Marstrand's theorem and tangent measures

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1 Introduction

The aim of this notes is to give a complete and self-contained proof of the following result.

Theorem 1.1 (Marstrand)**.** *Let μ be a locally finite Borel measure on $\mathbb{R}^n$, $\alpha \geq 0$ and $E \subseteq \mathbb{R}^n$ a Borel set s.t. $\mu(E) > 0$. Assume that*

$$0 < \Theta_{*\alpha}(\mu, x) = \Theta^*_\alpha(\mu, x) < +\infty \quad \textit{for } \mu\textit{-a.e. } x \in E.$$

Then α is an integer.

This beautiful theorem was first proved by Marstrand in [2]; in deed, the author proved a much stronger result, which provides important information on the measures μ that satisfies the assumptions of theorem 1.1. Moreover, it is the starting point of the Preiss' regularity theory (see [4]). It is well known that, given E a locally $\mathcal{H}^d$-finite and d-rectifiable set in $\mathbb{R}^n$, the measure $\mu = \mathcal{H}^d \llcorner E$ has d-dimensional density 1 for $\mathcal{H}^d$-a.e. $x \in E$ (see [3]). The Preiss' regularity theory goes in the opposite direction. The first part of the statement above is the Marstrand's theorem.

Theorem 1.2 (Preiss)**.** *Given a Borel locally finite measure μ s.t. $\Theta_\alpha(\mu, x)$ exists, it is finite and positive for μ-a.e. $x \in E$, then α is integer the support of μ can be covered μ-a.e. by an α-rectifiable set.*

Our proof of theorem 1.1 is based on the notion of tangent measures: given μ as in the Marstrand's theorem, a "blow-up" procedure provides the existence of a second (non trivial) measure ν, with the property of being α-uniform (see 1.22).

The presentation given has been strongly inspired by that of chapter 14 of [2] and that of chapter 3 of the [1].

1.1 Preliminaries

We briefly recall some preliminaries and well known notions; the following can be found in any book of Geometric Measure Theory (for instance, see [3]).

1.1.1 α-dimensional density

Definition 1.3 (α-density)**.** Let μ be a locally finite Borel measure in $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\alpha \geq 0$. We define the lower α-dimensional density of μ at x as

$$\Theta_{*\alpha}(\mu, x) \coloneqq \liminf_{r\to 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha};$$

similarly, we define the upper α-dimensional density of μ at x as

$$\Theta^*_\alpha(\mu, x) \coloneqq \limsup_{r\to 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}.$$

If $\Theta_{*\alpha}(\mu, x) = \Theta^*_\alpha(\mu, x)$, we denote the common value as $\Theta_\alpha(\mu, x)$. We say that μ admits α-dimensional density at x.

Remark 1.4. The constant ω_α in 1.3 is only needed as a normalization factor: if α is not integer, we can freely assume $\omega_\alpha = 1$; if α is integer, we set ω_α to be the α-dimensional volume of the unit ball in $\mathbb{R}^\alpha$.

1.1.2 Convergence in the sense of measure

Definition 1.5 (Weak* convergence of measures)**.** Given $(\mu_n)_n, \mu_\infty$ of locally finite Borel measures on $\mathbb{R}^n$, we say that $(\mu_n)_n$ converges to μ_∞ locally in the sense of measure (or simply $\mu_n \overset{*}{\rightharpoonup} \mu_\infty$) if the following holds true:

$$\lim_{n\to+\infty} \int_{\mathbb{R}^n} g(y)\ d\mu_n(y) = \int_{\mathbb{R}^n} g(y)\ d\mu_\infty(y) \quad \forall g \in C_c(\mathbb{R}^n).$$

Remark 1.6. In other words, the convergence in the sense of measure that we will consider is the one induced by duality with continuous functions compactly supported. In deed, this notion of convergence makes sense, since we deal with locally finite Borel measures.

The following properties hold true.

Proposition 1.7. *Let $(\mu_n)_n$ be a sequence of locally finite Borel measures that converges weakly* to a locally finite Borel measure μ. Then, for all lower semicontinuous and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$ there holds*

$$\int_{\mathbb{R}^n} g(y)\ d\mu(y) \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^n} g(y)\ d\mu_n(y).$$

In particular, the followings hold true:

- *for all open set A there holds*

$$\mu(A) \leq \liminf_{n\to+\infty} \mu_n(A);$$

- *for all closed set C there holds*

$$\mu(C) \geq \limsup_{n\to+\infty} \mu_n(C);$$

- *for all Borel set E s.t. $\mu(E) = 0$, there holds*

$$\mu(E) = \lim_{n\to+\infty} \mu_n(E).$$

Theorem 1.8 (Compactness of measures). *Given a sequence $(\mu_n)_n$ of locally finite Borel measures on $\mathbb{R}^n$, assume that $(\mu_n)_n$ is locally uniformly bounded, i.e. for all $r > 0$ there holds*

$$\sup_{n\in\mathbb{N}} \mu_n(B_r) < +\infty.$$

Then, up to subsequences, $(\mu_n)_n$ converges weakly to a locally finite Borel measure.*

1.1.3 Besicovitch's covering theorem

Definition 1.9 (Besicovitch's covering). Let $E \subseteq \mathbb{R}^n$; let $\mathcal{F}$ be a family of balls in $\mathbb{R}^d$ s.t.

$$\inf\{r \mid B(x,r) \in \mathcal{F}\} = 0 \quad \forall x \in E.$$

We say that $\mathcal{F}$ is a Besicovitch's covering of E.

Theorem 1.10 (Besicovitch's covering theorem - 1). *Let μ be a Borel, locally finite measure on $\mathbb{R}^n$. Let $E \subseteq \mathbb{R}^n$ be a Borel set s.t. $\mu(E) < +\infty$. Let $\mathcal{F}$ be a family of closed balls which is a Besicovitch's covering of E (see 1.9). Then, for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ disjoint s.t.*

- *$\mathcal{F}'$ is disjoint, at most countable and covers μ-a.a. of E;*
- *$\sum_{B\in\mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$.*

Theorem 1.11 (Besicovitch's covering theorem - 2). *Let μ be a Borel, locally finite measure on $\mathbb{R}^n$. Let $E \subseteq \mathbb{R}^n$ be a Borel set s.t. $\mu(E) < +\infty$. Let $\mathcal{F}$ be a family of closed balls which is a Besicovitch's covering of E (see 1.9). Then, for all $\varepsilon > 0$ there exists $\mathcal{F}' \subseteq \mathcal{F}$ disjoint s.t.*

- *$\mathcal{F}'$ covers μ-a.a. of E;*
- *$\sum_{B\in\mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$.*

Theorem 1.12 (Besicovitch's differentiation theorem). *Let μ be a locally finite Borel measure $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$ be a nonnegative measure. Then, for μ-a.e. $x \in \mathbb{R}^n$ the following holds true:*

$$\lim_{r\to 0} \fint_{B_r(x)} |f(x) - f(y)| \ d\mu(y) = 0;$$

in particular, for μ-a.e. $x \in \mathbb{R}^n$, f is L^1 approximately continuous at x, that is

$$f(x) = \lim_{r\to 0} \fint_{B_r(x)} f(y) \ d\mu(y).$$

1.2 Blow-up of a measure in a point

Blowing up a measure μ in a point x means looking at the behaviour of μ in very small neighbourhoods of x. Despite being very simple and intuitive, this idea is surprisingly powerful and it gives important information on the measure itself. We describe this procedure.

We introduce the following notation, which will be extremely useful.

Definition 1.13. Let μ be a locally finite Borel measure in $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $r > 0$. We denote as $\mu_{x,r}$ the locally finite Borel measure on $\mathbb{R}^n$ defined by

$$\mu_{x,r}(A) := \mu(x + rA) \quad \forall A \subseteq \mathbb{R}^n \text{ Borel.}$$

Remark 1.14. Given μ, x, r as in 1.13, we denote as $T_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ the affine map

$$T_{x,r}(y) := \frac{y - x}{r}.$$

By definition 1.13, there holds that

$$\mu_{x,r}(A) = \mu(T_{x,r}^{-1}(A)) \quad \forall A \subseteq \mathbb{R}^n \text{ Borel.}$$

In other words, $\mu_{x,r}$ is the push-forward of μ according to $T_{x,r}$. Then, given $f \in L^1(\mathbb{R}^n, \mu)$, we deduce that $f \in L^1(\mathbb{R}^n, \mu_{x,r})$; moreover, there holds

$$\int_{\mathbb{R}^n} f(y)\ d\mu_{x,r}(y) = \int_{\mathbb{R}^n} f \circ T_{x,r}(y)\ d\mu(y) = \int_{\mathbb{R}^n} f\left(\frac{y - x}{r}\right)\ d\mu(y).$$

Definition 1.15 (Tangent measure)**.** Let μ be a locally finite Borel measure in $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $r > 0$. We denote as $\mathrm{Tan}_\alpha(\mu, x)$ the set of all measure ν for which there exists a sequence of positive radii $r_i \downarrow 0$ s.t.

$$\frac{\mu_{x,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu.$$

Remark 1.16. The notion of tangent measures given in 1.15 is not the most general possible. In deed, this notion was first introduced by Preiss in [4], where all the weak* limit of sequence of the form $c_i\mu_{x,r_i}$ are considered. In the following, we will only deal with the definition of tangent measures given in 1.15, since it carries all the information needed.

Remark 1.17. Clearly, the tangent measures of definition 1.15 are locally finite and Borel. As remarked in 1.14, the convergence in the definition 1.15 can be stated as follows:

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \frac{1}{r^\alpha} g\left(\frac{y - x}{r}\right)\ d\mu(y) = \int_{\mathbb{R}^n} g(y)\ d\nu(y) \quad \forall g \in C_c(\mathbb{R}^n).$$

Remark 1.18. Blowing-up the measure μ blowing up at the point x means that we want to study the limiting behaviour of $\frac{\mu_{x,r}}{r^\alpha}$ as $r \downarrow 0$. By definition 1.13, it is immediate to see that

$$\frac{\mu_{x,r}(B_1)}{r^\alpha} = \frac{\mu(B_r(x))}{r^\alpha}.$$

As $r \downarrow 0$, the numerator goes to the measure of the point x and the denominator blows up to ∞. In other words, when r is very small, $\mu_{x,r}$ spreads the measure μ of the ball

$B_r(x)$ in the unit ball. In this sense, the measure $\mu_{x,r}$ zooms in the measure μ in a very small neighbourhood of the point x (which becomes the "new" origin the zoomed euclidean space). As for the denominator, if r^α is the right scaling factor, the limiting behaviour of $\frac{\mu_{x,r}}{r^\alpha}$ might be a (non trivial) measure carrying useful information on the geometry of the support of μ in a neighbourhood of x.

In the following, we describe the blow up procedure in the classical case. This will show as the notion of the tangent measure can be seen as a suitable definition of the concept of tangent planes to a C^1 submanifold of $\mathbb{R}^n$.

Proposition 1.19 (Tangent measures to a C^1 submanifold)**.** *Let Σ be k-submanifold of $\mathbb{R}^n$ of class C^1 (without boundary). Letting $\mu := \mathcal{H}^k \llcorner \Sigma$, the followings hold true:*

- *for all $x \in \Sigma$ for all $r > 0$ we have*

$$\frac{\mu_{x,r}}{r^k} = \mathcal{H}^k \llcorner \left(\frac{\Sigma - x}{r} \right);$$

- *for all $x \in \Sigma$, we have*

$$\mathcal{H}^k \llcorner \left(\frac{\Sigma - x}{r} \right) \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \, Tan_x\Sigma \quad as\ r \downarrow 0,$$

  where $Tan_x\Sigma$ is the tangent plane of Σ at x.

Remark 1.20. In the framework of proposition 1.19, for all $x \in \Sigma$, we deduce that $\mathcal{H}^k \llcorner \mathrm{Tan}_x\Sigma$ is the unique tangent measure to $\mathcal{H}^k \llcorner \Sigma$ at x. In some sense, this is not surprising: as we zoom in the neighbourhood of x the manifold Σ looks almost like the tangent plane $\mathrm{Tan}_x\Sigma$; similarly, the k-dimensional Hausdorff measure on Σ looks almost like the Lebesgue measure on a k-dimensional linear space.

Proof of 1.19. Fix $x \in \Sigma$. As for the first statement, given $r > 0$, by the rescaling property of $\mathcal{H}^k$, for all $A \subseteq \mathbb{R}^n$ Borel we have

$$\begin{aligned}
\frac{\mu_{x,r}(A)}{r^k} &= \frac{1}{r^k}\mathcal{H}^k((x + rA) \cap \Sigma) \\
&= \mathcal{H}^K \left(\left(\frac{x}{r} + A \right) \cap \frac{\Sigma}{r} \right) \\
&= \mathcal{H}^k \left(A \cap \frac{\Sigma - x}{r} \right) \\
&= \mathcal{H}^k \llcorner \left(\frac{\Sigma - x}{r} \right)(A).
\end{aligned}$$

We denote as B^k_δ the k-dimensional ball centered at the origin in $\mathbb{R}^k$ of radius δ. As for the second statement, we can make the following assumptions:

- $x = 0$;
- $\mathrm{Tan}_x\Sigma = \mathrm{Span}(e_1, \dots, e_k) = \mathbb{R}^k$, where $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$;
- there exist $\delta > 0$ and a C^1 map $\Phi : B^k_\delta \to B^{n-k}_\delta$ s.t. $\Gamma \cap (B^k_\delta \times B^{n-k}_\delta)$ is the graph of Φ, that is

$$\Gamma \cap (B^k_\delta \times B^{n-k}_\delta) = \{(x, \Phi(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid x \in B^k_\delta\}.$$

Let g be a test function in $C_c(\mathbb{R}^n)$; we have to check that

$$\lim_{r\to 0}\int_{\mathbb{R}^n} g\ d\mathcal{H}^k \llcorner \left(\frac{\Sigma}{r}\right) = \int_{\mathbb{R}^n} g\ d\mathcal{H}^k \llcorner (\mathrm{Tan}_x\Sigma). \tag{1}$$

Under our assumptions, (1) becomes

$$\lim_{r\to 0}\int_{\Sigma/r} g(y)\ d\mathcal{H}^k(y) = \int_{\mathbb{R}^k} g(x,0)\ dx \tag{2}$$

Notice that Σ/r can be parameterized in $B_{\delta/r}\times B^{n-k}_{\delta/r}$ as follows:

$$\Sigma/r\cap(B^k_{\delta/r}\times B^{n-k}_{\delta/r}) = \left\{\left(z,\frac{\Phi(rz)}{r}\right)\in\mathbb{R}^k\times\mathbb{R}^{n-k}\ \middle|\ z\in B^k_{\delta/r}\right\}.$$

Since we are interested in the limiting behaviour as r approaches 0, there exists $\eta>0$ s.t. for all $r\in(0,\eta)$ there holds

$$\mathrm{supp}(g)\subseteq B^k_{\delta/r}\times B^{n-k}_{\delta/r}.$$

Given $x\in B^k_\delta$, denote by $J\Phi(x)$ the jacobian determinant of Φ; by the area formula, for $r\in(0,\eta)$ there holds

$$\int_{\Sigma/r} g(y)\ d\mathcal{H}^k(y) = \int_{B^k_{\delta/r}} g\left(z,\frac{\Phi(zr)}{r}\right)\sqrt{1+J\Phi(rz)}\ dx. \tag{3}$$

Under our assumptions, notice that the map Φ as differential $d\Phi$ that vanishes at zero; hence, $J\Phi(0)=0$. As $r\downarrow 0$, for all $z\in\mathbb{R}^k$, there holds that

$$\lim_{r\to 0}\frac{\Phi(zr)}{r} = d\Phi_0(z) = 0.$$

Since Φ is a map of class C^1, we have that

$$\lim_{r\to 0} J\Phi(rz) = 0.$$

It is easy to show that the pointwise limit above are are uniformly bounded with respect to $r\in(0,\eta)$ and z in any compact set of $\mathbb{R}^k$. Then, by the dominated convergence theorem, we deduce that

$$\lim_{r\to 0}\int_{B^k_{\delta/r}} g\left(z,\frac{\Phi(zr)}{r}\right)\sqrt{1+J\Phi(rz)}\ dx = \int_{\mathbb{R}^k} g(x,0)\ dx. \tag{4}$$

□

Another important (and intuitive) fact is that the notion of tangent measure is completely local, as explained by the proposition above.

Proposition 1.21 (Locality of Tan_α)**.** *Let μ be a locally finite Borel measure on $\mathbb{R}^n$, $\alpha\geq 0$ and $f\in L^1_{loc}(\mathbb{R}^n,\mu)$ s.t. $f(x)\geq 0$ for μ-a.e. $x\in\mathbb{R}^n$. Then, for μ-a.e. $\mathbb{R}^n$ there holds*

$$Tan_\alpha(f\cdot\mu,x) = f(x)\,Tan_\alpha(\mu,x). \tag{5}$$

In particular, for any Borel set $B\subseteq\mathbb{R}^n$ the following holds true for μ-a.e. B:

$$Tan_\alpha(\mu\llcorner B,x) = Tan_\alpha(\mu,x). \tag{6}$$

Proof. We claim that (5) holds for every $x \in B_1$, where Ω is defined as follows:

$$\Omega := \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)| \; d\mu(y) = 0 \right\}.$$

The conclusion follows by the fact that $\mu(\mathbb{R}^n \setminus \Omega) = 0$; in deed, it is well knows that $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$ implies that f is L^1-approximately continuous at μ-a.e. $x \in \mathbb{R}^n$.

Fix $x \in \Omega$ and $\nu \in \text{Tan}_\alpha(\mu, x)$. By definition 1.15, there exists $r_i \downarrow 0$ s.t.

$$\nu_i := \frac{\mu_{x,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu.$$

For all $i \in \mathbb{N}$ define

$$\nu_i' := \frac{(f \cdot \mu)_{x,r_i}}{r_i^\alpha}.$$

We claim that $\nu_i' \to f\nu$; this would imply that

$$\text{Tan}_\alpha(f \cdot \mu, x) \subseteq f(x)\text{Tan}_\alpha(\mu, x).$$

Given $\rho > 0$, if we apply 1.17 we can compute

$$\begin{aligned}
(f(x)\nu_i - \nu_i')(B_\rho) &= \frac{1}{r_i^\alpha} \left(f(x)\mu_{x,r_i} - (f \cdot \mu)_{x,r_i} \right)(B_\rho) \\
&= \frac{1}{r_i^\alpha} \left[f(x)\mu(B_{r_i\rho}(x)) - \int_{B_\rho} f(y) \; d\mu_{x,r_i}(y) \right] \\
&= \frac{1}{r_i^\alpha} \left[f(x)\mu(B_{r_i\rho}(x)) - \int_{B_{r_i\rho}(x)} f(y) \; d\mu_{x,r_i}(y) \right] \\
&= \frac{1}{r_i^\alpha} \int_{B_{r_i\rho}(x)} [f(x) - f(y)] \; d\mu_{x,r_i}(y). \qquad (7)
\end{aligned}$$

Taking the total variation in the ball B_ρ, (7) leads to

$$\begin{aligned}
|f(x)\nu_i - \nu_i'| (B_\rho) &\leq \frac{1}{r_i^\alpha} \int_{B_{r_i\rho}} |f(x) - f(y)| \; d\mu(y) \\
&= \frac{\mu(B_{r_i\rho}(x))}{r_i^\alpha} \fint_{B_{r_i\rho}(x)} |f(x) - f(y)| \; d\mu(y). \qquad (8)
\end{aligned}$$

Since $x \in \Omega$, if we show that the ratio

$$\frac{\mu(B_{r_i\rho}(x))}{r_i^\alpha} = \frac{\mu_{x,r_i}(B_\rho)}{r_i^\alpha} = \nu_i(B_\rho)$$

is uniformly bounded with respect to i, from (8) we immediately deduce that

$$\lim_{i \to +\infty} |f(x)\nu_i - \nu_i'| (B_\rho) = 0. \qquad (9)$$

In other words, $(f(x)\nu_i - \nu_i')_i$ converges to 0 in total variation in any balls; in particular, $(f(x)\nu_i - \nu_i')_i$ converges to 0 locally in the sense of measures. Since $(f(x) \cdot \nu_i)_i$ converges weak* to $f(x)\nu$, we deduce that $(\nu_i')_i$ converges weak* to $f(x)\nu$, as desired. Having said that, take $g \in C_c(\mathbb{R}^n)$ s.t. g takes values in $[0,1]$ and $g \equiv 1$ in B_ρ. From the convergences in the sense of measures it follows that

$$\limsup_{i\to+\infty} \nu_i(B_\rho) \le \limsup_{i\to+\infty} \int_{\mathbb{R}^n} g(y)\ d\nu_i(y) = \int_{\mathbb{R}^n} g(y)\ d\nu(y) < +\infty.$$

Thus, we have shown that

$$\mathrm{Tan}_\alpha(f\cdot\mu, x) \subseteq f(x)\mathrm{Tan}_\alpha(\mu, x).$$

As for the reverse inclusion, we can argue in a similar way.

If we take $f := \mathbb{1}_B$, for some Borel set $B \subseteq \mathbb{R}^n$, immediately obtain (6) from (5). $\square$

1.3 α-uniform measures

We introduce the notion of α-uniform measure; we will prove that in the hypothesis of theorem 1.1, there are very interesting tangent measures, which have the property of being α-uniform. This fact will play a crucial role in the proof of Marstrand's theorem.

Definition 1.22 (α-uniform measure)**.** Let μ be a locally finite Borel measure on $\mathbb{R}^n$ and $\alpha \ge 0$. We say that μ is α-uniform if the following holds:

$$\mu(B_r(x)) = \omega_\alpha r^\alpha \quad \forall x \in \mathrm{supp}(\mu)\ \forall r > 0, \tag{10}$$

where ω_α is defined as in 1.4. We denote by $\mathcal{U}_\alpha(\mathbb{R}^n)$ the set of the α-uniform measures ν s.t. $0 \in \mathrm{supp}(\nu)$.

Remark 1.23.
- The fact that the support of measure $\mu \in \mathcal{U}_\alpha(\mathbb{R}^n)$ must contain 0 is simply needed to exclude the zero measure from $\mathcal{U}_\alpha(\mathbb{R}^n)$.
- From (10), there follows immediately the same property for closed balls. In deed, given μ α-uniform, $x \in \mathrm{supp}(\mu)$ and $r > 0$, for all $\varepsilon > 0$ there holds

  $$\mu(\partial B_r(x)) \le \mu(B_{r+\varepsilon}(x)) - \mu(B_r(x)) = \omega_\alpha[(r+\varepsilon)^\alpha - r^\alpha] = o(\varepsilon).$$

  Thus, we obtain $\mu(\partial B_r(x)) = 0$, which implies that

  $$\mu(\overline{B}_r(x)) = \omega_\alpha r^\alpha \quad \forall x \in \mathrm{supp}(\mu)\ \forall r > 0. \tag{11}$$

The huge symmetry properties of a measure $\mu \in \mathcal{U}_\alpha(\mathbb{R}^n)$ yields a useful change of variable formula.

Proposition 1.24. *Let μ be an α-uniform measure in $\mathbb{R}^n$, for some $\alpha \ge 0$. Let $\varphi : [0,+\infty) \to \mathbb{R}$ be a Borel function s.t. $\varphi(|\cdot|) \in L^1(\mathbb{R}^n, \mu)$. Then, for all $y \in supp(\mu)$ there holds $\varphi(|\cdot - y|) \in L^1(\mathbb{R}^n, \mu)$; furthermore, there holds*

$$\int_{\mathbb{R}^n} \varphi(|z|)\ d\mu(z) = \int_{\mathbb{R}^n} \varphi(|z-y|)\ d\mu(z). \tag{12}$$

Proof. Assume that $\varphi = \mathbb{1}_{[0,r)}$; then, (12) is an immediate consequence of (10). By linearity, (12) holds true if φ is a step function of the type

$$\varphi = \sum_{i=1}^N a_i \mathbb{1}_{[0,r_i)}.$$

By approximation and Beppo Levi's theorem, we deduce that (12) holds true if φ is a nonnegative functions; hence, we immediately extend (12) to the case in which φ is real-valued and $\varphi(|\cdot|) \in L^1(\mathbb{R}^n, \mu)$. $\square$

We can easily characterize the α-uniform measure in $\mathbb{R}^n$ for $\alpha \geq n$.

Proposition 1.25. *Given $\alpha \geq n$, take μ an α-uniform measure in $\mathbb{R}^n$. Then μ is absolutely continuous with respect to $\mathscr{L}^n$. Moreover, if $\alpha > n$, then $\mathcal{U}_\alpha(\mathbb{R}^n) = \emptyset$; if $\alpha = n$, then $\mathcal{U}_\alpha(\mathbb{R}^n) = \{\mathscr{L}^n\}$.*

Proof. **Step 1**: Take $E \subseteq \mathbb{R}^n$ a Borel set s.t. $\mathscr{L}^n(E) = 0$; we have to show that $\mu(E) = 0$. Without loss of generality, we can assume that $E \subseteq \operatorname{supp}(\mu)$: in deed, $\mu(E \setminus \operatorname{supp}(\mu)) = 0$. Then, by Besicovitch's covering theorem, for all $\varepsilon > 0$ we can cover E with at most countably many balls $(B_i)_i$ centered in E of radii at most 1 s.t.

$$\sum_i \mathscr{L}^n(B_i) \leq \varepsilon.$$

For simplicity, denote $B_i = B_{r_i}(x_i)$; recall that $r_i \leq 1$ and $x_i \in \operatorname{supp}(\mu)$ for all i. Since $\alpha \geq n$, there holds

$$\begin{aligned}
\varepsilon &\geq \sum_i \mathscr{L}^n(B_i) = \sum_i \omega_n r_i^n \\
&= \frac{\omega_n}{\omega_\alpha} \sum_i \omega_\alpha r_i^\alpha r_i^{n-\alpha} \geq \frac{\omega_n}{\omega_\alpha} \sum_i \omega_\alpha r_i^\alpha \\
&= \frac{\omega_n}{\omega_\alpha} \sum_i \mu(B_i) \geq \frac{\omega_n}{\omega_\alpha} \mu(E).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\mu(E) = 0$.

Step 2: We have that μ and $\mathscr{L}^n$ are both σ-finite measure (in deed, they are locally finite). So, the Radon-Nikodym theorem provides the existence of $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mathscr{L}^n)$ nonnegative s.t. $\mu = f \cdot \mathscr{L}^n$. From the Besicovitch's differentiation theorem and the fact that μ is α uniform, we deduce that for $\mathscr{L}^n$ −a.e. $x \in \mathbb{R}^n$ there holds

$$f(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_n r^n} = \lim_{r \to 0} \frac{\omega_\alpha}{\omega_n} r^{\alpha - n}. \tag{13}$$

If $\alpha > n$, then (13) yields $f(x) = 0$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$, that is $\mu \equiv 0$; however, this is against the fact that $0 \in \operatorname{supp}(\mu)$, as in definition 1.22. Then, $\mathcal{U}_\alpha(\mathbb{R}^n) = \emptyset$ if $\alpha > n$.

If $n = \alpha$, then (13) implies that $f(x) = 1$ for all $x \in \operatorname{supp}(\mu)$. Since $\mu \in \mathcal{U}_n(\mathbb{R}^n)$ (in particular, $0 \in \operatorname{supp}(\mu)$), for all $r > 0$ there holds

$$\omega_n r^n = \mu(B_r) = \int_{B_r} f(y)\, d\mathscr{L}^n(y) = \mathscr{L}^n(B_r \cap \operatorname{supp}(\mu)). \tag{14}$$

Since $\operatorname{supp}(\mu)$ is closed, we deduce that $B_r \operatorname{supp}(\mu)$ (recall that every non empty open set in $\mathbb{R}^n$ has positive Lebesgue measure). Since r is arbitrary, we conclude that $\operatorname{supp}(\mu) = \mathbb{R}^n$, that is $\mu = \mathscr{L}^n$. $\square$

The simple characterization given in 1.25 yields the corollary below, which will be extremely useful in the following. It goes in the direction of studying α-uniform measures concentrated on specific subsets of the euclidean space.

Corollary 1.26. *Let μ be measure in $\mathcal{U}_m(\mathbb{R}^n)$, where m, n are positive integer. Assume that supp$(\mu) \subseteq V$, where V is an m-dimensional affine subspace in $\mathbb{R}^n$. Then, $\mu = \mathcal{H}^m \llcorner V$.*

Proof. Clearly, we can see μ as an m-uniform measure measure in $V \simeq \mathbb{R}^m$. Proposition 1.25 implies that μ is the m-dimensional Lebesgue measure in $\mathbb{R}^m$, that is $\mu = \mathcal{H}^m \llcorner V$ (under the identification of V and $\mathbb{R}^m$). $\square$

As explained in 1.4, ω_α is only a normalization constant in definition 1.22. However, its choice for α integer guarantees that $\mathcal{H}^k \llcorner V$ is a k-uniform measure in $\mathbb{R}^n$ for all k-dimensional affine plane $V \subseteq \mathbb{R}^n$. In deed, this fact is obvious. We just mention the extremely interesting fact that, if k is an integer less than n, there exist k-uniform measures in $\mathbb{R}^n$ which are not of the form $\mathcal{H}^k \llcorner V$, for some k-dimensional affine plane $V \subseteq \mathbb{R}^n$.

2 Proof of Mastrand's theorem

We are now in the position to give a complete proof of theorem 1.1. For the reader convenience, we split this long proof in some parts.

The Mastrand's theorem is an immediate consequence of the following propositions.

Proposition 2.1. *Let μ be a measure as in theorem 1.1. Then, for μ-a.e. $x \in E$ there holds*

$$\emptyset \neq Tan_\alpha(\mu, x) \subseteq \{\Theta_\alpha(\mu, x)\nu \mid \nu \in \mathcal{U}_\alpha(\mathbb{R}^n)\}. \tag{15}$$

Proposition 2.2. *If $\mathcal{U}_\alpha(\mathbb{R}^n) \neq \emptyset$, then α is an integer less or equal than n.*

If we assume propositions 2.1 and 2.2, the proof of Marstrand follows easily.

Proof of theorem 1.1. Since E has positive measure, proposition 2.1 yields the existence of $x \in E$ s.t. (15) holds true. In particular, $\mathcal{U}_\alpha(\mathbb{R}^n) \neq \emptyset$; then, by proposition 2.2, we infer that α is an integer less or equal than n. $\square$

The following sections are devoted to prove propositions 2.1 and 2.2.

2.1 Proof of proposition 2.1

As for proposition 2.1, the proof is based on a very common "countable decomposition" argument.

Proof of proposition 2.1. Without loss of generality, we can assume that for all $x \in E$ $\Theta_\alpha(\mu, x)$ exists, it is positive and finite.

Step 1: Given $i, j, k \in \mathbb{N}$ consider the sets

$$E^{i,j,k} := \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{(j-1)\omega_\alpha}{i} \leq \frac{\mu(B_r(x))}{r^\alpha} \leq \frac{(j+1)\omega_\alpha}{i} \;\; \forall r \leq \frac{1}{k} \right\}.$$

By the assumption on E, we immediately see that

$$E \subseteq \bigcap_i \bigcup_{j,k} E^{i,j,k}.$$

Fix $i, j, k \in N$; we claim that for μ-a.e. $x \in E^{i,j,k}$ the following holds:

$$|\nu(B_r(y)) - \Theta_\alpha(\mu, x)\omega_\alpha r^\alpha| \leq \frac{2\omega_\alpha r^\alpha}{i} \quad \forall \nu \in \mathrm{Tan}_\alpha(\mu \llcorner E^{i,j,k}, x), \; \forall y \in \mathrm{supp}(\nu), \; \forall r > 0. \tag{16}$$

By the locality property of tangent measures (see 1.21) and (16), we obtain that for μ-a.e. $x \in E^{i,j,k}$ the following holds:

$$|\nu(B_r(y)) - \Theta_\alpha(\mu, x)\omega_\alpha r^\alpha| \le \frac{2\omega_\alpha r^\alpha}{i} \quad \forall \nu \in \mathrm{Tan}_\alpha(\mu, x),\ \forall y \in \mathrm{supp}(\nu),\ \forall r > 0. \quad (17)$$

Fix $i \in \mathbb{N}$; since $E \subseteq \bigcup_{j,k} E^{i,j,k}$, we deduce (17) holds for μ-a.e. $x \in E$. Then, we have that for μ-a.e. $x \in E$ the following holds:

$$|\nu(B_r(y)) - \Theta_\alpha(\mu, x)\omega_\alpha r^\alpha| \le \frac{2\omega_\alpha r^\alpha}{i} \quad \forall \nu \in \mathrm{Tan}_\alpha(\mu, x),\ \forall y \in \mathrm{supp}(\nu),\ \forall r > 0,\ \forall i \in \mathbb{N},$$

which immediately yields

$$\nu(B_r(y)) = \Theta_\alpha(\mu, x)\omega_\alpha r^\alpha \quad \forall \nu \in \mathrm{Tan}_\alpha(\mu, x),\ \forall y \in \mathrm{supp}(\nu),\ \forall r > 0. \quad (18)$$

In other words, $\frac{\nu}{\Theta_\alpha(\mu,x)}$ is an α-uniform measure; to conclude that $\nu' := \frac{\nu}{\Theta_\alpha(\mu,x)} \in \mathcal{U}_\alpha(\mathbb{R}^n)$, it suffices to show that $0 \in \mathrm{supp}\left(\frac{\nu}{\Theta_\alpha(\mu,x)}\right)$, that is $0 \in \mathrm{supp}(\nu)$. In deed, by the convergence in the sense of measures, for all $\rho > 0$, there holds

$$\begin{aligned}
\frac{\nu(\overline{B}_\rho)}{\rho^\alpha} &\ge \frac{1}{\rho^\alpha} \limsup_{i \to +\infty} \frac{\mu_{x,r_i}(\overline{B}_\rho)}{r_i^\alpha} \\
&\ge \limsup_{i \to +\infty} \frac{\mu(B_{\rho r_i}(x))}{(\rho r_i)^\alpha} \\
&\ge \omega_\alpha \Theta_\alpha(\mu, x) > 0.
\end{aligned}$$

Letting $\rho \uparrow r$, we deduce that

$$\nu(B_r) \ge \omega_\alpha \Theta_\alpha(\mu, x) > 0 \quad \forall r > 0. \quad (19)$$

Hence, $0 \in \mathrm{supp}(\nu)$.

Step 2: We are left with the task of proving that for all $i, j, k \in \mathbb{N}$ (16) holds for μ-a.e. $x \in E^{i,j,k}$. Since i, j, k are fixed, set $F = E^{i,j,k}$; define

$$\begin{aligned}
F_1 &:= \left\{ x \in F \;\middle|\; \lim_{r \to 0} \frac{\mu(B_r(x) \setminus F)}{r^\alpha} = 0,\ \mathrm{Tan}_\alpha(\mu, x) = \mathrm{Tan}_\alpha(\mu \llcorner F, x) \right\} \\
&= \{ x \in F \mid \Theta^*_\alpha(\mu \llcorner F^c, x) = 0,\ \mathrm{Tan}_\alpha(\mu, x) = \mathrm{Tan}_\alpha(\mu \llcorner F, x) \}.
\end{aligned}$$

Recall that $\mu(F \setminus F_1) = 0$; so, it suffices to show (16) for all $x \in F_1$. Fix $x \in F_1$, $\nu \in \mathrm{Tan}_\alpha(\mu \llcorner F, x) = \mathrm{Tan}_\alpha(\mu, x)$ and $r_i \downarrow 0$ s.t.

$$\nu_i := \frac{(\mu \llcorner F)_{x,r_i}}{r_i^\alpha} \stackrel{*}{\rightharpoonup} \nu.$$

We claim that for all $y \in \mathrm{supp}(\nu)$ there exists $(x_i)_i \subseteq F$ s.t.

$$y_i := \frac{x_i - x}{r_i} \to y.$$

By the convergence in the sense of measures, for all $\rho > 0$ we have that

$$0 < \nu(B_\rho(y)) \le \liminf_{i \to +\infty} \nu_i(B_\rho(y)) = \liminf_{i \to +\infty} \frac{\mu(B_{\rho r_i}(x + r_i y) \cap F)}{r_i^\alpha}.$$

In particular, $\mu(B_{\rho r_i}(x+r_i y)\cap F)>0$ for all i large enough. Then, it is clear that there exists $N(\rho)$ s.t. for $i>N(\rho)$ there exists $x_{i,\rho}\in B_{\rho r_i}(x+r_i y)\cap F$. So, we have that

$$\left|\frac{x_{i,\rho}-x}{r_i}-y\right|\le\rho.$$

At this point, with a diagonal argument, we built a sequence $(x_i)_i\subseteq F$ s.t.

$$\lim_{i\to+\infty}\left|\frac{x_i-x}{r_i}-y\right|=0.$$

We claim that there exists $S\subseteq\mathbb{R}^+$ at most countable s.t. for all $\rho\in\mathbb{R}^+\setminus S$ there holds

$$\lim_{i\to+\infty}\nu_i(B_\rho(y_i))=\nu(B_\rho(y)). \tag{20}$$

Notice that the center of the balls in (20) are not the same. So, we define $\xi_i\coloneqq(\nu_i)_{y_i-y,1}$. Then, (20) becomes

$$\lim_{i\to+\infty}\xi_i(B_\rho(y))=\nu(B_\rho). \tag{21}$$

Assume that $\xi_i\overset{*}{\rightharpoonup}\nu$; the existence of such $S\subseteq\mathbb{R}^+$ follows from the fact that

$$\lim_{i\to+\infty}\xi_i(A)=\nu(A)$$

for all A open s.t. $\nu(\partial A)=0$; since ν is locally finite, there exist at most countably many radii ρ s.t. $\nu(\partial B_\rho(y)$ is positive. Having said that, it is immediate to check that $\xi_i\overset{*}{\rightharpoonup}\nu$: in deed, this is obvious by the facts that $\nu_i\overset{*}{\rightharpoonup}\nu$ and $y_i\to y$.

Fix $\rho\in\mathbb{R}^+\setminus S$; let us compute

$$\lim_{i\to+\infty}\nu^i(B_\rho(y_i))=\lim_{i\to+\infty}\frac{\mu(B_{\rho r_i}(x_i)\cap F)}{r_i^\alpha}. \tag{22}$$

Since $\frac{x_i-x}{r_i}\to y_i$, there exists a constant $C>0$ s.t. $|x_i-x|\le Cr_i$ for all i. So, by the fact that $x\in F_1$, we obtain that

$$\lim_{i\to+\infty}\frac{\mu(B_{\rho r_i}(x_i)\setminus F)}{r_i^\alpha}\le\lim_{i\to+\infty}\frac{\mu(B_{(C+\rho)r_i}(x)\setminus F)}{r_i^\alpha}=0. \tag{23}$$

By (20), (22), (23) and the fact the $x\in F=E^{i,j,k}$, we deduce that

$$\nu(B_\rho(y))=\lim_{i\to+\infty}\nu_i(B_\rho(y_i))=\lim_{i\to+\infty}\frac{\mu(B_{\rho r_i}(x))}{r_i^\alpha}\in\left(\frac{(j-1)\omega_\alpha\rho^\alpha}{i},\frac{(j+1)\omega_\alpha\rho^\alpha}{i}\right). \tag{24}$$

Notice, that by definition of F, the quantity $\Theta_\alpha(\mu,x)\omega_\alpha\rho^\alpha$ belongs to the same interval. Then, we deduce that for all $\rho\in\mathbb{R}^+\setminus S$ there holds

$$|\nu(B_\rho(y))-\Theta_\alpha(\mu,x)\omega_\alpha\rho^\alpha|\le 2\frac{\omega_\alpha\rho^\alpha}{i}. \tag{25}$$

Since S is at most countable, by continuity, we deduce that (25) holds true for all $\rho\in\mathbb{R}^+$, which proves (16).

Step 3: We have shown that $\mathrm{Tan}_\alpha(\mu, x) \subseteq \Theta_\alpha(\mu, x)\mathcal{U}_\alpha(\mathbb{R}^n)$ for μ-a.e. $x \in E$. To conclude, we show that $\mathrm{Tan}_\alpha(\mu, x) \neq \emptyset$ for μ-a.e. $x \in E$. Fix any $x \in E$ s.t. $\Theta_\alpha(\mu, x) < +\infty$. Then, for all $\rho > 0$, we have that

$$\sup_{r>0} \frac{\mu(B_{\rho r}(x))}{r^\alpha} = \sup_{r>0} \frac{\mu_{x,r}(B_\rho)}{r^\alpha} < +\infty.$$

Hence, the family of measures $(r^{-\alpha}\mu_{x,r})_{r\leq 1}$ is locally uniformly bounded; by the compactness of measures with respect to the weak* convergence (in duality with $C_c(\mathbb{R}^n)$), we deduce that there exist a subsequence $r_i \downarrow 0$ and a locally finite Borel measure ν s.t.

$$\frac{\mu_{x,r}}{r^\alpha} \overset{*}{\rightharpoonup} \nu.$$

In other words, $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ by definition. □

2.2 Proof of proposition 2.2

We highlight the main steps of the proof of the proposition 2.2, which can be divided in some lemmas.

Sketch of proof of 2.2.

1. We already know that $\mathcal{U}_\alpha(\mathbb{R}^n) = \emptyset$ if $\alpha > n$ (see 1.25).
2. The fundamental step consists in showing that, if $\alpha < n$, then $\mathcal{U}_\alpha(\mathbb{R}^n) \neq \emptyset$ implies that $\mathcal{U}_\alpha(\mathbb{R}^{n-1}) \neq \emptyset$.
3. By iteration of the previous argument, we obtain that $\mathcal{U}_\alpha(\mathbb{R}^{[\alpha]}) \neq \emptyset$. Assume by contradiction that α is not integer; then $\alpha > [\alpha]$ and $\mathcal{U}_\alpha(\mathbb{R}^{[\alpha]})$ should be empty by the first of these steps. Hence, we find a contradiction.

□

We study the tangent measures to an α-uniform measure. Having in mind the heuristic description given in 1.18 and the definition 1.22, it would not by surprising that the tangent measures to an α-uniform measure are still α-uniform measures, as shown in the lemma below. The proof is very similar to that of proposition 2.1

Lemma 2.3. *Let $\alpha \geq 0$, $\mu \in \mathcal{U}_\alpha(\mathbb{R}^n)$ and $x \in supp(\mu)$. Then*

$$\emptyset \neq Tan_\alpha(\mu, x) \subseteq \mathcal{U}_\alpha(\mathbb{R}^n).$$

Remark 2.4. Given a measure μ as in theorem 1.1, proposition 2.1 guarantees that tangent measures are (up to multiplicative factors) α-uniform measures μ-a.e. In some sense, lemma 2.3 is the analogous of the following: take Σ submanifold of class C^1 in $\mathbb{R}^n$ and $x \in \Sigma$; then $\mathrm{Tan}_x(\mathrm{Tan}_x(x + \mathrm{Tan}_x\Sigma) = \mathrm{Tan}_x\Sigma$.

Proof of 2.3. The argument given in the third step of the proof of proposition 2.1 shows that $\mathrm{Tan}_\alpha(\mu, x) \neq \emptyset$ for all $x \in \mathrm{supp}(\mu)$: since μ is α-uniform, at every point $x \in \mathrm{supp}(\mu)$ there holds that $\Theta_\alpha(\mu, x) = 1$.

Now fix $x \in \mathrm{supp}(\mu), \nu \in \mathrm{Tan}_\alpha(\mu, x)$ and $r_i \downarrow 0$ s.t.

$$\nu_i := \frac{\mu_{x,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu.$$

Given $y \in \mathrm{supp}(\nu)$, arguing as in the second step of the proof of proposition 2.1, we can easily check the following facts:

- there exists a sequence $(x_i)_i \in \mathrm{supp}(\mu)$ s.t.

$$y_i \coloneqq \frac{x_i - x}{r_i} \to y;$$

- there exists a set $S \subseteq \mathbb{R}^+$ at most countable s.t. for all $\rho \in \mathbb{R}^+ \setminus S$ there holds

$$\nu(B_\rho(y)) = \lim_{i\to+\infty} \frac{\mu_{x,r_i}(B_\rho(x_i))}{r_i^\alpha} = \lim_{i\to+\infty} \frac{\mu(B_{\rho r_i}(x))}{r_i^\alpha} = \omega_\alpha \rho^\alpha,$$

  where we use also use the fact that μ is α-uniform.

- Since S is at most countable, by continuity, we deduce that

$$\nu(B_r(y)) = \omega_\alpha r^\alpha \quad \forall r > 0.$$

We only have to check that $0 \in \mathrm{supp}(\nu)$. Fix $\rho > 0$; by the weak* convergence and the fact that μ is α-uniform, it follows that

$$\nu(\overline{B}_\rho) \geq \limsup_{i\to+\infty} \frac{\mu_{x,r_i}(B_\rho)}{r_i^\alpha} = \limsup_{i\to+\infty} \frac{\mu(B_{\rho r_i}(x))}{r_i^\alpha} = \omega_\alpha \rho^\alpha > 0.$$

By approximation, we deduce that every open ball centered at the origin has positive measure ν. $\square$

The following is an elementary geometric remark.

Lemma 2.5. *Take $0 \leq \alpha < n$ and $\mu \in \mathcal{U}_\alpha(\mathbb{R}^n)$. There exists $y \in$ supp(μ) and a system of coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ s.t.*

$$\textit{supp}(\nu) \subseteq \{x_1 \geq 0\} \quad \forall \nu \in \textit{Tan}_\alpha(\mu, x). \tag{26}$$

Proof. Set $E \coloneqq \mathrm{supp}(\mu)$. Since $\alpha < n$, we claim that $E \neq \mathbb{R}^n$; in deed, we show that B_1 is not contained in E. Assume by contradiction that $B_1 \subseteq E$; then, we can use the Besicovitch's covering theorem to cover $\mathscr{L}^n$-a.a. on B_1 with at most countably many disjoint balls $(B_i)_i$ of radii at most 1. For all i, we set $B_i = B_{r_i}(x_i)$; recall that $x_i \in B_1 \subseteq E$ and $r_i < 1$. Since μ is α-uniform and $\alpha < n$, we have that

$$\mu(B_1) \geq \sum_i \mu(B_{r_i}(x_i)) = \sum_i \omega_\alpha r_i^\alpha > \sum_i \frac{\omega_\alpha}{\omega_n} \mathscr{L}^n(B_{r_i}(x_i)) = \omega_\alpha \frac{\mathscr{L}^n(B_1)}{\omega_n} = \omega_\alpha. \tag{27}$$

Then $\mu(B_1) > \omega_\alpha$; since $0 \in \mathrm{supp}(\mu)$, this yields a contradiction.

Having shown that $E \neq \mathbb{R}^n$, fix $y \notin E$; since E ia a non empty closed set, there exists $z \in E$ s.t. $\mathrm{dist}(y, E) = |y - z| \coloneqq a$. We can choose coordinates $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ s.t. $z = 0$ and $y = (-a, 0, \ldots, 0)$. Set

$$\tilde{E} \coloneqq \mathbb{R}^k \setminus B_a(y);$$

by definition of distance from a subset, we have that $E \subseteq \tilde{E}$ (see figure 1). We claim that z fulfils the required properties. Fix $\nu \in \mathrm{Tan}_\alpha(\mu, 0)$ and a sequence $r_i \downarrow 0$ s.t.

$$\nu_i = \frac{\mu_{0,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} 0.$$

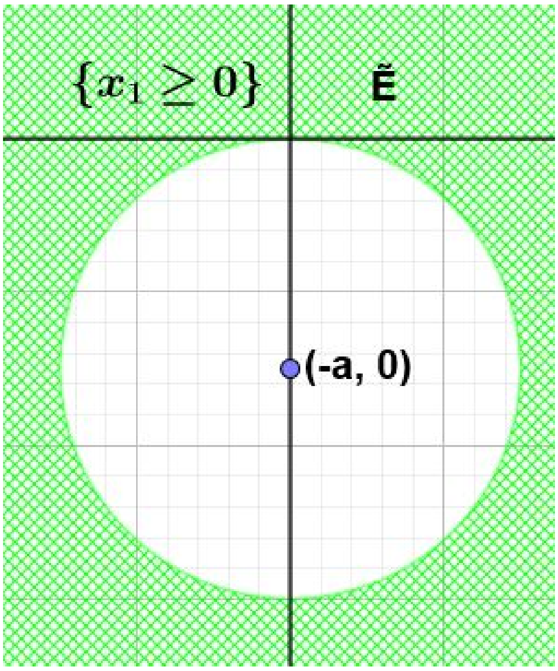

Figure 1

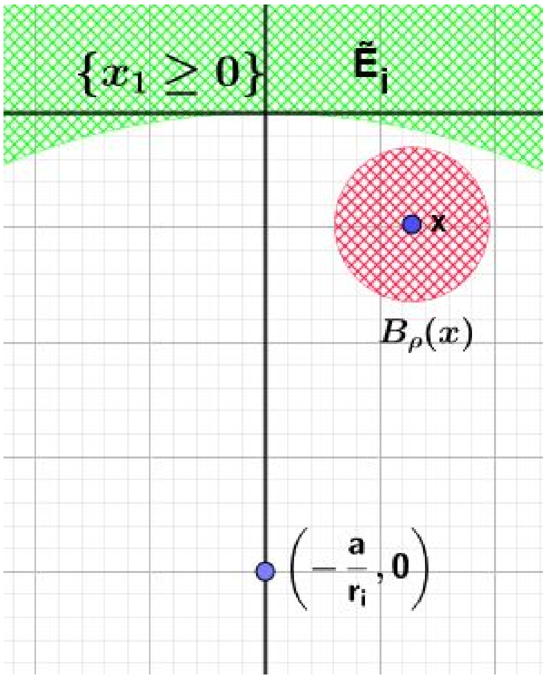

Figure 2

Clearly, the support of ν_i is contained in the set

$$\tilde{E}_i := \mathbb{R}^n \setminus B_{a/r_i}\left(\frac{y}{r_i}\right).$$

Notice that for any x in the half space $\{x_1 < 0\}$ there exist $r_0 > 0$ and $N \in \mathbb{N}$ s.t. $B_r(x) \cap \tilde{E}_i$ for all $i \geq N$ for all $r \leq r_0$ (see figure 2). Then, we obtain that $\nu_i(B_r(x)) = 0$ for all $i \geq N$ for all $r \leq r_0$. Thus, $\nu(B_r(x)) = 0$ for all $r < r_0$. This is enough to conclude that $x \notin \operatorname{supp}(\nu)$. □

The next lemma is the key step in the proof of proposition 2.2.

Lemma 2.6. *Take $0 \leq \alpha < n$ and $\mu \in \mathcal{U}_\alpha(\mathbb{R}^n)$. If $supp(\nu) \subseteq \{x_1 \geq 0\}$, then for all $\tilde{\nu} \in Tan_\alpha(\nu, 0)$ there holds*

$$supp(\tilde{\nu}) \subseteq \{x_1 = 0\}.$$

Remark 2.7. In some sense, the statement of lemma 2.6 is the analogous of the fact that the tangent space to the half space is an hyperplane of codimension 1.

Proof of 2.6. Take $\tilde{\nu} \in \operatorname{Tan}_\alpha(\nu, 0)$; given $r > 0$, define the quantities

$$b(r) := \omega_\alpha \fint_{B_r} z \, d\nu(z), \quad c(r) := \omega_\alpha \fint_{B_r} z \, d\tilde{\nu}(z),$$

where the integrals are defined component-wise. Up to the multiplicative factor ω_α, $b(r)$ and $c(r)$ are the baricenters of the measure $\nu \llcorner B_r$ and $\tilde{\nu} \llcorner B_r$, respectively. Denote $b(r) = (b_1(r), \ldots, b_n(r))$ and $c(r) = (c_1(r), \ldots, c_n(r))$. Since $\operatorname{supp}(\nu) \subseteq \{x_1 \geq 0\}$, we have that $b_1(r) \geq 0$ for all $r > 0$; similarly, since $\operatorname{supp}(\tilde{\nu}) \subseteq \operatorname{supp}(\nu) \subseteq \{x_1 \geq 0\}$, then $c_1(r) \geq 0$. If we prove that $c_1(r) = 0$, then we conclude immediately that

$$\operatorname{supp}(\tilde{\nu} \llcorner B_r) \subseteq \{x_1 = 0\} \tag{28}$$

Since $\operatorname{supp}(\tilde{\nu} \llcorner B_r) \subseteq \operatorname{supp}(\tilde{\nu}) \cap \overline{B}_r$, then the validity of (28) for all $r > 0$ yields

$$\operatorname{supp}(\tilde{\nu}) \subseteq \{x_1 = 0\}.$$

Hence, we have to show that $c_1(r) = 0$ for all $r > 0$; in deed, we will show that $c(r) = 0$ for all $r > 0$. By the α-uniform properties of $\nu, \tilde{\nu}$ ($\tilde{\nu} \in \mathcal{U}_\alpha(\mathbb{R}^n)$ because of 2.3), it follows that

$$b(r) = \frac{1}{r^\alpha} \int_{B_r} z \; d\nu(z), \quad c(r) = \frac{1}{r^\alpha} \int_{B_r} z \; d\tilde{\nu}(z). \tag{29}$$

We have to show that $c(r) = 0$ for all $r > 0$. The idea is to study the limiting behaviour of $b(r)$ when $r \downarrow 0$.

Step 1: We will check that there exists a constant $C(\alpha) > 0$, depending only on α, s.t. for all $r > 0$ for all $y \in \operatorname{supp}(\nu) \cap B_{2r}$ there holds

$$|< b(r), y >| \leq C(\alpha) \, |y|^2 . \tag{30}$$

Now, we show how to conclude the proof. Fix $r_i \downarrow 0$ s.t.

$$\nu_i := \frac{\nu_{0,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} 0.$$

By the weak* convergence of measures, it follows that there exists $S \subseteq \mathbb{R}^+$ at most countable s.t. for all $\rho \in \mathbb{R}^+ \setminus S$ there holds

$$\lim_{i \to +\infty} b(r_i \rho) = c(\rho); \tag{31}$$

in deed, we have

$$b(\rho r_i) = \frac{1}{\rho^\alpha r_i^\alpha} \int_{B_{\rho r_i}} z \; d\nu(z) = \frac{1}{\rho^\alpha} \int_{B_\rho} z \; d\nu_i(z); \tag{32}$$

by the weak* convergence of measure, we deduce that

$$\lim_{i \to +\infty} \frac{1}{\rho^\alpha} \int_{B_\rho} z \; d\nu_i(z) = \frac{1}{\rho} z \; d\tilde{\nu}(z)$$

for all $\rho \in \mathbb{R}^+$ s.t. $\tilde{\nu}(\partial B_\rho) = 0$. Since $\tilde{\nu}$ is locally finite, the set of the radii ρ for which $\tilde{\nu}(\partial B_\rho) > 0$ is at most countable. Then, we obtain (31) for all $\rho \in \mathbb{R}^+ \setminus S$.

Take $\rho \in \mathbb{R}^+ \setminus S$ and $z \in B_\rho \cap \operatorname{supp}(\tilde{\nu})$. As in the second step of the proof of proposition 2.1, we can check that there exists a sequence $(y_i)_i \in \operatorname{supp}(\nu)$ s.t.

$$z_i := \frac{y_i}{r_i} \to z. \tag{33}$$

Clearly, we can assume that $|y_i| \leq 2 r_i \rho$ for all i; since $y_i \in \operatorname{supp}(\nu) \cap B_{2\rho}$, we can use (31), (33) and the estimate proved in (30) to obtain the followings:

$$|< c(\rho), z >| = \lim_{i \to +\infty} \frac{|< b(\rho r_i), y_i >|}{r_i} = C(\alpha) \lim_{i \to 0} \frac{|y_i|^2}{r_i} = 0. \tag{34}$$

To resume, we have shown that

$$< c(\rho), z >= 0 \quad \forall \rho \in \mathbb{R}^+ \setminus S \; \forall z \in B_\rho \cap \operatorname{supp}(\tilde{\nu}); \tag{35}$$

Given $\rho \in \mathbb{R}^+ \setminus S$, (35) immediately yields

$$0 = \frac{1}{\rho^\alpha} \int_{B_\rho} < c(\rho), z > \ d\tilde{\nu}(z) =< c(\rho), \frac{1}{\rho^\alpha} \int_{B_\rho} z \ d\tilde{\nu}(z) >= |c(\rho)| \, .$$

Since S is countable, by continuity, we deduce that $c(\rho) = 0$ for all $\rho > 0$. So, the proof is concluded, modulo checking (30).

Step 2: We are left with the task of showing that there exists a constant $C(\alpha) > 0$, depending only on α, s.t. for all $r > 0$ for all $y \in \operatorname{supp}(\nu) \cap B_{2r}$ (30) holds true. We start with the trivial identity

$$2 < x, y >= |y|^2 + (r^2 - |x-y|^2) - (r^2 - |x|^2) \quad \forall x, y \in \mathbb{R}^n \ \forall r > 0. \tag{36}$$

Then, take $r > 0$ and $y \in \operatorname{supp}(\nu) \cap B_{2r}$; we have

$$\begin{aligned}
2 \left|< b(r), y >\right| &= r^{-\alpha} \left| \int_{B_r} 2 < x, y > \ d\nu(x) \right| &(37)\\
&= r^{-\alpha} \left| |y|^2 \nu(B_r) + \int_{B_r} (r^2 - |x-y|^2) \ d\nu(x) - \int_{B_r} (r^2 - |x|^2) \ d\nu(x) \right| &(38)\\
&\leq \omega_\alpha |y|^2 + r^{-\alpha} \left| \int_{B_r} (r^2 - |x-y|^2) \ d\nu(x) - \int_{B_r(y)} (r^2 - |x-y|^2) \ d\nu(x) \right| \\
&\leq \omega_\alpha \left|y^2\right| + r^{-\alpha} \int_{B_r \setminus B_r(y)} \left|r^2 - |x-y|^2\right| \ d\nu(x) \\
&\quad + r^{-\alpha} \int_{B_r(y) \setminus B_r} \left|r^2 - |x-y|^2\right| \ d\nu(x) &(39)\\
&\leq \omega_\alpha |y|^2 + 4r^{-\alpha+1} |y| \left[\nu(B_r \setminus B_r(y)) + \nu(B_r(y) \setminus \nu(B_r))\right] \\
&= \omega_\alpha |y|^2 + 4r^{-\alpha+1} |y| \, \nu((B_r \setminus B_r(y)) \cup (B_r(y) \setminus \nu(B_r)). &(40)
\end{aligned}$$

In (37) we used (36); in (38) we used the fact that $\nu \in \mathcal{U}_\alpha(\mathbb{R}^n)$ and the change of variable formula for α-uniform measures stated in 1.24 (it applies since $y \in \operatorname{supp}(\nu)$); in (39) we used the following facts: since $y \in B_{2r}$, for all $x \in B_r \setminus B_r(y)$ there holds

$$0 \leq |x-y|^2 - r^2 \leq |x-y|^2 - |x|^2 = (|x-y| - |y|)(|x-y| + |y|) \leq 4r\,|y| \, ;$$

similarly, for $x \in B_r(y) \setminus B_r$, there holds

$$0 \leq r^2 - |x-y|^2 \leq 4r\,|y| \, .$$

At this point, two cases may occur.

- If $|y| < r$, then we have that

$$(B_r \setminus B_r(y)) \cup (B_r(y) \setminus \nu(B_r) \subseteq B_{r+|y|} \setminus B_{r-|y|}.$$

Then, for $|y| < r$ and $y \in \operatorname{supp}(\nu)$, (40) yields

$$\begin{aligned}
2 \left|< b(r), y >\right| &\leq \omega_\alpha |y|^2 + 4\,|y|\, r^{1-\alpha} [\nu(B_{r+|y|}) - \nu(B_{r-|y|})] \\
&= \omega_\alpha |y|^2 + 4\omega_\alpha |y|\, r^{1-\alpha} [(r+|y|)^\alpha - [r - |y|]^\alpha] \\
&= \omega_\alpha |y|^2 + 4\omega_\alpha |y|\, r \left[\left(1 + \frac{|y|}{r}\right)^\alpha - \left(1 - \frac{|y|}{r}\right)^\alpha \right] &(41)
\end{aligned}$$

If $\alpha \in [0,1]$, the function $\psi(s) := s^\alpha$ is α-Hölder continuous in $[0,+\infty)$. Denote by $C(\alpha)$ a positive contant that may change from line to line, depending only by α. Then, (41) immediately implies that

$$\begin{aligned}
2\left|< b(r), y >\right| &\leq \omega_\alpha |y|^2 + 4C(\alpha)\omega_\alpha |y|\, r \left(\frac{2|y|}{r}\right)^\alpha \\
&= \omega_\alpha |y|^2 + C(\alpha)\frac{|y|^2}{r^{\alpha-1}} \\
&\leq \omega_\alpha |y|^2 + C(\alpha)|y|^2 \\
&= C(\alpha)|y|^2 . \qquad (42)
\end{aligned}$$

If $\alpha > 1$, the computation is very similar: recall that the function $\psi(s) := s^\alpha$ is convex $[0,+\infty)$; hence, there holds

$$\left(1 + \frac{|y|}{r}\right)^\alpha - \left(1 - \frac{|y|}{r}\right)^\alpha \leq \left(\sup_{s\in(0,2)} \psi'(s)\right) 2\frac{|y|}{r} = C(\alpha)\frac{y}{r}.$$

Then, (41) yields

$$2\left|< b(r), y >\right| \leq \omega_\alpha |y|^2 + C(\alpha)|y|\, r\frac{|y|}{r} = C(\alpha)|y|^2 .. \qquad (43)$$

Hence, for all $\alpha \geq 0$, for all $r > 0$ for all $|y| < r$ s.t. $y \in \mathrm{supp}(\nu)$, we obtain (30).
Step 2: If $r \leq |y| \leq 2r$, we have that

$$(B_r \setminus B_r(y)) \cup (B_r(y) \setminus \nu(B_r) \subseteq B_{r+|y|}.$$

Then, (30) yields

$$\begin{aligned}
2\left|< b(r), y >\right| &\leq \omega_\alpha |y|^2 + 4|y|\, r^{1-\alpha}\nu(B_{r+|y|}) \\
&= \omega_\alpha |y|^2 + 4\omega_\alpha |y|\, r^{1-\alpha}(r + |y|)^\alpha \\
&= \omega_\alpha |y|^2 + C(\alpha)|y|\, r \left(1 + \frac{|y|}{r}\right)^\alpha \\
&\leq \omega_\alpha |y|^2 + C(\alpha)|y|\, r \\
&\leq C(\alpha)|y|^2 .
\end{aligned}$$

We have shown that (30) holds true for all $r > 0$ for all $y \in B_{2r} \cap \mathrm{supp}(\nu)$.

Hence, the proof is concluded. □

Now, we are in the position to give a complete proof of proposition 2.2.

Proof of 2.2. We have already noticed in 1.25 that α as to be at most n. However, if $\alpha = n$, the proof is concluded; so, we can assume that $\alpha < n$.
We claim that $\mathcal{U}_\alpha(\mathbb{R}^n) \neq \emptyset$ implies that $\mathcal{U}_\alpha(\mathbb{R}^{n-1}) \neq \emptyset$.

- Take $\mu \in \mathcal{U}_\alpha(\mathbb{R}^n)$; by lemma (2.5), we can find $y \in \mathrm{supp}(\mu)$ and a system of coordinates $(x_1, \dots, x_n)$ on $\mathbb{R}^n$ s.t. $\mathrm{supp}(\mu) \subseteq \{x_1 \geq 0\}$.
- Take $\nu \in \mathrm{Tan}_\alpha$; by lemma (2.3), we deduce that $\nu \in \mathcal{U}_\alpha(\mathbb{R}^n)$. It is immediate to check that $\mathrm{supp}(\nu) \subseteq \mathrm{supp}(\mu) \subseteq \{x_1 \geq 0\}$.

- Finally, consider $\tilde{\nu} \in \mathrm{Tan}_\alpha(\nu, 0)$; by lemma 2.3, $\tilde{\nu} \in \mathcal{U}_\alpha(\mathbb{R}^n)$; moreover, lemma 2.6 implies that $\mathrm{supp}(\tilde{\nu}) \subseteq \{x_1 = 0\}$. Then, $\tilde{\nu}$ can be naturally seen as an element of $\mathcal{U}_\alpha(\mathbb{R}^{n-1})$.

At this point, we show that α is integer. By iteration of the previous argument, we obtain that $\mathcal{U}_\alpha(\mathbb{R}^{[\alpha]}) \neq \emptyset$. Assume by contradiction that α is not integer; then $\alpha > [\alpha]$ and $\mathcal{U}_\alpha(\mathbb{R}^{[\alpha]})$ should be empty by the first of these steps. Hence, we find a contradiction. $\square$

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