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Cardinal Characteristics and Large Cardinals

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Disclaimer

This notes have been typeset in LaTeX "on the fly" during the course on Cardinal Characteristics and Large Cardinals held by Andrew Brooke-Taylor at the University of Leeds in the fall of 2017/2018, and they have not been reviewed yet. They are primarily intended for personal use, and in particular they are *not* the official notes of the course. As a consequence, they can be *very* inaccurate, messy, and they may contain serious errors. Emails pointing out errors, mistakes, etc. are very welcome.

Deliberate omissions are marked [like this], while MISSING denotes that I was unable to transcribe something (which can be a single word, an entire theorem, etc.)

Info

You can find this notes on `http://poisson.phc.dm.unipi.it/~mennuni/Mennuni_ccalc_notes.pdf` (but they could be moved; in case, check my Leeds webpage[1]). You can contact me at `mmrm@leeds.ac.uk`. This version has been compiled on December 11, 2017. To get the source code click on the paper clip.

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[1]Which does not exist yet, otherwise I would have linked that.

Chapter 1

02/10

Assumptions are color coded: black (white on the board) means κ regular, red means $\kappa^{<\kappa} = \kappa$ and blue means κ inaccessible.

Cardinal characteristics of the continuum have been studied a lot, but there is still work ongoing. E.g. it was recently shown that $\mathfrak{p} = \mathfrak{t}$, and there is a recent preprint with 10 different cardinals in Chicoń's diagram.

This course is about generalisation to higher cardinals: replace ω with κ and finite with $< \kappa$.

We are going to start from scratch from cardinal characteristics of the continuum in a uniform approach for what will come later.

1.1 Good References

- For classical cardinal characteristics of the continuum, Blass's article inside *Handbook of set theory.*
- For large cardinals, Kanamori's book.

1.2 Bounding and Dominating Number

Definition 1.1 (κ regular)**.** For functions $f, g\colon \kappa \to \kappa$, write $f \leq^* g$ (f is *eventually dominated* by g) to mean

$$\exists \alpha < \kappa \, \forall \beta \geq \kappa \; f(\beta) \leq g(\beta)$$

Remark 1.2. As κ is regular, this is equivalent to ask that $f \leq g$ on all but $< \kappa$ many points.

Another reason for choosing κ to be regular is because otherwise the increasing functions wouldn't be dense (cofinal) in this preorder.

Definition 1.3. We define

$$\mathfrak{b}_\kappa := \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \kappa^\kappa \wedge \forall g\colon \kappa \to \kappa\ \exists f \in \mathcal{F}\ f \not\leq^* g\}$$
$$\mathfrak{d}_\kappa := \min\{|\mathcal{G}| \mid \mathcal{G} \subseteq \kappa^\kappa \wedge \forall f\colon \kappa \to \kappa\ \exists g \in \mathcal{G}\ f \leq^* g\}$$

In other words, $\mathfrak{b}_\kappa$ is the least size of an unbounded set, while $\mathfrak{d}_\kappa$ is the least size of a dominating set.

Remark 1.4. $\not\leq^*$ means $\neg(\leq^*)$. Later in the course we will also consider $(\neg \leq)^*$, which is a different object.

Remark 1.5. Every dominating set is unbounded. In particular, $\mathfrak{b}_\kappa \leq \mathfrak{d}_\kappa$.

These notions can be generalised:

Definition 1.6. Suppose $(\mathbb{P}, \leq)$ is a preorder such that[1] $\forall\, p \in \mathbb{P}\ \exists q \in \mathbb{P}\ q > p$. Then U is an *unbounded set* iff $\forall q \in \mathbb{P}\ \exists p \in U\ p \not\leq q$, and D is a *dominating set* iff $\forall p \in \mathbb{P}\ \exists q \in D\ p \leq q$. We define

$$\mathfrak{b}(\mathbb{P}) := \min\{|U| \mid U \text{ unbounded}\} \qquad \mathfrak{d}(\mathbb{P}) := \min\{|D| \mid D \text{ dominating}\}$$

Example 1.7 (κ-meagre sets)**.** The *generalised Baire space* is κ^κ with the *box topology*, generated by sets of the form

$$[s] = \{f \in \kappa^\kappa \mid f \restriction |s| = s\}$$

as s varies in $\kappa^{<\kappa}$. Similarly, the *generalised Cantor space* is 2^κ with the box topology.

Remark 1.8. In κ^κ and 2^κ

- The intersection of fewer than κ many open sets is open[2].
- There is an open base of size κ, because $\kappa^{<\kappa} = \kappa$.
- In the ω case, *the* Baire space ω^ω is *a* Baire space[3] (definition later).

Definition 1.9. In a topological space,

- A set X is *nowhere dense* iff for any open set V there is an open subset $U \subseteq V$ such that $U \cap X = \emptyset$.
- X is *κ-meagre* iff it is a union of κ-many nowhere dense sets. Let $\mathcal{M}_\kappa$ be the set of κ-meagre subsets of the topological space at hand. If κ is clear from context we may just say *meagre*.

[1] Otherwise you get boring stuff: the singleton a maximal element is a dominating set, and there are no unbounded sets.

[2] This only works because κ is regular. Also, the box topology has a universal property similar to the one enjoyed by the product topology, but subject to this requirement.

[3] Apparently people manage to avoid confusion even in languages with no articles.

Remark 1.10. $\mathcal{M}_\kappa$ is a κ-ideal, since subsets of a nowhere dense sets are nowhere dense, and the union of κ-many meagre sets is κ-meagre.

Example 1.11. Consider $(\mathcal{M}_\kappa, \subseteq)$. What are $\mathfrak{b}$ and $\mathfrak{d}$ for this partial order?

$$\mathfrak{b}(\mathcal{M}_\kappa, \subseteq) = \min\{|\mathcal{U}| \mid \mathcal{U} \subseteq \mathcal{M}_\kappa \wedge \forall Y \in \mathcal{M}_\kappa \, \exists X \in \mathcal{U} \; X \not\subseteq Y\}$$

In other words, it is the least cardinality of a set of meagre sets whose union is not meagre. This is known as the *additivity* $\operatorname{add}(\mathcal{M}_\kappa)$ of the meagre ideal. Dually, $\mathfrak{d}(\mathcal{M}_\kappa, \subseteq)$ is the least cardinality of a cofinal subset of $\mathcal{M}_\kappa$, and is denoted with $\operatorname{cof}(\mathcal{M}_\kappa)$. Under the "red" assumptions[4], $\operatorname{add}(\mathcal{M}_\kappa) \leq \operatorname{cof}(\mathcal{M}_\kappa)$.

Remark 1.12. The things above apply to both 2^κ and κ^κ. But let's say[5] we are working in 2^κ.

Proposition 1.13. Let $(\mathbb{P}, \leq)$ be a preorder such that $\forall p \, \exists q \; q > p$. Then

$$\mathfrak{b}(\mathbb{P}) = \operatorname{cf}(\mathfrak{b}(\mathbb{P})) \leq \operatorname{cf}(\mathfrak{d}(\mathbb{P})) \leq \mathfrak{d}(\mathbb{P}) \leq |\mathbb{P}|$$

Proof. If B is unbounded with $|B| = \mathfrak{b}(\mathbb{P})$ but the latter is singular, then we can write $B = \bigcup_{\alpha < \operatorname{cf} \mathfrak{b}(\mathbb{P})} B_\alpha$, where $\forall \alpha \; |B_\alpha| < \mathfrak{b}$. Then we can choose q_α such that $p \leq q_\alpha$ for all $p \in B_\alpha$, and $\{q_\alpha \mid \alpha \in \operatorname{cf}(\mathfrak{b}(\mathbb{P}))\}$ would be unbounded, contradicting minimality of $|B|$.

The rest of the proof is left as an exercise. $\square$

[4] Also we need the non-existence of maximal elements.

[5] Actually, if κ is not weakly compact, the two spaces are homeomorphic.

Chapter 2

03/10

2.1 Singular Dominating Numbers

Question 2.1. Can $\mathfrak{d}(\mathbb{P})$ be singular?

Let's elaborate on that with an example.

Example 2.2. Let β, δ be infinite cardinals such that[1] $\mathrm{cf}(\beta) = \beta \leq \mathrm{cf}(\delta) \leq \delta = \delta^{<\beta}$. Consider the partial order $\mathbb{Q}$ with underlying set $\beta \times [\delta]^{<\beta}$ and $(\rho, x) \leq (\sigma, y)$ iff $\rho \leq \sigma$ and $x \subseteq y$.

Claim. $\mathfrak{b}(\mathbb{Q}) = \beta$ and $\mathfrak{d}(\mathbb{Q}) = \delta$.

Proof. If $B \subseteq \mathbb{Q}$ and $|B| < \beta$, take $\sigma := \sup\{\rho \mid \exists x \ (\rho, x) \in B\}$ and let $y := \bigcup\{x \mid \exists p \ (p, x) \in B\}$. Then (σ, y) is an upper bound for B, so $\mathfrak{b}(\mathbb{Q}) \geq \beta$. To show equality, notice that $\{(\alpha, \emptyset) \mid \alpha < \beta\}$ is unbounded.

Now suppose $D \subseteq \mathbb{Q}$ is a dominating set such that $|D| < \delta$. Consider $X := \bigcup\{x \mid (\rho, x) \in D\}$. If δ is regular, then obviously $|X| < \delta$. Otherwise, by the previous Proposition, $|X| \leq |D| \cdot \beta < \delta$. Take $\gamma \in \delta \setminus X$. Then $(0, \{\gamma\})$ is not dominated by any element of D, and this shows $\mathfrak{d}(\mathbb{Q}) \geq \delta$. But $|\mathbb{Q}| = \beta \times \delta^{<\beta} = \delta$. □

Definition 2.3. A function $f\colon \mathbb{P} \to \mathbb{Q}$ is a *cofinal embedding* iff

- $\forall p, p' \in \mathbb{P} \ p \leq p' \iff f(p) \leq_{\mathbb{Q}} f(p')$, and
- $\forall q \in \mathbb{Q} \ \exists p \in \mathbb{P} \ (q \leq f(p))$.

Lemma 2.4. If $f\colon \mathbb{P} \to \mathbb{Q}$ is a cofinal embedding, then $\mathfrak{b}(\mathbb{P}) = \mathfrak{b}(\mathbb{Q})$ and $\mathfrak{d}(\mathbb{P}) = \mathfrak{d}(\mathbb{Q})$.

Proof. Chase around unbounded or dominating sets. □

[1] E.g. under GCH let $\beta = \aleph_1$ and $\delta = \aleph_{\aleph_{\omega_2}}$.

So we may try to embed our contrived example above into a more natural object.

Theorem 2.5 (Hechler)**.** In the case ω, if $\mathbb{P}$ is such that every countable subset of $\mathbb{P}$ has an upper bound, then there is a forcing extension of the universe in which $\mathbb{P}$ cofinally embeds into $(\omega^\omega, \leq^*)$.

Theorem 2.6 (Cummings, Shelah, $\kappa = \kappa^{<\kappa}$)**.** Suppose $\mathbb{P}$ is a well-founded poset with $\mathfrak{b}(\mathbb{P}) \geq \kappa^+$. Then there is a forcing $\mathbb{D}(\kappa, \mathbb{P})$ such that

1. $\mathbb{D}(\kappa, \mathbb{P})$ is κ-closed and κ^+-c.c. In particular it preserves cardinals and cofinalities.
2. $V^{\mathbb{D}(\kappa,\mathbb{P})} \models \mathbb{P}$ cofinally embeds into $(\kappa^\kappa, \leq^*)$.
3. If $V \models \mathfrak{b}(\mathbb{P}) = \beta$, then $V^{\mathbb{D}(\kappa,\mathbb{P})} \models \mathfrak{b}_\kappa = \beta$
4. If $V \models \mathfrak{d}(\mathbb{P}) = \delta$, then $V^{\mathbb{D}(\kappa,\mathbb{P})} \models \mathfrak{d}_\kappa = \delta$

Lemma 2.7. Every poset has a well-founded dominating subset.

Proof. Just keep on choosing elements by induction. □

Since then the inclusion map will be a cofinal embedding, the well-foundedness hypothesis in the Theorem above is not really restrictive.

2.2 Beyond Preorders: Galois-Tukey Connections

Consider triples $\mathbb{A} = (A_-, A_+, A)$, where A is a binary with domain A_- and codomain A_+, i.e. $A \subseteq A_- \times A_+$.

Definition 2.8. The *norm* $\|A\|$ of A is defined as

$$\|A\| = \min\{|Y| \mid Y \subseteq A_+ \wedge \forall x \in A_- \; \exists y \in Y \; (x \; A \; y)\}$$

So, basically, $\|A\|$ is $\mathfrak{d}$ for A. In fact, another notation is $\mathfrak{d}(A)$. What about $\mathfrak{b}$? The nice thing about Galois-Tukey connections is that they allow you to dualise things:

Definition 2.9. The *dual* of $\mathbb{A}$ is $\mathbb{A}^\perp := (A_+, A_-, \neg\check{A})$, where $y \; \check{A} \; x \equiv x \; A \; y$.

Pictorially, the dual of R is $\not{\text{Я}}$. Now we have, by spelling out the definitions,

$$\|A^\perp\| = \min\{|Y| \mid Y \subseteq A_- \wedge \forall x \in A^+ \; \exists y \in Y \; \neg(y \; A \; x)\}$$

and that's exactly $\mathfrak{b}(A)$. This is the sense in which $\mathfrak{b}$ and $\mathfrak{d}$ are dual.

Definition 2.10. A *morphism* $\Phi\colon \mathbb{A} \to \mathbb{B}$ is a pair of functions $\Phi = (\Phi_-, \Phi_+)$ such that

- $\Phi_+\colon A_+ \to B_+$
- $\Phi_-\colon B_- \to A_-$
- $\forall a \in A_+\ \forall b \in B_-\ \Phi_-(b)\ A\ a \Longrightarrow b\ B\ \Phi_+(a)$.

Terminology of Vojtáš: a Galois-Tukey connection from $\mathbb{B}$ to $\mathbb{A}$ is a morphism[2] from $\mathbb{A}$ to $\mathbb{B}$.

Exercise 2.11. If there is a morphism $\mathbb{A} \to \mathbb{B}$ (we write that as $\mathbb{A} \preceq \mathbb{B}$), then $\|\mathbb{A}\| \geq \|\mathbb{B}\|$ and $\|\mathbb{A}^\perp\| \leq \|\mathbb{B}^\perp\|$, i.e. $\mathfrak{d}(\mathbb{A}) \geq \mathfrak{d}(\mathbb{B})$ and $\mathfrak{b}(\mathbb{A}) \leq \mathfrak{b}(\mathbb{B})$.

Remark 2.12. This is easier to apply than cofinal embeddings: the condition is an "if… then", not an "if and only if".

Exercise 2.13. Express the least cardinality $\mathrm{non}(\mathcal{M}_\kappa)$ of a non-meagre set as $\mathfrak{b}$ of something and the least number $\mathrm{cov}(\mathcal{M}_\kappa)$ of meagre sets require to cover all of κ^κ as $\mathfrak{d}$ of something.

[2] Yes, these things do form a category.

Chapter 3

09/10

3.1 Examples of Triples and Morphisms

Example 3.1. $\mathcal{D} \coloneqq (\kappa^\kappa, \kappa^\kappa, \leq^*)$

Example 3.2. Let $\mathrm{Cof}(\mathcal{M}_\kappa) \coloneqq (\mathcal{M}_\kappa, \mathcal{M}_\kappa, \subseteq)$. Then $\mathfrak{d}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa)$ and $\mathfrak{b}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{M}_\kappa)$.

Solution of Exercise 2.13. Let $\mathrm{Cov}(\mathcal{M}_\kappa) \coloneqq (2^\kappa, \mathcal{M}_\kappa, \in)$. Then $\mathfrak{d}(\mathrm{Cov}(\mathcal{M}_\kappa))$ equals

$$\min\{|\mathcal{U}| \mid \mathcal{U} \subseteq \mathcal{M}_\kappa \wedge \forall x \in 2^\kappa \; \exists X \in \mathcal{U} \; x \in X\}$$

i.e. the least size of a set of meagre sets that covers 2^κ, i.e. $\mathrm{cov}(\mathcal{M}_\kappa)$.

On the other hand, $\mathfrak{b}(\mathrm{Cov}(\mathcal{M}_\kappa))$ is the least size of a non meagre set, i.e. $\mathrm{non}(\mathcal{M}_\kappa)$, as can be seen by writing it as

$$\min\{|Y| \mid Y \subseteq 2^\kappa \wedge \forall X \in \mathcal{M}_\kappa \; \exists y \in Y \; y \notin X\}$$ □

Proposition 3.3. There is a morphism $\Phi\colon \mathrm{Cof}(\mathcal{M}_\kappa) \to \mathrm{Cov}(\mathcal{M}_\kappa)$

Proof. We have to find maps

$$\Phi_+\colon \mathcal{M}_\kappa \to \mathcal{M}_\kappa \qquad \Phi_-\colon 2^\kappa \to \mathcal{M}_\kappa$$

such that if $\Phi_-(x) \subseteq Y$ then $x \in \Phi_+(Y)$. Take $\Phi_+ = \mathrm{id}_{\mathcal{M}_\kappa}$ and $\Phi_-(x) = \{x\}$. □

From this and Exercise 2.11 we immediately get

Corollary 3.4. $\mathfrak{b}(\mathrm{Cof}) \leq \mathfrak{b}(\mathrm{Cov})$ and $\mathfrak{d}(\mathrm{Cof}) \geq \mathfrak{d}(\mathrm{Cov})$. In other words, $\mathrm{add}(\mathcal{M}_\kappa) \leq \mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa) \geq \mathrm{cov}(\mathcal{M}_\kappa)$.

Exercise 3.5. Try to proof the above inequalities directly from the definitions. It should boil down to the morphism above.

Proposition 3.6. There is a morphism[1] $\Psi\colon \mathrm{Cof}(\mathcal{M}_\kappa) \to \mathrm{Cov}(\mathcal{M}_\kappa)^\perp$.

Proof. We have to find maps

$$\Psi_+\colon \mathcal{M}_\kappa \to 2^\kappa \qquad \Psi_-\colon \mathcal{M}_\kappa \to \mathcal{M}_\kappa$$

such that if $\Psi_-(X) \subseteq Y$ then $X \not\ni \Psi_+(Y)$. Let $\Psi_- = \mathrm{id}_{\mathcal{M}_\kappa}$ and let $\Psi_+(Y)$ be any element[2] $y \in 2^\kappa \setminus Y$. □

We therefore have the following picture, where arrows mean $\leq$:

$$\begin{array}{ccc} \mathrm{non}(\mathcal{M}_\kappa) & \longrightarrow & \mathrm{cof}(\mathcal{M}_\kappa) \\ \uparrow & \mathfrak{b} \longrightarrow \mathfrak{d} & \uparrow \\ \mathrm{add}(\mathcal{M}_\kappa) & \longrightarrow & \mathrm{cov}(\mathcal{M}_\kappa) \end{array}$$

Example 3.7. Let $\mathcal{E} = (\kappa^\kappa, \kappa^\kappa, \neq^*)$, where for $f, g\colon \kappa \to \kappa$ we say that f is *eventually different from* g, written $f \neq^* g$, if $\exists \alpha < \kappa\, \forall \beta \geq \alpha\; f(\beta) \neq g(\beta)$.

Remark 3.8. $\neq^*$ is symmetric, but here we are thinking of it in a "partial order" sense. Distinguishing left and right in this context is very important.

We have

$$\left\|\mathcal{E}^\perp\right\| = \mathfrak{b}(\neq^*) = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \kappa^\kappa \wedge \forall g \in \kappa^\kappa\; \exists f \in \mathcal{F}\; \neg f \neq^* g\}$$

Recall that $\neg f \neq^* g$ means $\forall \alpha < \kappa\; \exists \beta \geq \alpha\; f(\beta) = g(\beta)$. Also

$$\|\mathcal{E}\| = \mathfrak{d}(\neq^*) = \min\{|\mathcal{G}| \mid \mathcal{G} \subseteq \kappa^\kappa \wedge \forall f \in \kappa^\kappa\; \exists g \in \mathcal{G}\; f \neq^* g\}$$

Proposition 3.9. $\mathcal{D} \preceq \mathcal{E}$.

Proof. One morphism is given by $\Phi_+ \coloneqq \kappa^\kappa \to \kappa^\kappa$ defined as $d \mapsto (\Phi_+(d)(\alpha) \coloneqq d(\alpha)+1)$ and $\Phi_-\colon \kappa^\kappa \to \kappa^\kappa$ the identity. If $\Phi_-(e) \leq^* d$ then $e \neq^* \Phi_+(d)$. □

Proposition 3.10. $\mathcal{D} \preceq \mathcal{E} \preceq \mathrm{Cov}(\mathcal{M}_\kappa)$

Proof. We want $\Phi_+\colon \kappa^\kappa \to \mathcal{M}_\kappa$ and $\Phi_-\colon \kappa^\kappa \to \kappa^\kappa$ such that if $\Phi_-(x) \neq^* g$ then $x \in \Phi_+(g)$. Let $\Phi_- = \mathrm{id}_{\kappa^\kappa}$, and define

$$\Phi_+(f) \coloneqq \{g \mid g \neq^* f\}$$

[1] Recall that $\mathrm{Cov}(\mathcal{M}_\kappa)^\perp = (\mathcal{M}_\kappa, 2^\kappa, \not\ni)$.

[2] Here we hare using the $\kappa^{<\kappa} = \kappa$, because if 2^κ turned out to be meagre...

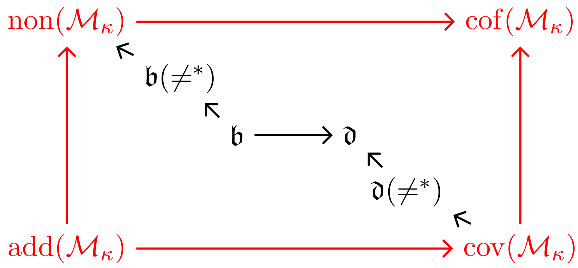
The point is that for every $f \in \kappa^\kappa$ the set $\{g \mid g \neq^* f\}$ is meagre. The reason for this is that

$$\{g \mid g \neq^* f\} = \bigcup_{\alpha<\kappa} \{g \mid \forall\beta \geq \alpha\ g(\beta) \neq f(\beta)\}$$

And each of the sets we're taking the union of, i.e. for fixed α, is nowhere dense, because if $s \in \kappa^{<\kappa}$ defines an open set, extend s to $t \in \kappa^\kappa$ taking the value $f(\beta)$ on some $\beta \geq \alpha$. □

Remark 3.11. Pay attention to the last step in the proof above, since we are going to use similar tricks often.

As a result of the Proposition, the diagram becomes



Spoiler 3.12. We will show later that $(2^\kappa, \mathcal{M}_\kappa, \in) \equiv (\kappa^\kappa, \mathcal{M}_\kappa, \in)$.

Chapter 4

10/10

4.1 κ^κ vs 2^κ

Claim. Meagre sets in κ^κ are "basically the same" as meagre sets in 2^κ. More precisely, there is an homeomorphic embedding of κ^κ into 2^κ with comeagre image.

Proof. Consider the function $\varphi\colon \kappa^\kappa \to 2^\kappa$ sending f to $f(0)$ many 0's, then $1+f(1)$, many 1's, then $1+f(2)$ many 0's etc. More formally, define $\varphi(f) := \bigcup_{\alpha<\kappa} s_f(\alpha)$, where $s_f\colon \kappa \to 2^{<\kappa}$, $s_f(\beta) \supseteq s_f(\alpha)$ for $\beta \geq \alpha$ is defined by recursion by letting $s_f(\beta)$ be $\bigcup_{\alpha<\beta} s_f(\alpha)$ followed by $1+f(\beta)$ many 0's if β is even and nonzero, and $(1+f(\beta))$ many 1's if β is odd, or $f(0)$ many 0's if $\beta = 0$.

This is an homeomorphism to its range. To see this, consider that the open base set $[t]$, for $t \in \kappa^\kappa$ maps to $[s_t(|t|) \frown r]$, where r is 0 if $|t|$ is even and 1 if $|t|$ is odd. So our map is open. To see it is continuous, notice that anyting in $2^{<\kappa}$ is of the form $s_t(|t|) \frown r$, where r is α many 0's or 1's. So, for $t \in \kappa^{<\kappa}$, this has inverse image $\bigcup_{1+\beta\geq\alpha}[t \frown \beta]$. Since, clearly, the map is injective, it's an homeomorphism to its range.

We now show that $2^\kappa \setminus \mathrm{Ran}(\varphi)$ is meagre; to see this, let C be the set of $x \in 2^\kappa$ such that x eventually stops alternating. We have

$$C = \bigcup_{\alpha<\kappa} \{x \in 2^\kappa \mid \forall\beta \geq \alpha\ x(\beta) = 0\} \cup \bigcup_{\alpha<\kappa} \{x \in 2^\kappa \mid \forall\beta \geq \alpha\ x(\beta) = 1\}$$

and each of the sets we are taking the union of is nowhere dense: just extend something beyond α forcing it to be out of the set.

Therefore, up to a meagre set κ^κ is the same as 2^κ. □

Remark 4.1. There is another encoding one could use: use 1's as separators and put $f(\alpha)$ many 0's each time. This may even be easier to work with.

Corollary 4.2. $(2^\kappa, \mathcal{M}_\kappa^{2^\kappa}, \in) \equiv (\kappa^\kappa, \mathcal{M}_\kappa^{\kappa^\kappa}, \in)$

Proof. To see $\preceq$, let $\Phi_+ \colon \mathcal{M}_\kappa^{2^\kappa} \to \mathcal{M}_\kappa^{\kappa^\kappa}$ be φ^{-1}, and let $\Phi_- \colon \kappa^\kappa \to 2^\kappa$ be φ. If $\varphi(f) \in X$ then $f \in \varphi^{-1}(X)$, so this is a morphism.

The morphism in the other direction is given by $\Phi_+ \colon \mathcal{M}_\kappa^{\kappa^\kappa} \to \mathcal{M}_\kappa^{2^\kappa}$ being[1] $X \mapsto \varphi\text{"}X \cup C$ and $\Phi_- \colon 2^\kappa \to \kappa^\kappa$ being φ^{-1} if defined, arbitrary otherwise. If $\Phi_-(x) \in Y$, then $x \in \Phi_+(Y)$, so we are done. □

The objects above were called $\mathrm{Cov}(\mathcal{M}_\kappa)$. What about $\mathrm{Cof}(\mathcal{M}_\kappa)$?

Corollary 4.3. $(\mathcal{M}_\kappa^{2^\kappa}, \mathcal{M}_\kappa^{2^\kappa}, \subseteq) \equiv (\mathcal{M}_\kappa^{\kappa^\kappa}, \mathcal{M}_\kappa^{\kappa^\kappa}, \subseteq)$

Proof. To see $\preceq$, let Φ_+ be φ^{-1} and Φ_- be φ". Clearly, if $\varphi\text{"}X \subseteq Y$ then $X \subseteq \varphi^{-1}Y$.

For the other direction, let Φ_+ be $C \cup \varphi$" and $\Phi_- := \varphi^{-1}$. If $\varphi^{-1}(Y) \subseteq X$, then $Y \subseteq \varphi\text{"}X \cup C$, so we are done. □

4.2 Baire's Category Theorem

We were actually tacitly using the following result, which we are now going to prove:

Theorem 4.4 (Baire's Category Theorem)**.** Every meagre set has empty interior.

Proof. Work in[2] 2^κ. Let X be meagre, as witnessed by writing $X = \bigcup_{\alpha<\kappa} X_\alpha$ with X_α nowhere dense, and let $\emptyset \neq U \subseteq 2^\kappa$ be open. We want to show that $U \setminus X \neq \emptyset$.

Since X_0 is nowhere dense, take $s_0 \in 2^{<\kappa}$ such that $[s_0] \subseteq U \setminus X_0$. Take $s_1 \in 2^{<\kappa}$ strictly extending s_0, such that $[s_1] \subseteq [s_0] \setminus X_1$. Go on like this for successor steps, and for limit λ take s_λ strictly extending $\bigcup_{\alpha<\lambda} s_\alpha$ such that $[s_\lambda] \subseteq \left[\bigcup_{\alpha<\lambda} s_\alpha\right] \setminus X_\lambda$. Then take $x = \bigcup_{\alpha<\kappa} s_\alpha$. Then $x \in U \setminus X$. □

4.3 Interval Partitions

Definition 4.5. Let $(i_\alpha \mid \alpha < \kappa)$ be a strictly increasing, continuous sequence of ordinals less than κ. Then $([i_\alpha, i_{\alpha+1}) \mid \alpha < \kappa)$ is an *interval partition.* Denote the set of all interval partitions by IP.

Definition 4.6. For interval partitions $I = (I_\alpha \mid \alpha < \kappa)$ and $J = (J_\alpha \mid \alpha < \kappa)$, say that I *dominates* J, written $J \leq^* I$ iff for some $\gamma < \kappa$ and all $\alpha \geq \gamma$ there is a $\beta \in \kappa$ such that $J_\beta \subseteq I_\alpha$.

In other words, eventually each I_α is big enough to contain some J_β.

[1] C is the complement of the range of φ.

[2] Note that to do something similar to the classical case ("complete metric spaces") one should figure out what "metric" means.

Proposition 4.7. $\mathcal{D} \equiv (\mathrm{IP}, \mathrm{IP}, \leq^*)$ (recall that $\mathcal{D} := (\kappa^\kappa, \kappa^\kappa, \leq^*)$).

Proof. Consider $\Psi_1 \colon \mathrm{IP} \to \kappa^\kappa$ sending

$$([i_\alpha, i_{\alpha+1})) \mapsto (\gamma \mapsto i_{\alpha+2} \text{ for the } \alpha \text{ such that } \gamma \in [i_\alpha, i_\alpha + 1))$$

Then let $\Psi_2 \colon \kappa^\kappa \to \mathrm{IP}$ be defined as

$$f \mapsto \text{some } J = ([j_\alpha, j_{\alpha+1})) \text{ such that } \gamma < j_\alpha \implies f(\gamma) < j_{\alpha+1}$$

Exercise 4.8. These work as Φ_+ and Φ_- for both directions.

□

Chapter 5

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5.1 Interval Partitions and Meagreness

Definition 5.1. A *κ-chopped function* is a pair (x, I) with $x \in 2^\kappa$ and I an interval partition. We say that $y \in 2^\kappa$ *matches* (x, I) iff for cofinally many $\alpha \in \kappa$ we have $y \restriction I_\alpha = x \restriction I_\alpha$.

The idea is that matching is the negation of $\neq^*$, but in chunks.

Definition 5.2. Let

$$\mathrm{Match}(x, I) := \{y \in 2^\kappa \mid y \text{ matches } (x, I)\}$$

Call $M \subseteq 2^\kappa$ *combinatorially meagre* iff there is some κ-chopped (x, I) such that $M \cap \mathrm{Match}(x, I) = \emptyset$.

Basically, we are thinking of $\mathrm{Match}(x, I)$ as the basic combinatorially comeagre sets. The reason is the following. Consider

$$2^\kappa \setminus \mathrm{Match}(x, I) = \bigcup_{\alpha<\kappa} \{y \mid \forall \beta \geq \alpha \; y \restriction I_\beta \neq x \restriction I_\beta\}$$

Claim. Each set in that union is nowhere dense.

Proof. For any open set, go a little bit further and make it match some $x \restriction I_\beta$. □

Corollary 5.3. Combinatorially meagre sets are meagre.

Question 5.4. Does the other implication hold?

Proposition 5.5 (Blass, Hyttinen, Zhang)**.** If κ is strongly inaccessible or $\kappa = \omega$, then meagre implies combinatorially meagre.

Proof. Suppose that A is meagre, as witnessed by $A = \bigcup_{\alpha<ka} A_\alpha$, with each A_α nowhere dense. We can WLOG assume the union is increasing, i.e. $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, because as κ is inaccessible or ω, in particular $\kappa^{>\kappa} = \kappa$. We want to construct a κ-chopped function (x, I) not matched by any member of A.

Construct a continuous, strictly increasing sequence of ordinals i_α, which will give us the interval partition I, and a sequence σ_α, for $\alpha < \kappa$, such that $\sigma_\alpha \colon [i_\alpha, i_{\alpha+1}) \to 2$. Then the concatenation (union) of the σ_α will be our x.

Because κ is inaccessible or ω, we can just choose $i_{\alpha+1}$ and σ_α such that for *all* $\tau \in 2^{i_\alpha}$ we have $\tau^\frown\sigma_\alpha \cap A_\alpha = \emptyset$. E.g. enumerate $2^{i_\alpha} = \{\tau_0, \tau_1, \tau_2, \ldots\}$, then extend τ_0 by $\sigma_{\alpha 0}$ to avoid A_α, extend $\tau_1{}^\frown\sigma_{\alpha 0}$ by $\sigma_{\alpha 1}$ to avoid A_α, etc, and let $\sigma_\alpha \coloneqq \sigma_{\alpha 0}{}^\frown\sigma_{\alpha 1}{}^\frown\sigma_{\alpha 2}{}^\frown\ldots$. By construction, $A \cap \mathrm{Match}(x, I) = \emptyset$. □

Theorem 5.6. If κ is regular, but not strongly inaccessible and not ω, then there is a meagre set that is not combinatorially meagre.

Proof. By hypothesis, there is some $\mu < \kappa \leq 2^\mu$. Say that y *repeats* at α if $\forall \xi < \alpha\ y(\xi) = y(\alpha + \xi)$. Recall that an ordinal γ is *indecomposable* iff γ cannot be written as $\alpha + \beta$ for $\alpha, \beta < \gamma$. In other words, γ is of the form ω^α, or 0. Defin

$$X \coloneqq \{y \in 2^\kappa \mid y \text{ repeats at an indecomposable } \alpha \in [\mu, \kappa)\}$$

We now show that $2^\kappa \setminus X$ is meagre but not combinatorially meagre. In fact, X is open dense: given any sequence, extend up to the next indecomposable ordinal and then repeat. To show that, for every (x, I), we have $X \not\supseteq \mathrm{Match}(x, I)$, for every (x, I) we are going to construct some $y \in \mathrm{Match}(x, I) \setminus X$. First note that if J is coarser than I, then y matching (x, J) implies that y matches (x, I), so WLOG we can thin out the i_α.

The i_α form a club, and the indecomposables $\geq \mu$ form another club. Therefore, WLOG every i_α other than $i_0 = 0$ is an indecomposable $\geq \mu$. Proceed by induction: for the base case, on $I_0 \cup I_1$ set $y(\xi)$ to be 1 iff $\xi = 0$, and 0 otherwise. This ensures that we do not get repetitions at indecomposables in $I_0 \cup I_1$. To define y on $[i_{2\beta}, i_{2\beta+1})$ and $[i_{2\beta+1}, i_{2\beta+2})$, first let $y \restriction [i_{2\beta+1}, i_{\beta+2}) = x \restriction [i_{2\beta+1}, i_{\beta+2})$, to ensure matching. Then we use the bit on $[i_{2\beta}, i_{2\beta+1})$ to ensure there are no repetitions at indecomposables: if $\alpha \in I_{2\beta}$ is indecomposable, set $y(\alpha) = 0$ to prevent repetitions at α (because $y(0) = 1$); this takes care of the indecomposables in $[i_{2\beta}, i_{2\beta+1})$, but what about the ones in $[i_{2\beta+1}, i_{\beta+2})$? We have not defined y yet on $(i_{2\beta}, i_{2\beta} + \mu)$; by indecomposability, $i_{2\beta+\mu}$ will not be indecomposable[1]. For α an indecomposable in $I_{2\beta+1}$, define $f_\alpha \colon \mu \to 2$ as

$$f_\alpha(x) = y(\alpha + i_{2\beta} + 1 + \xi)$$

[1] Recall that i_1 is already $\geq \mu$.

There are at most $|i_{2\beta+2}| < \kappa \leq 2^\mu$ of these, so we can choose $g\colon \mu \to 2$ different from every f_α. Then define $y(i_{2\beta} + 1 + \xi) := g(\xi)$, and define y arbitrarily on other elements of $I_{2\beta}$.

We are now left to check that for every α indecomposable in $I_{2\beta+1}$ we do not have repetition at α. Indeed, for ξ with $g(\xi) \neq f_\alpha(\xi)$ we have

$$y(\alpha + i_{2\beta} + 1 + \xi) = f_\alpha(\xi) \neq g(\xi) = y(i_{2\beta} + 1 + \xi) \qquad \square$$

Chapter 6

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6.1 Two Lemmas, One Lovely, One Not

Recall that we had $\mathcal{D} \preceq \mathcal{E} \preceq \mathrm{Cov}(\mathcal{M}_\kappa)$, so

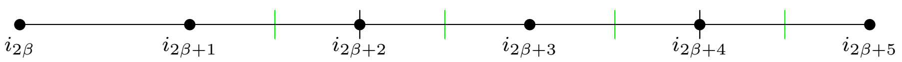
$$\mathfrak{b}_\kappa \leq \mathfrak{b}_\kappa(\neq^*) \leq \mathrm{non}(\mathcal{M}_\kappa)$$
$$\mathfrak{d}_\kappa \geq \mathfrak{d}_\kappa(\neq^*) \geq \mathrm{cov}(\mathcal{M}_\kappa)$$

Also, recall that if I, J are interval partitions, then $I \leq^* J$ means that for all but $< \kappa$ many α there is a β such that $J_\alpha \supseteq I_\beta$.

Note that there is an asymmetry between $\mathcal{D}$ and interval partitions: $\leq$ is a total order, $\subseteq$ is not. But we can get around that:

Lemma 6.1. Suppose that I, J are interval partitions, and let I' be the interval partition $(I_{2\beta} \cup I_{2\beta+1} \mid \beta < \kappa)$. If $\neg(I' \geq^* J)$, then for cofinally many α there is a β such that $I_\beta \subseteq J_\alpha$.

Proof. $\neg(I' \geq^* J)$ means that cofinally many I'_β do *not* contain a J_α.



If no j_α is in $[i_{2,\gamma}, i_{2\gamma+2})$ we are done. If it contains one j_α, we're done anyway (look at the picture). $\square$

Definition 6.2. Let $\mathrm{Fn}(\kappa, 2, \kappa)$ be the set of partial functions $\kappa \to 2$ with domain of size $< \kappa$ (not necessarily an initial segment).

Lemma 6.3. There are functions $\Phi_- \colon \mathrm{CF} \times \mathrm{IP} \to ((\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa})^\kappa$, where CF stands for "chopped functions", and $\Phi_+ \colon \mathrm{IP} \times ((\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa})^\kappa \to 2^\kappa$ such that if

- $(x, I) \in \mathrm{CF}$
- $J \in \mathrm{IP}$
- $y \in ((\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa})^{\kappa}$
- cofinally many J_α contain an I_β, (i.e. $\neg(I' \geq^* J)$)
- $\Phi_-((x, I), J)(\beta) = y(\beta)$ for cofinally many β, i.e. $\neg\Phi_-((x, I), J) \neq^* y)$

then $\Phi_+(J, y)$ matches (x, I).

Spoiler 6.4. We will use this to show that $\mathrm{non}(\mathcal{M}_\kappa) \leq \mathfrak{b}(\neq^*)$ and $\mathrm{cov} \geq \mathfrak{d}(\neq^*)$ (so that will be equalities, since we already know the opposite inequalities.).

Proof. First, construct Φ_-. Suppose $I, J \in \mathrm{IP}$ are such that for cofinally many α we have $J_\alpha \supseteq I_\beta$ for some β. Let $A = \{\alpha_\gamma \mid \gamma < \kappa\}$ be the increasing enumeration of these α. For each $\gamma < \kappa$, let δ_γ be such that $J_{\alpha_\gamma} \supseteq I_{\delta_\gamma}$. Define

$$\Phi_-((x, I), J)(\beta) := (x \restriction I_{\delta_\gamma} \mid \gamma < \omega_{\beta+1})$$

(replace $\omega_{\beta+1}$ with $\beta + 1$ in the ω case). For other I, J, define Φ_- arbitrarily.

We define Φ_+ recursively, defining $\Phi_+(J, y) \restriction$ a subset of J_α for at most one α at every stage. At stage $\beta < \kappa$:

- if $y(\beta)$ is a sequence of length $\omega_{\beta+1}$ (or $\beta + 1$ in the ω case) of partial functions, all of whose domains are included in distinct J_α's, then choose such an α that has not been considered yet[1]; say $J_\alpha \supseteq \mathrm{dom}(y(\beta)(\gamma))$. Let

$$\Phi_+(J, y) \restriction \mathrm{dom}(y(\beta)(\gamma)) := y(\beta)(\gamma)$$

- if not, do nothing.

At the end, extend $\Phi_+(J, y)$ arbitrarily to get a total function in 2^κ.

Let's now check that these actually work. Suppose we have $(x, I), J, y$ as in the hypotheses, and fix β such that $\Phi_-((x, I), J)(\beta) = y(\beta)$ (by assumption, there's cofinally many of them). Then $y(\beta)$ is, by definition, a length[2] $\omega_{\beta+1}$ of partial functions $(x \restriction I_{\delta_\gamma})$ all of whose domains are contained in distinct J_α's. So, for some γ dependent on β,

$$\Phi_+(J, y) \restriction I_{\delta_\gamma} = y(\beta)(\gamma) = x \restriction I_{\delta_\gamma}$$

and different β give different α, therefore different γ. So $\Phi_+(J, y)$ matches (x, I). □

[1] This is ok because $|\beta| \leq \omega_\beta < \omega_{\beta+1}$.
[2] $\beta + 1$ in the ω case.

Remark 6.5. In the proof above, we only needed κ to be closed under the $\aleph$ function, so it also works for weakly inaccessible κ. Anyway, the next Corollary requires strong inaccessibility.

Corollary 6.6.

1. (Blass, Hyttinen, Zhang) $\mathrm{non}(\mathcal{M}_\kappa) = \mathfrak{b}(\neq^*)$
2. (Landver) $\mathrm{cov}(\mathcal{M}_\kappa) = \mathfrak{d}(\neq^*)$

Proof.

1. As we already know $\geq$, it suffices to show $\leq$. Suppose $\mathcal{Y} \subseteq ((\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa})^\kappa$. By strong inaccessibility, we can identify $(\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa}$ with κ, and therefore the whole thing with κ^κ. Suppose $|\mathcal{Y}| = \mathfrak{b}_\kappa(\neq^*)$ is unbounded with respect to $\neq^*$. We will use this to construct a non-meagre set. Suppose $\mathcal{J}$ is a $(\leq^*)$-unbounded family of partitions of size $\mathfrak{b}_\kappa \leq \mathfrak{b}_\kappa(\neq^*)$.

   Claim. $M := \{\Phi_+(J, y) \mid J \in \mathcal{H}, y \in \mathcal{Y}\}$ is non-meagre.

   To prove the claim and conclude the proof of this point, if (x, I) is a chopped function, since combinatorially meagre is the same as meagre (by strong inaccessibility), take $J \in \mathcal{J}$ such that $\neg(J \leq^* I')$, which exists because $\mathcal{J}$ is unbounded. By Lemma 6.1 we know that J_α contains some I_β for cofinally many α. Take $y \in \mathcal{Y}$ such that $\Phi_-((x, I), J)(\beta) = y(\beta)$ for cofinally many β; this exists because $\mathcal{Y}$ is unbounded in $\neq^*$. By Lemma 6.3, we know that $\Phi_+(J, y)$ matches (x, I). So $M \not\subseteq \mathrm{Match}(x, I)^{\complement}$. Now, this is true for any (x, I), and since combinatorially meagre is the same as meagre, this tells us that M is non-meagre. As $|M| = \mathfrak{b}(\neq^*)$, we have $\mathrm{non}(\mathcal{M}_\kappa) \leq \mathfrak{b}(\neq^*)$.

2. We already know $\leq$. Suppose $\mathcal{X} \subseteq \mathrm{CF}$ is of size $< \mathfrak{d}(\neq^*) \leq \mathfrak{d}(\leq^*)$. In particular, we have

   $$|\{I' \mid (x, I) \in \mathcal{X}\}| < \mathfrak{d}(\leq^*) = \mathfrak{d}(\mathrm{IP}, \leq^*)$$

   So we can choose $J \in \mathrm{IP}$ such that J_α contains an I_β for cofinally many α. Identify $(\mathrm{Fn}(\kappa, 2, \kappa))^\kappa$ with κ. Then, modulo this identification,

   $$|\{\Phi_-((x, I), J) \in \kappa^\kappa \mid (x, I) \in \kappa\}| < d(\neq^*)$$

   so pick $y \in (\mathrm{Fn}(\kappa, 2, \kappa)^{<\kappa})^\kappa$ such that for all $(x, I) \in \mathcal{X}$ we have $\Phi_-((x, I), J)(\beta) = y(\beta)$ for cofinally many β.

   We are therefore in a position to apply Lemma 6.3, and so $\Phi_+(J, y) \in 2^\kappa$ matches (x, I). In particular, $\Phi_+(J, y) \notin \bigcup_{(x,I)\in\mathcal{X}} 2^\kappa \setminus \mathrm{Match}(x, I)$. This means that $\{2^\kappa \setminus \mathrm{Match}(x, I) \mid (x, I) \in \mathcal{X}\}$ does not cover 2^κ. This shows that $\mathrm{cov}(\mathcal{M}_\kappa) \geq \mathfrak{d}(\neq^*)$.

□

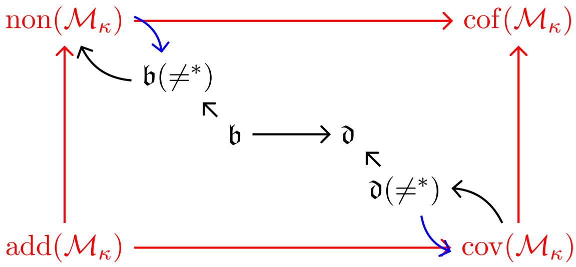
Chapter 7

24/10

7.1 $\mathfrak{b}_\kappa$ and $\mathfrak{b}_\kappa(\neq^*)$

[Proof of the second point of Corollary 6.6; written directly in the previous chapter]

Let's update our diagram:



Question 7.1. We have $\mathfrak{b}_\kappa \leq \mathfrak{b}_\kappa(\neq^*)$ and $\mathfrak{d}_\kappa \geq \mathfrak{d}_\kappa(\neq^*)$. Can the inequality be strict?

Fact 7.2. In the inequalities above,

1. If κ is ω then $<$ is consistent in both cases
2. (Baumhauer, Goldstern, Shelah, in preparation) If κ is supercompact, consistently $\mathfrak{b}_\kappa < \mathrm{non}(\mathcal{M}_\kappa)(= \mathfrak{b}(\neq^*))$.
3. (Shealah, preprint) If κ is supercompact, consistently, $(\mathfrak{d}(\neq^*) =) \,\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa$.

On the other hand,

Fact 7.3. [Hyttinen] If κ is a successor cardinal, then $\mathfrak{b}_\kappa = \mathfrak{b}_\kappa(\neq^*)$.

Note how this could interfere with the equalities we have in the "blue" case and the consistency results above, in the supercompact case.

Fact 7.4 (Matet, Shelah)**.** If κ is a successor and $2^{<\kappa} = \kappa$, then $\mathfrak{d}_\kappa = \mathfrak{d}_\kappa(\neq^*)$.

Proposition 7.5.

1. For any $\sigma \in 2^{<\kappa}$, the set A_σ of $y \in 2^\kappa$ with no occurrences of σ, i.e.

$$A_\sigma = \{y \in 2^\kappa \mid \forall \tau \in 2^{<\kappa}\ \tau \frown \sigma \not\subseteq y\}$$

   is nowhere dense.

2. (Landver) $2^{<\kappa} > \kappa$ implies that $\kappa^+ = \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{cov}(\mathcal{M}_\kappa)$,

3. (Blass, Hyttinen, Zhang) $\mathrm{non}(\mathcal{M}_\kappa) \geq 2^{<\kappa}$

Proof.

1. Immediate.

2. Any $2 \in 2^\kappa$ has only κ many $< \kappa$ substrings. If $\lambda < \kappa$ is such that $2^\lambda > \kappa$, take $\Sigma \subseteq 2^\lambda$ with $|\Sigma| = \kappa^+$. Then

$$\{A_\sigma \mid \sigma \in \Sigma\}$$

   is a κ^+-sized covering set.

3. $\mathrm{non}(\mathcal{M}_\kappa) \geq \kappa$ holds by definition, so we may assume $2^{<\kappa} > \kappa$. Let $X \subseteq 2^\kappa$ with $|X| < 2^{<\kappa}$. We want to show that X is meagre. Let $\lambda < \kappa$ be such that $|X| < 2^\lambda$. Then $X \subseteq A_\sigma$ for some $\sigma \in 2^\lambda$, which is nowhere dense.

□

This allows us to consistently break the equalities seen before: using this, we can get

Proposition 7.6. Consistently, $\mathfrak{b}_\kappa(\neq^*) < \mathrm{non}(\mathcal{M}_\kappa)$ and $\mathfrak{d}_\kappa(\neq^*) > \mathrm{cov}(\mathcal{M}_\kappa)$.

Proof. To force $\mathfrak{b}_\kappa(\neq^*) < \mathrm{non}(\mathcal{M}_\kappa)$ start with a model of GCH, let κ be a successor and force to add κ^{++}-many Cohen reals[1]. In $V[G]$ we have $2^{<\kappa} = \kappa^{++} = 2^\kappa$. So from the last point of the previous Proposition we get that $\mathrm{non}(\mathcal{M}_\kappa) = \kappa^{++}$. But by the Hyttinen result (Fact 7.3), $\mathfrak{b}_\kappa(\neq^*) = \mathfrak{b}_\kappa$. Since the forcing notion has c.c.c. it is κ^κ-bounding, i.e. any $g\colon \kappa \to \kappa$ in the extension is dominated by a $h\colon \kappa \to \kappa$ in the ground model; to see this, if $\dot{g}$ is a name for a function $\kappa \to \kappa$, for every $\gamma \in \kappa$ there is a maximal antichain of conditions p such that $p \Vdash \dot{g}(\check{\gamma}) = \check{\alpha}$, so we can just define $h(\gamma)$ to be the sup of these α's. Then $1 \Vdash \dot{g} \leq \hat{h}$. So if B is unbounded in the ground model, B remains unbounded int he extension. So

$$\mathfrak{b}(\neq^*)^{V[G]} = \mathfrak{b}_\kappa^{V[G]} = \kappa^+ < \kappa^{++} = \mathrm{non}(\mathcal{M}_\kappa)$$ □

It is open if this can be done with $2^{<\kappa} = \kappa$.

[1] Real reals, i.e. subsets of ω, not κ-reals.

Chapter 8

30/10

8.1 More on Combinatorially Meagre Sets

Proposition 8.1. $\mathrm{Match}(x, I) \subseteq \mathrm{Match}(y, J)$ if and only if for all but $< \kappa$ many intervals I_α of I there is $b\eta$ such that $J_\beta \subseteq I_\alpha$ and $x \upharpoonright J_\beta = y \upharpoonright J_\beta$.

Remark 8.2. Thinking of the sets in the first statement as as the “comeagre” sets, the statement in terms of the “meagre” ones is $2^\kappa \setminus \mathrm{Match}(y, J) \subseteq 2^\kappa \setminus \mathrm{Match}(x, I)$.

Proof.

($\Rightarrow$) Suppose there are κ many intervals I_{α_γ} such that for every J_β contained in I_{α_γ} we have $x \upharpoonright J_\beta \neq y \upharpoonright J_\beta$. Also, assume that successive I_{α_γ}'s have a J_β in between. Define

$$x'(\alpha) := \begin{cases} x(\alpha) & \text{if } \exists\gamma\ \alpha \in I_{\alpha_\gamma} \\ 1 - y(\alpha) & \text{otherwise} \end{cases}$$

To conclude, it is sufficient to show that $x' \in \mathrm{Match}(x, I) \setminus \mathrm{Match}(y, J)$. It is clear that x' matches x on I. For the other part, if J_β is contained in some I_{α_γ}, our assumption tells us that $x' \notin \mathrm{Match}(y, J)$. Otherwise, use the assumption above to find a J_β between two successive I_{α_γ}'s.

($\Leftarrow$) Suppose $z \in \mathrm{Match}(x, I)$. Then there are κ many I intervals $I\alpha_\gamma$ such that $z \upharpoonright I_{\alpha_\gamma} = x \upharpoonright I_{\alpha_\gamma}$. For κ many γ, WLOG for all γ there is β such that $J_\beta \subseteq I_\alpha$ and $y \upharpoonright J_\beta = x \upharpoonright J_\beta = z \upharpoonright J_\beta$. □

Definition 8.3. Say that (x, I) is *engulfed by* (y, J) iff[1] $\mathrm{Match}(x, I) \supseteq \mathrm{Match}(y, J)$.

We have seen that essentially $\mathrm{Cof}(\mathcal{M}_\kappa) = (\mathcal{M}_\kappa, \mathcal{M}_\kappa, \subseteq)$ is equivalent to $\mathrm{Cof}'(\mathcal{M}_\kappa) := (\mathrm{CF}, \mathrm{CF}, \text{ is engulfed by})$. The morphism from the former to

[1] So the complements, the “meagre” sets, are engulfed.

the latter is given by

$$\Phi_+ \colon M \mapsto \text{ some } (y, J) \text{ with } M \subseteq 2^\kappa \setminus \mathrm{Match}(y, J)$$
$$\Phi_- \colon (x, I) \mapsto 2^\kappa \setminus \mathrm{Match}(x, I)$$

While the morphism in the other direction is given by Φ_+ and Φ_- swapped: if $\Phi_{(}M)$ is less than some "bigger" (x, I) and is engulfed by (y, J), then $M \subseteq 2^\kappa \setminus \mathrm{Match}(y, J)$. This is a particular case of the following:

Exercise 8.4. If D is cofinal in $\mathbb{P}$, then $(D, D, \leq) \equiv (\mathbb{P}, \mathbb{P}, \leq)$.

Corollary 8.5. $\mathrm{Cof}(\mathcal{M}_\kappa) \preceq \mathcal{D}_\kappa$.

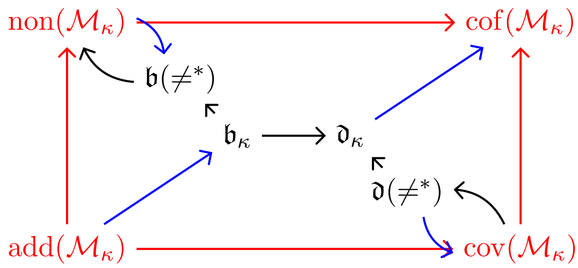
Proof. We know $\mathrm{Cof}(\mathcal{M}_\kappa) \equiv \mathrm{Cof}'(\mathcal{M}_\kappa)$ and $\mathcal{D}_\kappa \equiv \mathrm{IP}$. By Proposition 8.1, if (x, I) is engulfed by (y, J), then $I \leq^* J$. We can then take as morphism

$$\Phi_+ \colon (x, J) \mapsto J \qquad \Phi_i \colon I \mapsto (x, I) \text{ (some } x)$$

since what we just said say exactly that this maps give us a morphism. □

Corollary 8.6. $\mathrm{cof}(\mathcal{M}_\kappa) \geq \mathfrak{d}_\kappa$ and $\mathrm{add}(\mathcal{M}_\kappa) \leq \mathfrak{b}_\kappa$.

So we have the following picture



Also, [someone, I missed the name] claims in a preprint that the last arrows we added to the diagram can be black, i.e. are true just assuming regularity.

In the ω case, Chicon's diagram also involves other posets related to the ideal of Lebesgue null sets. The problem in the κ case is, for now, that nobody has still come up with a suitable generalisation of the Lebesgue null sets.

8.2 Slaloms

Definition 8.7. A *slalom* is a function $\varphi \colon \kappa \to [\kappa]^{<\kappa}$ such that $\forall \alpha\ \varphi(\alpha) \in [\kappa]^{\leq |\alpha|}$. If $h \colon \kappa \to \kappa$ is a function with $\lim_{\alpha \to \kappa} h(\alpha) = \kappa$, an *$h$-slalom* is a function $\varphi \colon \kappa \to [\kappa]^{<\kappa}$ such that $\forall \alpha\ \varphi(\alpha) \in [\kappa]^{\leq |h(\alpha)|}$.

Definition 8.8. For $f \in \kappa^\kappa$, we say that f *is localised at* φ, written $f \in^* \varphi$ iff for all but $< \kappa$ many α we have $f(\alpha) \in \varphi(\alpha)$.

Proposition 8.9 (Bartzynski, $\kappa = \omega$)**.** If $\mathcal{N}$ is the Lebesgue null ideal, $\text{add}(\mathcal{N}) = \mathfrak{b}(\in^*)$ and $\text{cof}(\mathcal{N}) = \mathfrak{d}(\in^*)$.

Definition 8.10. A *partial h-slalom* is a partial function $\varphi\colon \kappa \to [\kappa]^{<\kappa}$ with $|\text{dom}\,\varphi| = \kappa$ such that $\forall \alpha \in \text{dom}\,\varphi\ \varphi(\alpha) \in [\kappa]^{\leq |h(\alpha)|}$. We say that $f \in^*_{\text{p}} \varphi$ iff for all but $< \kappa$ many $\alpha \in \text{dom}(\varphi)$ we have $f(\alpha) \in \varphi(\alpha)$.

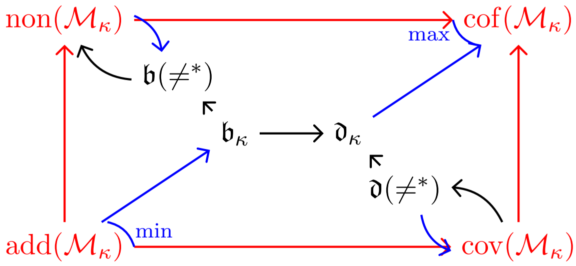
Spoiler 8.11. In the ω case, we have $\mathfrak{b}(\in^*) \to \mathfrak{b}_p(\in^*) \to \text{add}(\mathcal{M}_\omega)$. Also, $\mathfrak{p} = \mathfrak{t} \to \mathfrak{b}_{\text{p}}(\in^*)$.

Chapter 9

31/10

9.1

The goal of today is getting the diagram here:



For convenience, think of 2^κ as the group with coordinatewise addition modulo 2. Think of any $\sigma \in 2^{<\kappa}$ in 2^κ as σ on its domain and 0 elsewhere. With these conventions, $B + 2^{<\kappa}$ means $\{b + \sigma \mid b \in B, \sigma \in 2^{<\kappa}\}$, i.e. B modulo small differences.

Lemma 9.1 (κ regular, $2^{<\kappa} = \kappa$)**.** Denote with $\mathcal{NWD}_\kappa$ the collection of nowhere dense sets in 2^κ. There are functions

$$\Phi_+ \colon 2^\kappa \times \kappa^\kappa \,\mathrm{tp}\, \mathcal{M}_\kappa \qquad 2^\kappa \times \mathcal{NWD}_\kappa \to \kappa^\kappa$$

such that if $B \in \mathcal{NWD}_\kappa$, $x \in 2^\kappa$ and $f \in \kappa^\kappa$ are such that

- $\lim_{\alpha \to \kappa} f(\alpha) = \kappa$
- $x \notin B + 2^\kappa$
- $f \geq^* \Phi_-(x, B)$

then $B \subseteq \Phi_+(x, f)$.

Once we have the Lemma, we have

Corollary 9.2. The following hold:

1. $\text{add}(\mathcal{M}_\kappa) \geq \min\{\mathfrak{b}_\kappa, \text{cov}(\mathcal{M}_\kappa)\}$
2. $\text{cof}(\mathcal{M}_\kappa) \leq \max\{\mathfrak{d}_\kappa, \text{non}(\mathcal{M}_\kappa)\}$

Proof.

1. If $2^{<\kappa} > \kappa$, by Proposition 7.5 we have $\text{add}(\mathcal{M}_\kappa) = \text{cov}(\mathcal{M}_\kappa) = \kappa^+$. If $2^{<\kappa} = \kappa$, if $\mathcal{B} \subseteq \mathcal{NWD}_\kappa$ is such that $|\mathcal{B}| < \min\{\mathfrak{b}_\kappa, \text{cov}(\mathcal{M}_\kappa)\}$, we can find $x \in 2^\kappa \setminus (\bigcup \mathcal{B} + 2^{<\kappa})$ and then $f \geq^* \Phi_-(x, B)$ for all $B \in \mathcal{B}$. Then for all $B \in \mathcal{B}$ we have $B \subseteq \Phi_+(x, f)$, so $\bigcup \mathcal{B}$ is meagre.
2. Let $\mathcal{F} \subseteq \kappa^\kappa$ be dominating, $X \subseteq 2^\kappa$ be non-meagre. We are now going to show that $\{\Phi_+(x, f) \mid f \in \mathcal{F}, x \in X\}$ is cofinal in $\mathcal{M}_\kappa$. If M is meagre, say $M = \bigcup_{\alpha<\kappa} Y_\alpha$, choose $x \in X \setminus M$ and $f \geq^* \Phi_-(x, Y_\alpha)$ for all[1] α. Then $\forall \alpha\ Y_\alpha \subseteq \Phi_+(x, f)$, so $M \subseteq \Phi_+(x, f)$.

□

Remark 9.3. In the proof above, we used tacitly the fact that the functions in a dominating family can be chosen to be increasing.

Corollary 9.4. $\text{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}_\kappa, \text{cov}(\mathcal{M}_\kappa)\}$ and $\text{cof}(\mathcal{M}_\kappa) = \max\{\mathfrak{d}_\kappa, \text{non}(\mathcal{M}_\kappa)\}$ and

Proof of Lemma 9.1. Enumerate $2^{<\kappa}$ as $\{\sigma_\alpha \mid \alpha < \kappa\}$. For f such that $\lim_{\alpha\to\kappa} f(\alpha) = \kappa$, set

$$\Phi_+(x, f) := \bigcup_{\alpha<\kappa} \bigcap_{\beta\geq\alpha} 2^\kappa \setminus [(\sigma_\beta + x) \restriction f(\beta)]$$

We are now going to show that each of those intersections is nowhere dense. If $\tau \in 2^{<\kappa}$, choose σ_β such that $\sigma_\beta + x \restriction |\tau| = \tau$ and $f(\beta) \geq |\tau|$. Then $(\sigma_\beta + x) \restriction f(\beta)$ is an extension of τ. For other f's, let $\Phi_+(x, f)$ be arbitrary.

Let now $B \in \mathcal{NWD}_\kappa$ and $x \notin B + 2^{<\kappa}$. As every nowhere dense set is contained in a closed one, we may assume WLOG that B is closed. For such B and x $\Phi_-(x, B)(\alpha)$ to be an ordinal γ such that $B \cap [(\sigma_\alpha + x) \restriction \gamma] = \emptyset$. Let $\Phi(x, B)$ be arbitrary for other (x, B).

Assume x, B, f satisfy the hypotheses of the Lemma. Let $y \in B$. Then $y \notin [(\sigma_\alpha + x) \restriction \Phi_-(x, B)(\alpha)]$ by definition of Φ_-. Since $f \geq^* \Phi_-(x, B)$, there is α such that for all $\beta \geq \alpha$ we have $y \in 2^\kappa \setminus [(\sigma_\alpha + x) \restriction f(\beta)]$. But, by definition, this means $y \in \Phi_+(x, f)$. □

[1] There's only κ many of them

Chapter 10

06/11

10.1 On Slaloms

We would like to deal with something similar to the ideal of Lebesgue null sets, but no one has come up with a suitable generalisation of that ideal for general κ. So we talk about slaloms instead.

Definition 10.1. Let $\mathrm{Loc}_h = \{\varphi\colon \kappa \to [\kappa]^{<\kappa} \mid \forall \alpha < \kappa\ |\varphi(\alpha)| = |h(\alpha)|\}$.

Remark 10.2. In the ω case requiring $|\varphi(\alpha)| \leq |h(\alpha)|$ instead does not make a difference. But for now let us be cautious and work with the definition above.

Notation 10.3. $\forall^* \alpha < \kappa$ means "for all but $< \kappa$ many".

Definition 10.4. For $f\colon \kappa \to \kappa$, say $f \in^* \varphi$ iff $\forall^* \alpha < \kappa\ f(\alpha) \in \varphi(\alpha)$.

We are now going to consider $\mathfrak{b}_h(\in^*)$ and $\mathfrak{d}_h(\in^*)$.

Fact 10.5. In the ω case we have $\mathfrak{b}_{\mathrm{id}_\omega}(\in^*) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}_{\mathrm{id}_\omega}(\in^*) = \mathrm{cof}(\mathcal{N})$, where $\mathcal{N}$ is the ideal of Lebesgue null sets.

In the ω case, there is a famous result stating

Fact 10.6 (Bartoszyńsky, Raissonnier, Stern)**.** $\mathrm{Cof}(\mathcal{N}) \preceq \mathrm{Cof}(\mathcal{M})$

Unpacking the proof Gives that $\mathrm{Cof}(\mathcal{N}) \equiv \mathrm{LOC}_{\mathrm{id}_\omega} := (\omega^\omega, \mathrm{Loc}_{\mathrm{id}_\omega}, \in^*)$, and this induces a morphism from the latter to $\mathrm{Cof}(\mathcal{M})$.This *does* generalise, so we are going to look at it.

Definition 10.7. Call pLoc_h the set of partial h-slaloms, and denote $\mathrm{pLOC}_{\mathrm{id}_\omega} := (\omega^\omega, \mathrm{pLoc}_{\mathrm{id}_\omega}, \in^*)$

Proposition 10.8. $\mathrm{LOC}_h \preceq \mathrm{pLoc}_h \preceq \mathcal{D}_\kappa$

Proof. For the first morphism $\Phi_+\colon \mathrm{Loc}_h \to \mathrm{pLoc}_h$ is inclusion, and $\Phi_-\colon \kappa^\kappa \to \kappa^\kappa$ is the identity.

For the second one, $\Phi_+\colon \mathrm{pLoc}_h \to \kappa^\kappa$ is

$$\Phi_+(\varphi)(\alpha)\sup(\varphi(\text{least } \beta \geq \alpha \text{ in } \operatorname{dom}\varphi))$$

and $\Phi_-\colon \kappa^\kappa \to \kappa^\kappa$ is the identity. To check that this works we need to see that if $\Phi_-(f) \in^*_{\mathrm{p}} \varphi$ then $f \leq^* \Phi_+(\varphi)$, i.e. if $f \in^*_{\mathrm{p}} \varphi$ then $f \leq^* \sup(\varphi(\text{least } \beta \geq \alpha \text{ in } \operatorname{dom}\varphi))$. For f increasing this works. Using the fact that the increasing f are dense, the proof can be completed. □

Corollary 10.9. $\mathfrak{b}_h(\in^*) \leq \mathfrak{b}_h(\in^*_{\mathrm{p}}) \leq \mathfrak{b}_\kappa$ and $\mathfrak{d}_h(\in^*) \geq \mathfrak{d}_h(\in^*_{\mathrm{p}}) \geq \mathfrak{d}_\kappa$.

Remark 10.10. In the ω case, $\mathfrak{d}_h(\in^*_{\mathrm{p}})$ has a name too. We will come back to that.

Lemma 10.11. For $\kappa = \lambda^+$ we have $\mathcal{D}_\kappa \preceq \mathrm{LOC}_h$. So $\mathrm{LOC}_h \equiv \mathrm{pLOC}_h \equiv \mathcal{D}_\kappa$.

Proof. For $\kappa = \lambda^+$, $|h(\alpha)|$ is almost always equal to λ. Define $\Phi_+\colon \kappa^\kappa \to \mathrm{Loc}_h$ as

$$g \mapsto (\alpha \mapsto g(\alpha) + 1 \text{ (as a set of ordinals)})$$

This is $\varphi\colon \kappa \to [\kappa]^\lambda = [\kappa]^{|h|(\alpha)}$. Then take $\Phi_- := \mathrm{id}_{\kappa^\kappa}$, and we have that if $\Phi_-(f) = f \leq^* g$ then $f \in^* \Phi_+(g)$ (unpacking the definitions shows that this is equivalent to $f \leq^* g$). □

Proposition 10.12. Let $g, h\colon \kappa \to \kappa$ be such that $\lim_{\alpha\to\kappa} g(\alpha) = \kappa = \lim_{\alpha\to\kappa} h(\alpha)$. Then $\mathrm{pLOC}_g \equiv \mathrm{pLOC}_h$.

Proof. We will show $\mathrm{pLOC}_g \preceq \mathrm{pLOC}_h$, i.e. $(\kappa^\kappa, \mathrm{pLoc}_g, \in^*_{\mathrm{p}}) \preceq (\kappa^\kappa, \mathrm{pLoc}_h, \in^*_{\mathrm{p}}$). Choose a strictly increasing $(\alpha_\gamma)_{\gamma\in\kappa}$ subset of $\operatorname{dom} h = \kappa$ such that $h(\alpha_\gamma) \geq g(\gamma)$. Define $\Phi_-\colon \kappa^\kappa \to \kappa^\kappa$ by $\Phi_-(f)(\gamma) = f(\alpha_\gamma)$. Define $\Phi_+\colon \mathrm{pLoc}_g \to \mathrm{pLoc}_h$ by

$$\operatorname{dom}((\Phi_+)(\varphi)) := \{\alpha_\gamma \mid \gamma \in \operatorname{dom}\varphi\} \qquad \underbrace{\Phi_+(\varphi)(\alpha_\gamma)}_{\in[\kappa]^{|h(\alpha_\gamma)|}} \supseteq \underbrace{\varphi(\gamma)}_{\in[\kappa]^{|g(\gamma)|}}$$

by extending arbitrarily the set if need be. Now assume $\Phi_-(f) \in^* \varphi$, i.e. $\forall^*\gamma \in \operatorname{dom}\varphi\ \Phi_-(f)(\gamma) = f(\alpha_\gamma) \in \varphi(\gamma)$. Then $\forall^*\alpha \in \operatorname{dom}(\Phi_+(\varphi))\ f(\alpha) \in \Phi_+(\varphi)(\alpha)$, and $\forall^*\gamma \in \operatorname{dom}\varphi\ f_(\alpha_\gamma) \in \Phi_+(\varphi)(\alpha_\gamma)$, as $\varphi(\gamma) \subseteq \Phi_+(\varphi)(\alpha_\gamma)$. □

Chapter 11

07/11

11.1 Towards the κ-B.R.S. Theorem

We are aiming towards showing that pLOC $\preceq$ COF$(\mathcal{M}_\kappa)$.

Lemma 11.1 (Main Lemma)**.** Let $X \subseteq 2^\kappa$ be a non-empty open set, and let $\lambda < \kappa$. Then there is a family $\mathcal{Y}$ of open subsets of X such that

(i) $|\mathcal{Y}| \leq \kappa$

(ii) Every open dense subset of 2^κ includes a member of $\mathcal{Y}$ as a subset.

(iii) For any $\mathcal{Y}' \subseteq \mathcal{Y}$ with $|\mathcal{Y}'| \leq \lambda$ we have $\bigcap \mathcal{Y}' \neq \emptyset$.

[the proof was actually started in the previous lecture, but I have preferred to keep it all in one chapter]

Proof. Let $(\Sigma_\alpha)_{\alpha<\kappa}$ enumerate subsets of $2^{<\kappa}$ of size $< \kappa$. This can be done because, for each α, Σ_α is (induced by) a collection of $\sigma \in 2^{<\kappa}$, and by strong inaccessibility $(2^{<\kappa})^{<\kappa} = \kappa$, so there are κ many Σ_α at most. For each α let $X_\alpha = \bigcup_{\sigma\in\Sigma_\alpha}[\sigma]$, i.e. $(X_\alpha)_\alpha$ lists the union of basic open sets, relative to X. From now one, assume WLOG $X = 2^\kappa$. For $\beta < \kappa$, let

$$A_\beta = \{\alpha \mid \forall\sigma \in 2^\beta\ \exists\tau \in 2^{<\kappa}\ \tau \supseteq \sigma \wedge \tau \in \Sigma_\alpha\}$$

Now define

$$\mathcal{Y} = \left\{ \bigcup_{\zeta<\lambda^+} X_{\alpha_\zeta} \;\middle|\; \alpha_0 \in \kappa \wedge \alpha_\zeta \in A_{\beta_\zeta} \text{ for } \zeta > 0 \text{ where } \beta_\zeta = \bigcup_{\xi<\zeta}\bigcup_{\sigma\in\Sigma_{\alpha_\xi}} \operatorname{dom}\sigma \right\}$$

To help digesting what $\mathcal{Y}$ is, think of it as a recursive construction where $\alpha \in \kappa$ is arbitrary, $\alpha_\zeta \in A_{\beta_\zeta}$ for $\zeta > 0$, and $\beta_\zeta = \bigcup_{\xi<\zeta}\bigcup_{\sigma\in\Sigma_{\alpha_\xi}} \operatorname{dom}\sigma$ (think of the $\bigcup$ as a sup).

Note that $|\mathcal{Y}| \leq \kappa^{\lambda^+} = \kappa$, so we have the first point of the thesis. For the second one, let $D \subseteq 2^\kappa$ be open dense. Notice that, for any β,

$$\{\alpha \in A_\beta \mid X_\alpha \subseteq D\} \neq \emptyset$$

because, for any fixed β, for all $\sigma \in 2^\beta$ we can take $\tau_\sigma \supseteq \sigma$ such that $[\tau_\sigma] \subseteq D$ and then let α be such that $\Sigma_\alpha = \{\tau_\sigma \mid \sigma \in 2^\beta\}$. Note that if $\beta \leq \gamma$ then $A_\beta \supseteq A_\gamma$. Recursively, construct α_ζ, for $\zeta < \lambda^+$, such that $\alpha_\zeta \in A_{\beta_\zeta}$ and $X_{\alpha_\zeta} \subseteq D$. The member of $\mathcal{Y}$ for this construction is $\bigcup_{\zeta<\lambda^+} X_{\alpha_\zeta}$: as each X_{α_ζ} is included in D, so is their union.

For the last point, suppose $\mathcal{Y}' = \{Y_\delta \mid \delta < \lambda\}$ is given. We find a point in the intersection through diagonalisation as follows. Suppose that

$$Y_\delta = \bigcup_{\zeta<\lambda^+} X_{\alpha(\delta,\zeta)}$$

as per the recursive construction above, i.e. $\alpha(\delta, 0)$ is arbitrary in κ and $\alpha(\delta_\zeta) \in A_{\beta(\delta,\zeta)}$. Analogously, let

$$\beta(\delta,\zeta) = \bigcup_{\xi<\zeta} \bigcup_{\sigma\in\Sigma_{\alpha(\delta,\xi)}} \operatorname{dom}\sigma$$

Define a partial injective function $\eta \colon \lambda^+ \to \lambda$ recursively by

$$\eta(0) := \min\{\delta \mid \forall\varepsilon < \lambda\ \beta(\delta, 1) \leq \beta(\varepsilon, 1)\}$$
$$\eta(\zeta + 1) := \min\Big\{\delta \notin \{\eta(\xi) \mid \xi < \zeta\} \Bigm| \forall\varepsilon \notin \{\eta(\xi) \mid \xi < \zeta\}\ \beta(\delta, \zeta + 1) \leq \beta(\varepsilon, \zeta + 1)\Big\}$$

Eventually, we run out of δ's, so this is a function from a proper initial segment of λ^+ to λ. Specifically, if we let λ_0 be such that $\{\eta(xi) \mid \xi < \lambda_0\} = \lambda$, then η a bijection[1] $\lambda_0 \to \lambda$. We now sow that $\bigcap Y_\delta \neq \emptyset$ by recursively constructing $(\sigma_\zeta \in 2^{<\kappa} \mid \zeta < \lambda_0)$ such that

- $\sigma_0 = \langle\rangle$
- if $\xi < \zeta$ then $\sigma_\xi \subseteq \sigma_\zeta$
- and $\sigma_\zeta = \bigcup_{\xi<\zeta} \sigma_\xi$ for limit ζ
- $\sigma_{\zeta+1} \in \Sigma_{\alpha(\eta(\zeta),\zeta)}$
- $\operatorname{dom}\sigma_\xi \subseteq \bigcup_{\xi<\zeta} \beta(\eta(\xi), \xi + 1)$

Once this is done, just let $\sigma = \bigcup_{\zeta<\lambda_0} \sigma_\zeta$, and observe that

$$[\sigma] \subseteq \bigcap_\zeta X_{\alpha(\eta(\zeta),\zeta)} \subseteq \bigcap_\zeta Y_{\eta(\zeta)}$$

[1] Basically, the point of the all construction is that λ is the wrong ordering for $\mathcal{Y}'$, the correct one is λ_0.

To conclude, let's show that the construction above can actually be carried out. For this, notice that for $\xi < \zeta$ we have $\beta(\eta(\xi), \xi + 1) \leq \beta(\eta(\zeta), \xi + 1)$ by minimality of $\eta(\xi)$. But since β is increasing we have

$$\beta(\eta(\xi), \xi + 1) \leq \beta(\eta(\zeta), \xi + 1) \leq \beta(\eta(\zeta), \zeta) \leq \beta(\eta(\zeta), \zeta + 1)$$

Let's look at the recursion defining σ_ζ in the case $\zeta = 1$ for simplicity. Let $\sigma_1 \in \Sigma_{\alpha(\eta(0),0)}$ be arbitrary. So $\mathrm{dom}(\sigma_1) \subseteq \beta(\eta(0), 1)$ by definition of β. In the general successor case, assume we have σ_ζ as required, so

$$\mathrm{dom}(\sigma_\zeta) \subseteq \bigcup_{\xi<\zeta} \beta(\eta(\xi), \xi + 1)$$

RHS is at most $\beta(\eta(\zeta), \zeta)$ by (11.1). By definition, $\alpha(\eta(\zeta), \zeta) \in A_{\beta(\eta(\zeta),\zeta)}$. So we can find $\sigma_{\zeta+1} \in \Sigma_{\alpha(\eta(\zeta),\zeta)}$ extending σ_ζ. To conclude, just notice that by definition of β

$$\mathrm{dom}(\sigma_{\zeta+1}) \subseteq \beta(\eta(\zeta), \zeta + 1)$$

and that at limit stages the conditions are trivially satisfied. □

Chapter 12

13/11

12.1 The κ-B.R.S. Theorem

Theorem 12.1. pLOC $\preceq$ Cof$(\mathcal{M}_\kappa)$, i.e. there are $\Phi_- \colon \mathcal{M}_\kappa \to \kappa^\kappa$ and $\Phi_+ \colon \mathrm{pLoc} \to \mathcal{M}_\kappa$ such that if $\Phi_-(A) \in^* \varphi$ then $A \subseteq \Phi_+(\varphi)$.

Proof. Identify κ^β with κ; actually work with functions $f \colon \kappa \to \kappa^{<\kappa}$ with $f(\beta) \in \kappa^\beta$. So, instead of κ^κ, work with $[\kappa^{<\kappa}]^\kappa$ and partial slaloms $\varphi \colon \kappa \to [\kappa^{<\kappa}]^{<\kappa}$, where $\varphi(\beta) \in [\kappa^\beta]^{|\beta|}$.

Let $\langle X_\alpha \mid \alpha < \kappa \rangle$ be a base for the topology on 2^κ. For $\alpha, \beta < \kappa$, let $\mathcal{Y}_{\alpha,\beta} \coloneqq \{Y_\alpha, \beta, \gamma \mid \gamma < \kappa\}$ be given by the Main Lemma with X_α as X and $|\beta|$ as λ.

To define Φ_-, suppose A is meagre, as witnessed by $A = \bigcup_{\alpha<\kappa} A_\alpha$, each A_α nowhere dense, and WLOG[1] $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$. As said above, we want to define an element of $(\kappa^{<\kappa})^\kappa$, instead of one of κ^κ. Stipulate that[2]

$$A_\beta \cap Y_{\alpha,\beta,\Phi_-(A)(\beta)(\alpha)} = \emptyset$$

Such a $Y_{\alpha,\beta,\Phi_-(A)(\beta)(\alpha)}$ exists because $\mathcal{Y}_{\alpha,\beta}$ comes from the Lemma and A_β is nowhere dense, so its complement contains an open dense subset.

Given a partial slalom φ with $\varphi(\beta) \in [\kappa^\beta]^{|\beta|}$, put

$$\Phi_+(\varphi) \coloneqq 2^\kappa \setminus \Big(\bigcap_{\delta<\kappa} \bigcup_{\substack{\beta \geq \delta \\ \beta \in \operatorname{dom}\varphi}} \bigcup_{\alpha<\beta} \bigcap_{\sigma \in \varphi(\beta)} Y_{\alpha,\beta,\sigma(\alpha)} \Big)$$

Let's show this is meagre. $\bigcap_{\sigma\in\varphi(\beta)} Y_{\alpha,\beta,\sigma(\alpha)}$ is the intersection of $|\beta|$-many Y's from $\mathcal{Y}_{\alpha,\beta}$, so by the Main Lemma the intersection is a non-empty subset of X_α. Also, it's open, because each Y is and the open sets in this topology is stable under intersections of size $< \kappa$. So the set

$$\bigcup_{\substack{\beta \geq \delta \\ \beta \in \operatorname{dom}\varphi}} \bigcup_{\alpha<\beta} \bigcap_{\sigma \in \varphi(\beta)} Y_{\alpha,\beta,\sigma(\alpha)}$$

[1] Exercise: the union of $< \kappa$ nowhere dense subsets of 2^κ is nowhere dense.

[2] $\Phi_-(A)(\beta)$ should be a β-tuple, so we just need to define it on all the $\alpha < \beta$.

is open dense, as for each α, there is $\beta \in \varphi$ such that $\beta > \alpha$, and so the union meets X_α. It follows that $\Phi_+(\varphi)$ is meagre.

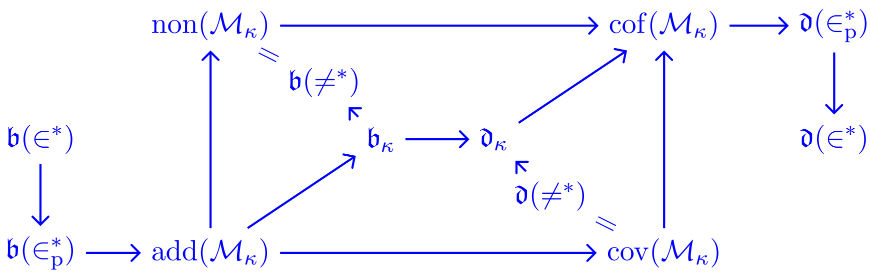
Now, assuming $\Phi_-(A) \in^* \varphi$, we need to show that $A \subseteq \Phi_+(\varphi)$. As $\Phi_-(A) \in^* \varphi$, there is β_0 such that for all $\beta \geq \beta_0$ we have $\Phi_-(A)(\beta) \in \varphi(\beta)$. Let $x \in A$, say $x \in A_\delta$ for some[3] $\delta \geq \beta_0$. Fix $\beta \in \operatorname{dom}\varphi$, $\beta \geq \delta$. For $\alpha < \beta$, we have $x \notin Y_{\alpha,\beta,\Phi_-(A)(\beta)(\alpha)}$ by choice of Φ_-. In particular, $x \notin \bigcap_{\sigma\in\varphi(\beta)} Y_{\alpha,\beta,\sigma(\alpha)}$. As this holds for all $\alpha < \beta$ and $\beta \geq \delta$, we have

$$x \notin \bigcup_{\substack{\beta\geq\delta \\ \beta\in\operatorname{dom}\varphi}} \bigcup_{\alpha<\beta} \bigcap_{\sigma\in\varphi(\beta)} Y_{\alpha,\beta,\sigma(\alpha)}$$

So x is not in the intersection as δ varies, i.e. $x \in \Phi_+(\varphi)$. □

Corollary 12.2. $\mathfrak{b}(\in^*_{\mathrm{p}}) \leq \operatorname{add}(\mathcal{M}_\kappa)$ and $\mathfrak{d}(\in^*_{\mathrm{p}}) \geq \operatorname{cof}(\mathcal{M}_\kappa)$.

So for inaccessibles we have



Question 12.3. Is $\mathfrak{b}(\in^*_{\mathrm{p}}) < \operatorname{add}(\mathcal{M}_\kappa)$ consistent? It is know to be in the case ω, but the proof uses a rank argument with Heckler forcing, that does not generalise well to the inaccessible case.

[3] As the union is increasing, then $x \in A_\beta$ for all $\beta \geq \delta$.

Chapter 13

14/11 – Stamatis Dimopoulos

13.1 Iterated Forcing – Basic Facts

We are going to assume familiarity with the basics of forcing.

Question 13.1. How to force GCH while preserving inaccessibles?

References:

1. Cummings[1], *Iterated forcing and elementary embeddings*, inside *Handbook of set theory.*
2. Baumgartner, *Iterated forcing*, Surveys in Set Theory. Beware of the fact that the notation here is oldish.

Definition 13.2. Let κ be an infinite cardinal, and $\lambda > \kappa$ an ordinal. *Cohen forcing* is defined as

$$\mathrm{Add}(\kappa, \lambda) := \{p \mid p \text{ partial function } \kappa \times \lambda \to 2, |p| < \kappa\}$$

ordered by reverse inclusion, i.e. $p \leq q$ iff $p \supseteq q$.

Another notation for $\mathrm{Add}(\kappa, \lambda)$, e.g. in Kunen's book, is $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2)$.

Definition 13.3 (Closure properties)**.** Let $\mathbb{P}$ be a forcing notion and κ an infinite cardinal. We say that

1. $\mathbb{P}$ is *κ-closed* iff every decreasing sequence of length $< \kappa$ has a lower bound.
2. $\mathbb{P}$ is *κ-directed closed* iff every downward directed subset of $\mathbb{P}$ of size $< \kappa$ has a lower bound.
3. $\mathbb{P}$ is *κ-distributive* iff for all generic filter G, for all $\lambda < \kappa$ every function $f : \lambda \to V$ in $V[G]$ exists already in V.

[1] Check his web page.

Remark 13.4. If $\mathbb{P}$ is separative, then $\mathbb{P}$ is κ-distributive if and only if the intersection of $< \kappa$-many open dense subsets of $\mathbb{P}$ is open dense.

Remark 13.5. In this list of properties of $\mathbb{P}$, each one implies the next one:

1. being κ-directed closed
2. being κ-closed
3. being κ-distributive
4. preserving cardinals $\leq \kappa$.

Moreover, the first two implications are strict.

Example 13.6. $\mathrm{Add}(\kappa, \lambda)$ is κ-directed closed.

Proposition 13.7. For κ infinite regular cardinal, $\mathrm{Add}(\kappa^+, 1)$ forces $2^\kappa = \kappa^+$.

Proof. $\mathrm{Add}(\kappa^+, 1)$ is κ^+-closed, so it does not add any new subset of κ. Let $G \subseteq \kappa^+$ be the new set added, i.e. the union of the generic filter. For any $A \subseteq \kappa$, it is dense to find a segment in G that looks like A. More formally, for any A this set is dense:

$$D_A \coloneqq \{p \in \mathbb{P} \mid \exists \alpha < \kappa^+ \ p \restriction [\alpha, \alpha +) \text{ codes } A\}$$

where "codes A" means that if you look at that function it is the characteristic function of A translated by α. As G intersects all of these, the function $f \colon \kappa^+ \to \mathscr{P}(\kappa)$ defined by $f(\alpha) = G \cap [\alpha, \alpha + \kappa)$ is surjective. □

Another way of showing this is proving that that forcing notion is isomorphic to $\mathrm{Add}(\kappa^+, 2^\kappa)$.

Remark 13.8. $\mathrm{Add}(\kappa, \lambda)$ is $(2^{<\kappa})^+$-c.c. If $\kappa^{<\kappa} = \kappa$, then $\mathrm{Add}(\kappa, \lambda)$ has the κ^+-c.c, so it preserves cardinals $\geq \kappa^+$.

Let's look at a two-step iteration: we want to do forcing a second time in the forcing extension; the point is that the poset we force with the second time may be in $V[G] \setminus V$, yet we want to be able to speak of this directly from the point of view of V.

Definition 13.9 (Two-Step Iteration)**.** Suppose $\mathbb{P}$ is a forcing notion, and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is a forcing notion. We define

$$\mathbb{P} * \dot{\mathbb{Q}} \coloneqq \{(p, \dot{q}) \mid p \in \mathbb{P}, \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\}$$

(pre[2])ordered in the following way

$$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \iff p_1 \leq p_2 \land p_1 \Vdash \dot{q}_1 \mathrel{\dot{\leq}} \dot{q}_2$$

[2]See later.

There is a variant where you replace $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$ with $p \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$, but they turn out the be equivalent.

There are some issues to address here, anyway:

1. $\mathbb{P} * \dot{\mathbb{Q}}$ can be a proper class. This is solved by choosing $\dot{q}$ as a representative for some equivalence class[3], e.g. the name with the least rank.

2. Actually, the $\leq$ we defined is not antisymmetric. This is solved by using preorders instead of posets[4].

Definition 13.10. $\mathbb{P}$ is an α-iteration, also denoted $\mathbb{P}_\alpha$, iff $\mathbb{P} = ((\mathbb{P}_\beta \mid \beta \leq \alpha), (\mathbb{Q}_\beta \mid \beta < \alpha))$ and for all $\beta < \alpha$

1. $\mathbb{P}_\beta$ is a forcing notion whose elements are β-sequences
2. if $p \in \mathbb{P}_\beta$ and $\gamma < \beta$, then $p \restriction \gamma \in \mathbb{P}_\gamma$
3. If $\beta < \alpha$, then $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta$ is a forcing notion
4. If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$, then $p(\gamma)$ is a $\mathbb{P}_\gamma$-name for an element of $\dot{\mathbb{Q}}_\gamma$
5. $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$ (the isomorphism is canonical)
6. for all $p, q \in \mathbb{P}_\beta$ we have $p \leq_{\mathbb{P}_\beta} q$ iff $\forall \gamma < \beta \; p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} p(\gamma) \leq_{\dot{\mathbb{Q}}_\gamma} q(\gamma)$
7. for all $\gamma \leq \beta$ we have[5] $\mathbb{1}_{\mathbb{P}_\beta}(\gamma) = \dot{\mathbb{1}}_{\mathbb{Q}_\gamma}$
8. if $p \in \mathbb{P}_\beta$, $\gamma < \beta$ and $q \leq_{\mathbb{P}_\gamma} p \restriction \gamma$ then $q ^\frown p \restriction [\gamma, \beta) \in \mathbb{P}_\beta$.

Remark 13.11. As a consequence of the definition, if $G \subseteq \mathbb{P}$ is a generic filter, then $G_\beta := \{p \restriction \beta \mid p \in G\}$ is a generic filter for $\mathbb{P}_\beta$ and $g_\beta := \{(p(\beta))_{G_\beta} \mid p \in G\}$ is a generic filter for $(\dot{\mathbb{Q}}_\beta)_{G_\beta}$.

Definition 13.12. If $p \in \mathbb{P}$, the *support* of p is defined by

$$\mathrm{supp}(p) := \{\beta < \alpha \mid p(\beta) \neq \dot{\mathbb{1}}_{\mathbb{Q}_\beta}\}$$

Definition 13.13. Suppose $\lambda \leq \alpha$ is a limit stage.

- $\mathbb{P}_\lambda$ is the *inverse limit* of $\{\mathbb{P}_\gamma \mid \gamma < \lambda\}$ iff

$$\mathbb{P}_\lambda = \{p \mid \; p \text{ is a } \lambda\text{-sequence}, \forall \gamma < \lambda \; p \restriction \gamma \in \mathbb{P}_\gamma\}$$

- $\mathbb{P}_\lambda$ is the *direct limit* of $\{\mathbb{P}_\gamma \mid \gamma < \lambda\}$ iff

$$\mathbb{P}_\lambda = \{p \mid \; p \text{ is a } \lambda\text{-sequence}, \forall \gamma < \lambda \; p \restriction \gamma \in \mathbb{P}_\gamma, \text{ and } \exists \beta < \lambda \; \forall \gamma \geq \beta \; p(\gamma) = \dot{\mathbb{1}}_{\mathbb{Q}_\gamma}\}$$

[3] The equivalence relation is "$\mathbb{1}$ forces the conditions to be equal"

[4] Or one could take quotients.

[5] In preorders we may have more equivalent maximal elements. We distinguish one.

- We say we use $< \kappa$-support iff inverse limits are taken at stages of cofinality κ and direct limits at cofinality $\geq \kappa$
- We say we use *Easton support* iff inverse limits are take at singular limit stages, and direct lmits are taken at regular limit stages.

Chapter 14

Stamatis Dimopoulos – 20/11

Proposition 14.1. Suppose $\mathbb{P}_\alpha = \mathbb{P}$ is the direct limit of $\{\mathbb{P}_\beta \mid \beta < \alpha\}$, κ regular $> \omega$. If

- $\forall \beta < \alpha$, $\mathbb{P}_\beta$ has the κ-c.c.
- if $\mathrm{cf}(\alpha) = \kappa$ then direct limits are taken at a stationary subset of α

Then $\mathbb{P}_\alpha$ has the κ-c.c.

Proposition 14.2. If $\mathbb{P}$ has the κ-c.c. and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ has the κ-c.c., then $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-c.c.

Proposition 14.3. Let κ be regular, $\kappa > \omega$, $\mathbb{P}_\alpha$ as in Definition 13.10. If

- $\forall \beta < \alpha \ \Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta$ is κ-directed closed
- all limits are either inverse or direct and inverse limits are taken at stages of cofinality $< \kappa$

then $\mathbb{P}_\alpha$ is κ-directed closed.

14.1 Factoring an iteration

Let $\beta < \alpha$. If $p \in \mathbb{P}_\alpha$, let $p^\beta = p \restriction \{\gamma \mid \beta \leq \gamma < \alpha\}$. Let $\mathbb{P}_{\beta\alpha} = \{p^\beta \mid p \in \mathbb{P}_\alpha\}$. If $G_\beta \subseteq \mathbb{P}_\beta$ is V-generic, then $p^\beta \leq q^\beta$ iff $\exists r \in G_\beta$ such that $r \cup p^\beta \leq_{\mathbb{P}_\alpha} r \cup q^\beta$. Let $\dot{\mathbb{P}}_{\geq\beta} \equiv \dot{\mathbb{P}}_{\beta\alpha} \equiv \dot{\mathbb{P}}_{[\beta,\alpha)}$ be a $\mathbb{P}_\beta$-name for $\mathbb{P}_{\beta\alpha}$.

Proposition 14.4. $\mathbb{P}_\alpha \cong \mathbb{P}_\beta * \dot{\mathbb{P}}_{\geq\beta}$.

Proposition 14.5. $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{P}}_{\geq\beta}$ is (isomorphic to) an $(\alpha - \beta)$-iteration (i.e. defines on $\{\gamma \mid \beta \leq \gamma < \alpha\}$)

Proposition 14.6. Let $\kappa > \omega$ be regular. If

- every $A \subseteq \mathrm{Ord}$ of size $< \kappa$ in the forcing extension by $\mathbb{P}_\beta$, is covered by a set $B \subseteq \mathrm{Ord}$, $B \in V$, $|B| < \kappa$
- $\forall \beta \leq \gamma < \alpha \Vdash_{\mathbb{P}_\gamma} \dot{\mathbb{Q}}_\gamma$ is κ-directed closed.
- inverse limits are taken at stages of cofinality $< \kappa$

then $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{P}}_{\beta\alpha}$ is κ-directed closed (also for κ-closed).

Proposition 14.7. If κ is inaccessible, $\mathbb{P}_\kappa$ is a κ-iteration and

- $\forall \alpha < \kappa\ \dot{\mathbb{Q}}_\alpha \in V_\kappa$
- a direct imit is taken at κ and at a stationary subset of stages $< \kappa$

then $\mathbb{P}_\kappa \subseteq V_\kappa$, $\mathbb{P}_\kappa$ is κ-c.c. and $\forall \alpha < \kappa$ for $\mathbb{P}_\kappa \cong \dot{\mathbb{P}}_\alpha * \dot{\mathbb{P}}_{\geq\alpha}$, $\dot{\mathbb{P}}_{\geq\alpha}$ is forced to be κ-c.c. and to have size κ.

Definition 14.8. The *GCH forcing* is the (class) iteration $\mathbb{P} = \langle\langle \mathbb{P}_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{Ord}\rangle\rangle$ with Easton support such that $\forall \alpha \in \mathrm{Ord}$, if $\mathbb{P}_\alpha$ has been defined and $\Vdash_{\mathbb{P}_\alpha} \alpha$ is a cardinal, then let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\alpha^+, 1)$; otherwise let $\dot{\mathbb{Q}}_\alpha$ name the trivial forcing[1].

Theorem 14.9. After forcing with $\mathbb{P}$, GCH holds and all inaccessible cardinals are preserved.

Proof. One should take care of the extra technicalities in class forcing; in this case everything works fine and we skip those details.

Let $G \subseteq \mathbb{P}$ be a V-generic filter. To see that GCH holds, let α be a cardinal in $V[G]$. Split $\mathbb{P} \cong \mathbb{P}_\alpha * \dot{\mathbb{P}}_{\geq\alpha}$, so $V[G_\alpha]$ is a sub-universe of $V[G]$. Now, α is still a cardinal in $V[G_\alpha]$. But then the next step forces GCH at α, i.e. $V[G_{\alpha+1}] \vDash 2^\alpha = \alpha^+$. By two of the previous propositions, $\dot{\mathbb{P}}_{\geq\alpha}$ is α^+-directed closed, hence α^+-distributive, so $2^\alpha = \alpha^+$ still holds in $V[G]$.

Now suppose κ is inaccessible in V. Suppose that κ is not regular in $V[G]$, and let $\lambda = \mathrm{cf}(\kappa) < \kappa$. Split $\mathbb{P} \cong \mathbb{P}_\lambda * \dot{\mathbb{P}}_{\geq\lambda}$. As $\mathbb{P}_\lambda$ has size $< \kappa$, it cannot change $\mathrm{cof}(\kappa)$, and as $\dot{\mathbb{P}}_{\geq\lambda}$ is λ^+-closed it cannot collapse $\mathrm{cof}(\kappa)$. This is a contradiction, so κ is still regular in $V[G]$. Suppose now that κ is not strong limit anymore in $V[G]$, and let $\lambda < \kappa$ be such that $2^\lambda \geq \kappa$. Split $\mathbb{P} \cong \mathbb{P}_\lambda * \dot{\mathbb{P}}_{\geq\lambda}$. Now $\mathbb{P}_\lambda$ is too small to force $2^\lambda \geq \kappa$, and $\dot{\mathbb{P}}_{\geq\lambda}$ is λ^+-closed, so it does not add any new subsets to λ, resulting in a contradiction. □

Remark 14.10. As being inaccessible is downward absolute, forcing cannot create new inaccessibles.

[1] The poset with just one element.

Chapter 15

21/11

Work with $\kappa = \omega$. Today we want to prove $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\in^*)$, where $\mathcal{N}$ is the idea of Lebesgue null sets. We need this fact:

Theorem 15.1. $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$.

Definition 15.2 (⚠ Beware: non-standard notation ⚠)**.** For this lecture[1], let a *converging series* be some $f\colon \omega \to \mathbb{Q}^{\geq 0}$ such that $\sum_{i\in\omega} f(i) < \infty$, and let $\mathfrak{h}$ be the least cardinality of a set of converging series such that no one converging series dominates (summand-wise in all but finitely often places) all of them.

Proposition 15.3. $\mathrm{add}(\mathcal{N}) \geq \mathfrak{h}$.

Proof. Take a family $\{G_\xi \mid \xi < \lambda < \mathfrak{h}\}$ of Lebesgue null sets. We want to show that $\bigcup_{\xi<\lambda} G_\xi$ is Lebesgue null. As G_ξ is Lebesgue null, it as a subset of

$$\bigcap_{n\in\omega} \bigcup_{m>n} I_m^\xi$$

where the I_m^ξ are some intervals with rational endpoints such that $\sum_{m=1}^{\infty} \mu(I_m^\xi) < \infty$. Fix an enumeration $(I_n)_{n\in\omega}$ of the intervals with rational endpoints and define

$$f_\xi(n) := \begin{cases} 1 & \text{if } \exists m\ I_n = I_n^\xi \\ 0 & \text{otherwise} \end{cases}$$

So we have

$$\sum_{n\in\omega} f_\xi(n) \cdot \mu(I_n) < \infty$$

As these are converging series and there are $\lambda < \mathfrak{h}$ of them, we can dominate (summand-wise, all but finite) all of these, and clearly we can assume that

[1] Usually both "series" and "$\mathfrak{h}$" mean something else.

the dominating series is the product of a $\{0,1\}$-function, say $f \in 2^\omega$, with $\mu(I_n)$. Take

$$G := \bigcap_{n\in\omega} \bigcup_{\substack{m>n \\ f(m)=1}} I_m$$

Then we have

$$G_\xi \subseteq \bigcap_n \bigcup_{m>n} I_n^\xi \subseteq G$$

and this shows $\mathfrak{h} \leq \operatorname{add}(\mathcal{N})$. $\square$

Chapter 16

27/11

[what follows was actually started in the previous lecture, but I have preferred to keep it all in one chapter]

We now want to show that $\mathfrak{h} \geq \mathrm{add}(\mathcal{N})$. We need the following fact.

Proposition 16.1. The following are equivalent:

1. $\kappa < \mathfrak{h}$
2. Any set of κ many functions $f\colon \omega \to \omega$ is localised by an $n \mapsto n^2$-slalom.
3. $\kappa < \mathfrak{b}$ and for any set of κ many functions $\omega \to \omega$ and any $g\colon \omega \to \omega$ such that $\sum_n \frac{1}{g(n)} < \infty$ dominating them all there is a slalom φ localising them all with $\sum_{n\in\omega} \frac{|\varphi(n)|}{g(n)} < \infty$.

Proof.

$2 \Rightarrow 1$ Let $F = \{f_\xi \mid \xi < \kappa\}$ be a set of converging series of size κ, i.e. for all $\xi < \kappa$ we have $f_\xi\colon \omega \to \mathbb{Q}^{>0}$ and $\sum_{n\in\omega} f_\xi(n) < \infty$. Define, for each ξ, a sequence $\langle n_k^\xi \mid k \in \omega\rangle$ such that

$$\forall k \sum_{i>n_k^\xi}^{\infty} f_\xi(i) < 2^{-k}$$

By assumption, there is $w\colon \omega \to \omega$ that dominates all of these sequences $k \mapsto n_k^\xi$. Define $f'_\xi(k) := f_\xi \restriction [w(k), w(k+1)) \in \omega^{<\omega}$. Identify $\omega^{<\omega}$ with ω, and use the hypothesis again to get a slalom φ such that for all k we have $|\varphi(k)| \leq k^2$ and for all $\xi < \kappa$ we have $f'_\xi \in^* \varphi$. Define $f\colon \omega \to \mathbb{Q}^{\geq 0}$ by

$$f(n) := \sup\left\{ s(n) \;\middle|\; s \in \varphi(k) \text{ for the } k \text{ s.t. } n \in [w(k), w(k+1)) \text{ and } \sum_{i=w(k)}^{w(k+1)-1} s(i) < 2^{-k} \right\}$$

(the idea is keeping track of the fact that n is in $[w(k), w(k+1))$). So

$$\sum_{n\in\omega} f(n) \leq \sum_{k\in\omega} \text{values in the } k\text{-inteval} \leq \sum_{k\in\omega} k^2 2^{-k} < \infty$$

(1 ⇒ 2) Suppose we have $\kappa < \mathfrak{h}$ many functions $\omega \to \omega$, say f_ξ for $\xi < \kappa$. Define $a_\xi \colon \omega \to \mathbb{Q}^{\geq 0}$ as

$$a_\xi(n) = \begin{cases} \max\{1/k^2 \mid f_\xi(k) = n\} & \text{if } \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Since $\kappa < \mathfrak{h}$, by definition there is $a(n)$ such that $\sum_n a(n) < \infty$ that eventually dominates every a_ξ. Assume WLOG that $\sum_n a(n) < 1$, and let $\varphi(k) = \{n \mid a(n) \geq k^{-2}\}$. As $\sum_n a(n) < 1$, for every k we have $|\varphi(k)| < k^2$, and so where a_ξ is dominated by a, f_ξ is guessed by φ.

(3 ⇒ 2) Take any set F of κ many functions $\omega \to \omega$. As $\kappa < \mathfrak{b}$ by hypothesis, there is $f \colon \omega \to \omega$ dominating everything in F. Let $(k_n)_{n\in\omega}$ be such that $\forall n\ k_n/f(n) = n^{-2}$. For $g \in \omega^\omega$, define $g' \in \omega^\omega$ by repeating $g(k_i)$ times the value $g(i)$: start with[1] k_1 times $g(1)$, then k_2 times $g(2)$, etc. As the elements of $\{e' \mid e \in F\}$ are all dominated by f' and $\sum_n 1/f(n) = \sum_{m\in\omega\setminus\{0\}} 1/m^2 < \infty$ we can apply our hypothesis and get a slalom φ with those properties. Take $\psi_m = \varphi(\ell)$ of least cardinality amongst those for ℓ in the k_m interval. Then we have

$$\infty > \sum_n \frac{|\varphi(n)|}{f'(n)} \geq \sum_n \frac{k_n|\psi_n|}{f(n)} = \sum_n \frac{|\psi_n|}{n^2}$$

In particular, we almost always have $|\psi_n|/n^2 < 1$.

(1 ⇒ 3) We will not see the proof of this part, as we are not going to need it in what follows. □

Corollary 16.2. $\mathfrak{h} = \mathfrak{b}_{n\mapsto n^2}(\in^*)$.

Proof. This is $1 \Leftrightarrow 2$ in Proposition 16.1. □

Proposition 16.3. If $\kappa < \operatorname{add}(\mathcal{N})$ then condition 3 in Proposition 16.1 holds.

Proof. By Theorem 15.1, we know $\kappa < \mathfrak{b}$. Take $F \subseteq \omega^\omega$ with $|F| = \kappa$ and f dominating everything in F with $\sum_n 1/f(n) < \infty$. Consider $X := \prod_{n\in\omega} f(n)$, where we think of $f(n)$ as the set of ordinals less than $f(n)$. Every $g \in X$ is by definition dominated by f, so we can define $H_g := \{x \in$

[1] We do not start with 0 because of $k_n/f(n) = n^{-2}$.

$X \mid \exists^\infty n\ x(n) = g(n)\}$. Equip each $f(n)$ with the equidistributed probability measure and let μ be the induced product measure on X. We have

$$\mu(H_g) = \mu\left(\bigcap_n \bigcup_{m>n} \{x \in X \mid x(m) = g(m)\}\right)$$
$$\leq \mu\left(\bigcup_{m>n} \{x \in X \mid x(m) = g(m)\}\right) \leq \sum_{m>n} \frac{1}{f(m)} \xrightarrow{n\to\infty} 0$$

Therefore[2] $\mu(H_g) = 0$. As $\bigcup_{e\in F} H_e$ is null, we can take a tree[3] T such that

its set of branches $[T]$ has positive measure above every node (16.1)

and $[T] \cap \bigcup_{e\in F} H_e = \emptyset$. Define $T(n) := \{x(n) \mid x \in [T]\}$ and $T_s := \{t \in T \mid s \leq t\}$.

Claim. $\forall e \in F\ \exists s \in T\ \forall n > h(s)\ e(n) \notin T_s(n)$

Suppose the Claim was false, as witnessed by e. Then there is $x \in [T]$ such that $\exists^\infty n\ x(n) = e(n)$. But then $x \in [T] \cap H_e$, contradicting the choice of T and proving the Claim.

For each $e \in F$, let $s \in T$ be given by the Claim. List the s's as $s_1, s_2, \ldots$, and denote $\varphi_n(m) = T_{s_n}(m)$. Then, by (16.1),

$$\prod_{m=1}^{\infty} \frac{|\varphi_n(m)|}{f(m)} > 0$$

Modify the first few $\varphi_n(m)$'s if necessary, to get

$$\prod_{m=1}^{\infty} \frac{|\varphi_n(m)|}{f(m)} > 1 - 2^{-n-1}$$

and let $\varphi(m) := \bigcap_n \varphi_n(m)$. We now have

$$\prod_{m=1}^{\infty} \frac{|\varphi(m)|}{f(m)} > 0$$

and $\psi_n := f(n) \setminus \varphi(n)$ is the slalom we were looking for. □

Corollary 16.4. $\mathrm{add}(\mathcal{N}) \leq \mathfrak{h}$.

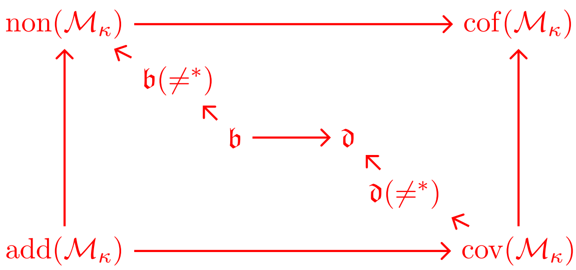
Proof. By $3 \Rightarrow 1$ in Proposition 16.1. □

[2] It is an instance of Borel-Cantelli.
[3] In X.

Chapter 17

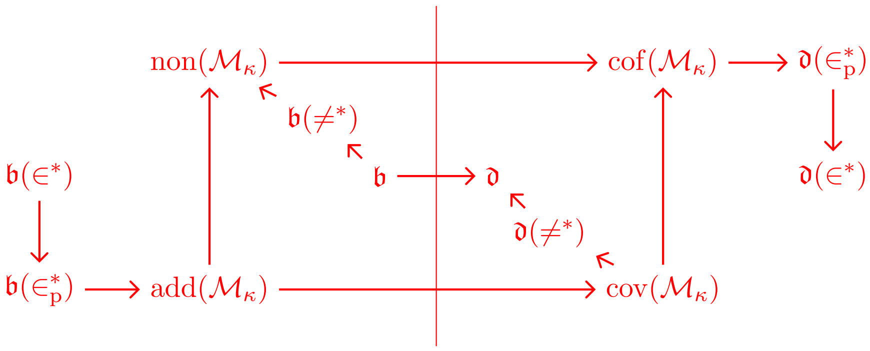
28/11

Remember that Chicon's diagram, without assuming inaccessibility, is



Today we want to see what happens to Chicon's diagram after Cohen forcing.

Theorem 17.1 ($\kappa = \kappa^{<\kappa}$)**.** If $\lambda > \kappa^+$ is such that $\lambda^\kappa = \kappa$, the poset $\mathrm{Add}(\kappa, \lambda)$ forces $\mathrm{non}(\mathcal{M}_\kappa) = \kappa^+$ and $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa = \lambda$. In particular, Chicon's diagram splits as follows, where everything in the left part is κ^+ and everything in the right part is $\lambda = 2^\kappa$



Before the proof, we need some preliminaries.

Recall that $\mathrm{Add}(\kappa, \lambda)$ is the poset of partial functions from $\kappa \times \lambda$ to κ with $|\mathrm{dom}| < \kappa$. Equivalently, it is a λ-fold product of $\mathrm{Add}(\kappa, 1)$ with $< \kappa$ support. As $\mathrm{Add}(\kappa, 1)$ is κ-directed-closed, it adds no new subsets of ordinals $< \kappa$.

Equivalently it is, up to forcing equivalence, a λ-length iteration of $\mathrm{Add}(\kappa, 1)$ with $< \kappa$ support.

Fact 17.2. $\mathrm{Add}(\kappa, \lambda)$ has the κ^+-c.c. (This uses $\kappa^{<\kappa} = \kappa$).

Proof. Exercise: re-read the Δ-system Lemma from Kunen (II-.1.6. in the original edition, 49 in some other one). □

Lemma 17.3. If $\mu < \lambda$ and $X \subseteq \mu$ in the $\mathrm{Add}(\kappa, \lambda)$-generic extension, then there is a subset B of λ of size at most μ such that X is already added by $\mathrm{Add}(\kappa, B)$.

Proof. Every such X has a "nice name" of the form

$$\bigcup_{\alpha<\mu} \{(\check{\alpha}, p) \mid p \in A_\alpha\}$$

where each A_α is an antichain. Each p has $|\mathrm{dom}(p)| < \kappa$, and $\mathrm{Add}(\kappa, \lambda)$ has the κ^+-c.c, so letting

$$B := \bigcup_{\alpha<\mu} \bigcup_{p \in A_\alpha} \mathrm{dom}(p)$$

we have $|B| \leq \mu$, and X is completely determined by the B coordinates of the forcing. □

Remark 17.4. If $\mu = \kappa$, since $\lambda^\kappa = \lambda$ there are only λ many such nice names, so $(2^\kappa)^{\mathrm{Add}(\kappa,\lambda)} \leq \lambda$. Also, each coordinate gives a different subset of κ, so $(2^\kappa)^{\mathrm{Add}(\kappa,\lambda)} \geq \lambda$.

Proof of Theorem 17.1. For any nowhere dense set $X \subseteq 2^\kappa$ there is $f\colon 2^{<\kappa} \to 2^{<\kappa}$ such that $\forall \sigma \in 2^{<ka}\ f(\sigma) \supseteq \sigma$ and

$$X \subseteq \{s \in 2^\kappa \mid \forall \sigma \in 2^{<\kappa}\ \underbrace{f(\sigma) \not\subseteq x}_{x \notin [f(\sigma)]}\} =: A_f$$

Let $f\colon 2^{<\kappa} \to 2^{<\kappa}$ be such that $\forall \sigma\ f(\sigma) \supseteq \sigma$ in the $\mathrm{Add}(\kappa, \lambda)$-generic extension[1]. By our assumptions $|2^{<\kappa}| = \kappa$, so by the previous Lemma there is a set B_f of size κ such that f is added by $\mathrm{Add}(\kappa, B_f)$. Moreover, for $\beta \notin B_f$, the β coordinate Cohen subset c_β of κ is *not* in A_f in the extension, by a genericity argument. Namely, split the poset as a product of B_f with all the rest and think of it as a two-step extension, and notice that it is dense for c_β to include some $f(\sigma)$. So now if we have $\mathcal{X}$ a set of nowhere dense sets of the form A_f in the $\mathrm{Add}(\kappa, \lambda)$-generic extension with[2] $|\mathcal{X}| < \lambda$, then

$$\underbrace{\left| \bigcup_{f \in \mathcal{X}} B_f \right|}_{=:\mathcal{B}} < \lambda$$

[1] Note that $2^{<\kappa}$ is unchanged in the generic extension.

[2] One can also show (exercise) that it is possible to find a name for $\mathcal{X}$ of cardinality $< \lambda$.

and therefore any $\beta \notin \mathcal{B}$ has $c_\beta \notin \bigcup_{f \in \mathcal{X}} A_f$. This shows that in the extension $\mathrm{cov}(\mathcal{M}_\kappa) \geq \lambda$, and as $2^\kappa = \lambda$ we have equality.

To conclude, we need to show that $\mathrm{non}(\mathcal{M}_\kappa) \leq \kappa^+$. We explicitly give a non-meagre set of size κ^+, namely[3]

$$\{c_\beta \mid \beta < \kappa^+\}$$

To see this is non-meagre, consider any κ many nowhere dense sets A_f in the extension. By the previous Lemma there is $B \subseteq \lambda$ adding all of them and with $|B| = \kappa$. Take $\beta \in \kappa^+ \setminus B$. Then $c_\beta \notin \bigcup A_f$, and so $\{c_\beta \mid \beta < \kappa^+\}$ is not contained in any (extension) meagre set. $\square$

[3] Or any κ^+-size subset of the λ-many Cohen reals we added.

Chapter 18

04/11

18.1 Hechler Forcing

Definition 18.1 (1-step version)**.** The conditions of $(\mathbb{H}, \leq)$ are pairs (s, f) such that

- $s \in \kappa^{<\kappa}$
- $f \in \kappa^{\kappa}$
- s is an initial segment of f; we denote this with $s \sqsubseteq f$

The order is $(s, f) \geq (t, g)$ iff[1] $t \sqsupseteq s$ and $\forall \alpha\ g(\alpha) \geq f(\alpha)$.

Remark 18.2. Note that in particular t dominates s on $\operatorname{dom} s$.

We can think of conditions as a "stem" s and a "promise" f.

Definition 18.3. A partial order $\mathbb{P}$ is

- $(1, <\kappa)$-*centred* iff every $<\kappa$ many conditions have a common extension;
- $(\lambda, <\kappa)$-*centred* iff $\mathbb{P} = \bigcup_{\alpha<\lambda} P_\alpha$, where each P_α is $(1, <\kappa)$-centred;
- κ-*centred* iff it is $(\kappa, <\kappa)$-centred.

Example 18.4. Hechler forcing at κ is κ-centred.

Proof. Each "stem" defined a P_α, i.e. for all $s \in \kappa^{<\kappa}$ the set $\{(s, f) \mid f \in \kappa^{\kappa}\}$ is $(1, <\kappa)$-centred: just take the supremum of the f's, which can be done as we have $<\kappa$ of them. □

Remark 18.5. If $\mathbb{P}$ is κ-centred, then $\mathbb{P}$ is κ^+-c.c.

[1] Again, this means that t is an initial segment of s.

The following notion is not needed in the ω case, but it is necessary in general to deal with small cofinality limit stages.

Definition 18.6. Assume $\mathbb{P}$ is $(<)\kappa$ closed and κ-centred, say $\mathbb{P} = \bigcup_{\gamma<\kappa} P_\gamma$, where each P_γ is $(1, < ka)$-centred. We say that $\mathbb{P}$ is *κ-centred with canonical lower bounds* iff there is $f_\mathbb{P} \colon \kappa^{<\kappa} \to \kappa$ such that whenever $\lambda < \kappa$ and $(p_\alpha \mid \alpha < \lambda)$ is a decreasing sequence from $\mathbb{P}$ with $p_\alpha \in P_{\gamma_\alpha}$, there is $p \in P_{f_\mathbb{P}(\gamma_\alpha|\alpha<\lambda)}$ such that for all $\alpha < \lambda$ we have $p \leq p_\alpha$.

Example 18.7. For Hechler forcing, if $p_\alpha = (s_\alpha, f_\alpha)$ and $p_\beta \leq p_\alpha$, then $s_\beta \sqsupseteq s_\alpha$, so we can take

$$f_\mathbb{H} \colon (s_0, s_1, s_2, \ldots, s_\alpha, \ldots \mid \alpha < \lambda) \mapsto \bigcup_{\alpha<\lambda} s_\alpha$$

Fact 18.8. Hechler forcing adds a function $h\kappa \to \kappa$ eventually dominating all ground model functions: it is dense for (s, f) to have $f \geq^* g$ for any given g, so we can just take $h = \bigcup_{(s,f)\in G} s$.

18.2 Slalom Forcing

Definition 18.9. Define $(\mathbb{S}_h, \leq)$ to as have conditions pairs $(s, \mathcal{F})$ such that

- there is $\lambda < \kappa$ such that $s \colon \lambda[\kappa]^{<\kappa}$ and $|s(\alpha)| \leq h(\alpha)$
- $\mathcal{F}$ is a set of functions $\kappa \to \kappa$ of size $h(\lambda)$

The order is $(s, \mathcal{F}) \geq (t, \mathcal{G})$ iff

- $t \supseteq s$, $\mathcal{G} \supseteq \mathcal{F}$, and
- $\forall\alpha \in \operatorname{dom} t \setminus \operatorname{dom} s \; \forall f \in \; mcF \; f(\alpha) \in t(\alpha)$.

Think of $\mathcal{F}$ as a "promise to localise all f in $\mathcal{F}$ hereafter". And in fact,

Fact 18.10. $\bigcup_{(s,\mathcal{F})\in G} s$ is a slalom localising all ground model functions.

Note that the requirement of $\mathcal{F}$ gets in the way of κ-centredness: the point is that the domain of a common extension of a family actually depends on the stems, and not just on their domains. This is where partial slaloms are more handy to manage.

Definition 18.11. *Partial h-slalom forcing* is defined analogously, except s can be partial and $\mathcal{F}$ can have any size $< \kappa$.

Proposition 18.12. This is κ-centred with canonical lower bounds.

Proof. You can now take the union of the promises and just keep the same stem: we can extend that later. □

Lemma 18.13. Suppose $(\mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \mu)$ is an iteration of κ-closed, κ-centred with canonical lower bounds forcings $\mathbb{Q}_\alpha$ with $< \kappa$ support and such that for each α the function $f_{\dot{\mathbb{Q}}_\alpha}$ is in the ground model[2] and $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha = \bigcup_{\gamma<\kappa} \dot{\mathbb{Q}}_{\alpha,\gamma}$. Then the set of conditions $p \in \mathbb{P}_\mu$ such that for all $\beta \in \operatorname{supp}(p)$ there is $\gamma < \kappa$ such that $p \restriction \beta \Vdash p(\beta) \in \mathbb{Q}_{\beta,\check{\gamma}}$ is dense.

In other words, it is dense that for everything in the support the stem lives in the ground model (or: it is dense to choose a stem).

Proof Sketch. Given $p \in \mathbb{P}$, list $\operatorname{supp}(p)$ as $(\beta_\delta \mid \delta < |\operatorname{supp}(p)|)$ such that each $\beta \in \operatorname{supp}(p)$ appears cofinally often[3]. Go through, at stage δ, extending to get $p_\delta(\beta_\delta)$ in a specific $Q_{\beta_\delta,\gamma}$. □

[2]The original ground model.

[3]Here we are assuming that the support is infinite. If it is not, extend arbitrarily. In the ω case, conditions have finite support, so take the maximum β in the support, [extend that?] and go backwards.

Chapter 19

05/12

19.1 Iterations of Centred Forcings

Lemma 19.1. Let $\mu < (2^\kappa)^+$ be an ordinal. Assume $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha$ is an iteration of length μ with $< \kappa$ supports of ($< \kappa$-closed) κ-centred with canonical lower bounds forcings $\mathbb{Q}_\alpha$ such that the functions $f_{\dot{\mathbb{Q}}_\alpha}$ are in the ground model. Then $\mathbb{P}_\mu$ is $< \kappa$-closed and (forcing equivalent to something) κ-centred (so, in particular, κ^+-c.c.).

Proof. κ-closure is standard. To see it is κ-centred, take an injection $f \colon \mu \to 2^\kappa$. Let $\mathcal{F}$ be the collection of all functions F such that there is $\delta_F < \kappa$ such that

- $\operatorname{dom} F \subseteq 2^\delta_F$
- $|\operatorname{dom} F| < \kappa$
- $\operatorname{codomain} F = \kappa$

These will correspond to the "stems", and partition our iteration. Since $\kappa^{<\kappa} = \kappa$, we have[1] $|\mathcal{F}| = \kappa$. Define the partition piece for F as

$$P_F := \{p \in \mathbb{P}_\mu \mid \forall \beta \in \operatorname{supp}(p)\ f(\beta) \restriction \delta_F \in \operatorname{dom} F \wedge p \restriction \beta \Vdash p(\beta) \in \dot{\mathbb{Q}}_{\beta, F(f(\beta)\restriction\delta_F)}\}$$

We now just need to show that

1. each P_F is $(1, < \kappa)$-centred, and
2. $\bigcup_{F \in \mathcal{F}} P_F$ is dense[2] in $\mathbb{P}_\mu$

[1] use that then $2^{\delta_F} \leq \kappa$.
[2] Which is enough up to forcing equivalence.

For the first part, assume we have $\lambda < \kappa$ many elements p_ξ of P_F. We find a common extension $p \restriction \beta$ by recursion in $\beta < \mu$. If $\forall \xi < \lambda\ \beta \notin \text{supp}(p_\xi)$, then take $p(\beta) = \mathbb{1}$. If $\beta \in \text{supp}(p_\xi)$, then[3]

$$p \restriction \beta \Vdash p_\xi(\beta) \in \dot{\mathbb{Q}}_{\beta, F(f(\beta) \restriction \delta_F)}$$

Since $\dot{\mathbb{Q}}_{\beta, F(f(\beta) \restriction \delta_F)}$ is $(1, < \kappa)$-centred, there is a (forced by $p \restriction \beta$ to be) common extension, call it $p(\beta)$. As we only had $\lambda < \kappa$ many p_ξ to consider and each had size $< \kappa$, the support of p has size $< \kappa$.

For the second part, let $p \in \mathbb{P}_\mu$; up to extending it, assume it WLOG to be as per Lemma 18.13. Since $|\text{supp}(p)| < \kappa$. By the identification given by f, think of this as $< \kappa$ many κ-length bit strings, all different, and find $\delta < \kappa$ such that $\forall \beta, \gamma \in \text{supp}(p)\ f(\beta) \restriction \delta \neq f(\gamma) \restriction \delta$. This is our δ_F. Let $F \in \mathcal{F}$ be the function with domain $\{f(\beta) \restriction \delta \mid \beta \in \text{supp}(p)\}$ such that $\forall \beta \in \text{supp}(p)\ F(f(\beta) \restriction \delta) \coloneqq \iota_\beta$, where $p \restriction \beta \Vdash p(\beta) \in \dot{\mathbb{Q}}_{\beta, \iota_\beta}$. Then $p \in P_F$. □

19.2 Iterations of Hechler Forcing

We saw that κ-Hechler forcing is $< \kappa$-closed and κ-centred with canonical lower bounds. We want to do a long iteration of it.

Let $\lambda \geq \kappa^+$ be regular, and consider a λ-length iteration of κ-Hechler forcing. If λ is big enough, it will not be κ-centred anymore, but it will still be κ^+-c.c: use Lemma 18.13 and a Δ system argument.

Exercise 19.2 (Prove this by the 12th of January as second part of the assessment for this course.)**.** Prove this.

[3] It is forced by p_ξ, and $p \restriction \beta$ is a common extension of all of them.

Chapter 20

11/12

20.1 Iterations of Hechler Forcing, continued

Take $\lambda \geq \kappa^+$ regular. Take a $<\kappa$-support iteration of Hechler forcing of length λ. We already said that this is κ-closed and κ^+-c.c.

Start with GCH and have $\lambda > \kappa^+$.

Proposition 20.1. This forcing makes $\mathrm{add}(\mathcal{M}_\kappa) = 2^\kappa = \lambda$.

Proof. We showed (Corollary 9.2) that $\mathrm{add}(\mathcal{M}_\kappa) \geq \min\{\mathrm{cov}(\mathcal{M}_\kappa), \mathfrak{b}_\kappa\}$. Notice that the αth Hechler κ-real, mod 2 componentwise, is a Cohen κ-real. So in the forcing we (cofinally) add λ many Cohens, so in the extension we have, by previous resulst, $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$.

The point of Hechler forcing is dealing with the $\mathfrak{b}_\kappa$ part, i.e. we want to show that $\mathfrak{b}_\kappa^{V[G]} = (2^\kappa)^{V[G]} = \lambda$. If B is a subset of κ^κ in $V[G]$ of size $< \lambda$ then, by what we saw in the previous lectures, B occurs after some initial segment of the forcing, and the next Hechler real dominates it. So $\mathfrak{b}_\kappa^{V[G]} = \lambda$. $\square$

Let now κ be inaccessible and $\lambda = \kappa^{++}$, and recall Lemma 19.1. We want to show that

Proposition 20.2. For any h in $V[G]$ we have $b(\in_h^*)^{V[G]} = \kappa^+$.

Question 20.3 (Open)**.** What happens with $\mathfrak{b}(\in_{\mathrm{p}}^*)$?

Lemma 20.4. Let κ be strongly inaccessible, $\mathbb{P}$ be κ-centred and $<\kappa$-closed, and $h \in \kappa^\kappa$. Assume $\dot{\varphi}$ is a $\mathbb{P}$-name for an h-slalom. Then there are h-slaloms φ_α, for $\alpha < \kappa$, in the ground model such that if $f \in (\kappa^\kappa)^V$ is not localised by any φ_α, then

$$\Vdash_{\mathbb{P}} \dot{\varphi} \text{ does not localise } \check{f}$$

Proof. Let $\mathbb{P} = \bigcup_{\alpha<\kappa} P_\alpha$; where each P_α is $(1, <\kappa)$-centred. Suppose $\dot{\varphi}$ is a $\mathbb{P}$-name for an h-slalom, and for $\alpha < \kappa$ define

$$\varphi_\alpha(\beta) \coloneqq \{\gamma \in \kappa \mid \exists p \in P_\alpha \; p \Vdash \check{\gamma} \in \dot{\varphi}(\check{\beta})\}$$

We claim that for every α, β we have $|\varphi_\alpha(\beta)| \leq h(\beta)$. In fact, if this does not happen we can take $h(\beta)^+$ many γ in $\varphi_\alpha(\beta)$ such that $p_\delta \in P_\alpha$ and $p_\delta \Vdash \check{\gamma}_\delta \in \dot{\varphi}(\check{\beta})$. But then[1] $\{p_\delta \mid \delta < h(\beta)^+\} \subseteq P_\alpha$ has cardinality $< \kappa$, so those conditions have a common extension q. By definition of $\varphi_\alpha(\beta)$, we have $q \Vdash |\dot{\varphi}(\check{\beta})| > \check{h}(\check{\beta})$. This contradicts the definition of φ, which was supposed to be a name for an h-slalom. Therefore every φ_α is an h-slalom.

If $f \in (\kappa^\kappa)^V$ is such that $\forall \alpha < \kappa \; \exists^\kappa \beta \; f(\beta) \notin \varphi_\alpha(\beta)$, fix $p \in \mathbb{P}$ and $\beta_0 < \kappa$. Let α be such that $p \in P_\alpha$. Take $\beta > \beta_0$ such that $f(\beta) \notin \varphi_\alpha(\beta)$, i.e. there is no $p' \in P_\alpha$ such that $p' \Vdash \check{f}(\check{\beta}) \in \dot{\varphi}(\check{\beta})$. In particular, $p \nVdash \check{f}(\check{\beta}) \in \dot{\varphi}(\check{\beta})$, and therefore there is $q \leq p$ such that $q \Vdash \neg \check{f}(\check{\beta}) \in \dot{\varphi}(\check{\beta})$. □

Proof of Proposition 20.2. For any h in $V[G]$, we know that h appears in an initial segment of the forcing say by stage α_0. Consider stage $\alpha_1 \coloneqq \alpha_0 + \kappa^+$. Then we have added κ^+ many Hechler[2] κ-reals "since" $V[G_{\alpha_0}]$, and a Hechler is not localised by any ground model slalom. These κ^+ many Hechlers are $\in^*$-unbounded in $V[G_{\alpha_1}]$, and by the previous Lemma they remain so in $V[G]$: any φ in $V[G]$ fails to localise them all because any φ in $V[G_{\alpha_1}]$ fails to localise more than κ many of them. To see why the last sentece is true, encode a slalom as a subset of κ, look at the stage where it appears and then consider the next Hechler. □

Dual arguments [with the same forcing?] apply to $\mathrm{cof}(\mathcal{M}_\kappa)$ and $\mathfrak{d}(\in^*)$.

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[1] As κ is inaccessible, $h(\beta)^+ < \kappa$. Also, $h(\beta)^+$ is still a cardinal in the generic extension by $<\kappa$-closure (the only thing we need is that κ does not collapse to $h(\beta)$).

[2] Maybe a similar argument works with Cohen κ-reals as well.