id_B$$

we have $\mathrm{dom}(f \circ f^{-1}) = B$. Since $\mathrm{ran}(f^{-1}) \subseteq \mathrm{dom}(f)$, we can apply Corollary 1.34, page 52 of [1]. Thus,

$$\mathrm{dom}(f \circ f^{-1}) = \mathrm{dom}(f^{-1}).$$

Then $\mathrm{dom}(f^{-1}) = B$.
F2. Let $(y, x_1), (y, x_2) \in f^{-1}$. Then

$$(x_1, y), (x_2, y) \in f.$$

Since $\mathrm{dom}(f^{-1}) = B$, we have $y \in B$. Then, there exists $x \in A$ such that
$$(y, x) \in f^{-1}.$$
Then
$$(x_1, x), (x_2, x) \in f^{-1} \circ f = id_A.$$
Then
$$x_1 = x \text{ and } x_2 = x.$$
Then $x_1 = x_2$.
(4) $\Rightarrow$ (1). Firstly, we show that
$$g \circ f = id_A \Rightarrow f \text{ INJ}$$
Given $x_1, x_2 \in A$ and $y \in B$ such that
$$(x_1, y), (x_2, y) \in f$$
there exists $z \in A$ such that
$$(y, z) \in g.$$
Then
$$(x_1, z), (x_2, z) \in g \circ f \Rightarrow x_1 = x_2 = z.$$
We show that
$$f \circ g = id_B \Rightarrow f \text{ SURJ}.$$
By Theorem 1.37 of the book
$$B = \mathrm{ran}(f \circ g) \subseteq \mathrm{ran}(f).$$
Since $\mathrm{ran}(f) \subseteq B$, we have $\mathrm{ran}(f) = B$. $\square$

References

1. Charles C. Pinter, *Set theory*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1971. MR 0284349 (44 #1577)