Amalgamating first-order structures

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What is this? This document contains (hastily written) notes for a mini-course, held on the 4th of December 2024 during the *Model Theory Workshop* at the EPFL in Lausanne. Most of the material follows very closely (parts of) chapters 7 and 8 of [Hod93]; for the Random Graph, see also [Cam97]. Other sources you can consult for model theory basics are [Poi00, TZ12, MT03, Mar02, CK90, Kir19]. There are also notes of a course I gave some years ago [Men22].

Info You can contact me at `R.Mennuni@posteo.net`. These notes are available at `https://poisson.phc.dm.unipi.it/~mennuni/amalgamation_notes.pdf`. This version has been compiled on the 5th December 2024. To get the source code click on the leftmost paper clip. The bibliography source file is in the rightmost one.

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Chapter 1

Finitely generated (sub)structures

Most of the material in this chapter is contained in [Hod93, Chapter 7].

1.1 What's an age again?

Question 1.1.1. Fix a countable structure M. How does the class K of finitely generated substructures of M look like?

These three properties of K are clear:

1. K is (at most) countable.

2. If $B \in K$ and $A \subseteq B$ is finitely generated, then $A \in K$.

3. If $A, B \in K$, then the generated substructure $\langle A, B \rangle \in K$.

Given a class K, we want to recognise whether K is the class of substructures of some countable structure. Can we hope that the properties above give a characterisation?

Well, property 3 is problematic, as the notion of "generated substructure" depends on the ambient structure M, which, if we start from a class of finitely generated structures K, is not given. So it is better to replace it by a different statement. Moreover, it is convenient to close K under isomorphism. This makes it a proper class, but makes certain things easier to state.[1] Let us start introducing some terminology.

Definition 1.1.2. A function $f\colon A \to M$ is an *embedding* of L-structures iff, for every atomic L-formula $\varphi(x_0, \ldots, x_n)$ and every $a_0, \ldots, a_n \in A$, we have

$$A \vDash \varphi(a_0, \ldots, a_n) \iff M \vDash \varphi(f(a_0), \ldots, f(a_n)) \tag{1.1}$$

[1] So for example one can say "$A \in K$" instead of "A is isomorphic to a structure in K". On the other hand, "K is countable" becomes "K is essentially countable", so this is really a matter of taste. Anyway, everything can be translated with no issue to the setting where K is a countable set, and not a proper class.

$$\begin{tikzcd}
B_0 \arrow[dr, dashed, "\exists g_0"] & \\
& C \\
B_1 \arrow[ur, dashed, "\exists g_1"'] &
\end{tikzcd}$$

Figure 1.1: The Joint Embedding Property.

Note that this is a strengthening of the definition of homomorphism, that only requires $\implies$.

Exercise 1.1.3. If f is an embedding, then (1.1) also holds for quantifier-free $\varphi(x_0, \ldots, x_n)$. Moreover, f is injective.

Definition 1.1.4. If M is an L-structure, we denote by $\mathrm{Age}(M)$ the class of finitely generated L-structures that can be embedded in M.

In other words, $A \in \mathrm{Age}(M)$ if and only if A is isomorphic to some finitely generated substructure of M.[2]

Definition 1.1.5. Let K be a class of structures.

1. K is *essentially countable* iff it contains at most countably many isomorphism types.
2. K has the *Hereditary Property* (HP) iff it is closed under finitely generated substructures and under isomorphism.
3. K has the *Joint Embedding Property* (JEP) iff, whenever $B_0, B_1 \in K$, there are $C \in K$ and embeddings $g_0 \colon B_0 \to C$ and $g_1 \colon B_1 \to C$, as in Figure 1.1.

Note that the JEP does not mention any ambient structure. These properties are in fact enough to characterise ages.

Theorem 1.1.6. Let K be a class of finitely generated L-structures. There is a countable M such that $K = \mathrm{Age}(M)$ if and only if K is essentially countable and has HP and JEP.

Proof. Left to right, essential countability and the HP are clear, and so is JEP if one considers the structure generated by $B_0 \cup B_1$.

Right to left, for each isomorphism type in K, fix a representative and list them as $(A_i \mid i < \omega)$. Define $B_0 = A_0$ and, inductively, let B_{n+1} be some structure in K containing both B_n and A_{n+1}: this B_{n+1} exists by the JEP. Let $M \coloneqq \bigcup_{i<\omega} B_i$. This is countable, and by construction $\mathrm{Age}(M) \supseteq K$. For the other inclusion, let $C \subseteq M$ be finitely generated. Its generators must lie in some B_n, and since B_n is a structure we have $C \subseteq B_n$. By HP we have $C \in K$, hence $\mathrm{Age}(M) \subseteq K$. $\square$

[2]In the alternative presentation, $\mathrm{Age}(M)$ would have simply been the set of finitely generated substructures of M.

1.2 Fraïssé's Theorem

The M constructed above is highly non-unique.

Example 1.2.1. Let $L = \{<\}$ and let K be the class of finite linear orders. Then $\mathrm{Age}(\mathbb{Z}) = K = \mathrm{Age}(\mathbb{Q})$. In fact, for *every* infinite linear order M, we have $K = \mathrm{Age}(M)$.

Can we single out a countable linear order that is, in some sense, *the* "limit" of the class of finite linear orders? For example, something unique up to isomorphism. We will do this by defining a property that allows isomorphisms to be built inductively. To understand what's happening, let us put ages aside for a moment, and let us contemplate a classical proof.

Definition 1.2.2. Let $L = \{<\}$, where $<$ is a binary relation symbol. The theory DLO of *dense linear orders without endpoints* has the following axioms:

1. $<$ is a *strict order*: an irreflexive, transitive relation;

2. $<$ is *linear*: $\forall x, y\ ((x < y) \vee (x = y) \vee (x > y))$;

3. $<$ has no *endpoints*: it has no maximum and no minimum;

4. $<$ is *dense*: $\forall x, y\ \big((x < y) \to (\exists z\ (x < z < y))\big)$.

This theory is consistent, as, clearly, $(\mathbb{Q}, <) \vDash \mathsf{DLO}$.

We will see a proof of the theorem below by *back-and-forth*. Legend has it that the first back-and-forth proof was by Cantor, who invented the method to prove what follows. Except this is false, and Cantor managed to prove it by only going "forth". Also, I have no idea whether the proof below is the first proof by back-and-forth ever written, but nowadays it is usually the first one people see. Anyway, here is the proof.

Theorem 1.2.3 (Cantor)**.** All countable dense linear orders with no endpoints are isomorphic (to $(\mathbb{Q}, <)$).

Proof. Let $(M, <)$ and $(N, <)$ be countable dense linear orders with no endpoints. Since they are dense (or, if you prefer, since they have no endpoints), M and N must both be infinite. Fix enumerations $(a_i)_{i<\omega}$ of M and $(b_j)_{j<\omega}$ of N. We build an isomorphism $f\colon M \to N$ inductively, by extending *partial isomorphisms*.

Start with f_0 being the empty function. If you prefer, f_0 is an isomorphism between the empty substructure of M and the empty substructure of N. We inductively define f_n in such a way that, for every $n \in \omega \setminus \{0\}$,

1. $f_n\colon A_n \to B_n$, where A_n is a finite substructure of M and B_n is a finite substructure of N;

2. $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$, and $f_n \subseteq f_{n+1}$;

3. f_n is an isomorphism of L-structures;

4. if $n = 2m$, then $a_m \in A_n$;

5. if $n = 2m + 1$, then $b_m \in B_n$.

Suppose we manage to do this for every $n \in \omega$. If you think about it for ≈ 30 seconds, you will realise that this is enough to conclude. But, to be more formal:

Because $A_n \subseteq A_{n+1}$, the union $\bigcup_{n\in\omega} \operatorname{graph}(f_n)$ is the graph of a function, call it f, with domain a subset of M and codomain N. In fact, by Item 4 its domain is the whole M, and its image is the whole of N by Item 5. If $m < m' < \omega$, then $a_m, a_{m'} \in A_{2m'}$ and by Item 3 we have

$$\begin{aligned} M \vDash a_m < a_{m'} \iff A_{2m'} \vDash a_m < a_{m'} \iff B_{2m'} \vDash f_{2m'}(a_m) < f_{2m'}(a_{m'}) \\ \iff N \vDash f_{2m'}(a_m) < f_{2m'}(a_{m'}) \iff N \vDash f(a_m) < f(a_{m'}) \end{aligned}$$

Therefore, $f\colon M \to N$ is an isomorphism of L-structures.

Let us do this inductive construction then. Suppose we have build an isomorphism $f_{n-1}\colon A_{n-1} \to B_{n-1}$ as above. Write $A_{n-1} = \{a_{i_0} < a_{i_1} < \ldots < a_{i_k}\}$ and $B_{n-1} = \{b_{j_0} < b_{j_1} < \ldots < b_{j_k}\}$, and recall that for all $i \le k$ we have $a_i \in M$ and $b_i \in N$. If n is even, say $n = 2m > 0$, we take care of the "forth" part, that is, we extend f_{n-1} to $A_n \coloneqq A_{n-1} \cup a_m$. We have four cases:

a) If we already have $a_m \in A_{n-1}$, do nothing. Or, more formally, set $A_n \coloneqq A_{n-1}$, $B_n \coloneqq B_{n-1}$, and $f_n \coloneqq f_{n-1}$.

b) $a_m < a_{i_0}$. In this case, since N has no endpoints, in particular it has no minimum, hence there must be some $b \in N$ with $N \vDash b < b_{i_0}$. Send a_m to b. Or, more formally, put $A_n \coloneqq A_{n-1} \cup \{a_m\}$, $B_n \coloneqq B_{n-1} \cup \{b\}$, and $f_n \coloneqq f_{n-1} \cup \{(a_m, b)\}$.

c) $a_m > a_{i_k}$. Similarly, N has no maximum, so it contains some $b > b_{i_k}$ where to send a_m. Or, more formally,... well, ok, you know what needs to be written here.

d) There is $\ell < k$ with $M \vDash a_{i_\ell} < a_m < a_{i_{\ell+1}}$. Because N is dense, there is $b \in N$ with $N \vDash b_{i_\ell} < b < b_{i_{\ell+1}}$. Send a_m to b.

This takes care of the "forth" part. The "back" part, that is, the odd stages of the construction, are handled in the same way, with the roles of M and N reversed; the only subtlety is that, for $n = 1$, there are no i_0, j_0. In that case, we start by simply choosing the preimage of b_0 arbitrarily, e.g. we can take $f_1(a_0) = b_0$. □

Corollary 1.2.4. DLO is a complete theory.

Proof. Take any two models of M_0, M_1 of DLO. By Löwenheim–Skolem, each M_i has a countable elementary substructure N_i. In particular, $M_i \equiv N_i$, hence, by Theorem 1.2.3,

$$M_0 \equiv N_0 \cong \mathbb{Q} \cong N_1 \equiv M_1$$

□

The proof of Theorem 1.2.3 that we just saw has been the source of much inspiration; in fact, many of the things we will see are in a sense "mined" from it.

To begin with, when we run the proof above with $M = N$, we discover that every partial isomorphism of $(\mathbb{Q}, <)$ with finite domain can be extended to an element of $\operatorname{Aut}(\mathbb{Q})$. This property has a name. This is also a good point to introduce a weaker property.

$$\begin{tikzcd}
B \arrow[dr, dashed, "\exists g"] & \\
A \arrow[u, "\subseteq"] \arrow[r, "f"'] & M
\end{tikzcd}$$

Figure 1.2: Weak homogeneity.

$$\begin{tikzcd}
& B_0 \arrow[dr, dashed, "\exists g_0"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & \circlearrowleft & C \\
& B_1 \arrow[ur, dashed, "\exists g_1"'] &
\end{tikzcd}$$

Figure 1.3: The Amalgamation Property.

Definition 1.2.5. A structure M is

1. *ultrahomogeneous* iff, whenever A, B are finitely generated substructures of M and $f\colon A \to B$ is an isomorphism, then there is $g \in \mathrm{Aut}(M)$ such that $f = g \upharpoonright A$;

2. *weakly homogeneous* iff, whenever A, B are finitely generated substructures of M and $A \subseteq B$, then every embedding $f\colon A \to M$ can be extended to an embedding $g\colon B \to M$, as in Figure 1.2

Back to classes of finitely generated structures, recall that we introduced JEP in order to substitute the notion of "generated substructure" in the absence of an ambient structure. In the definitions above, A and B interact as substructures of M, so we want a way to capture this. Let us cut to the chase.

Definition 1.2.6. 1. A class K of L-structures has the *Amalgamation property* (AP) iff, whenever $A, B_0, B_1 \in K$ and, for $i < 2$, there are embeddings $f_i\colon A \to B_i$, then there are $C \in K$ and embeddings $g_i\colon B_i \to C$ such that $g_0 \circ f_0 = g_1 \circ f_1$, as in Figure 1.3.

2. A *Fraïssé class* is a nonempty, essentially countable class of finitely generated L-structures, with L a countable language, that has the HP, JEP, and AP.

Theorem 1.2.7 (Fraïssé)**.** Let M be a ultrahomogeneous structure with $|L|, |M| \leq \aleph_0$. Then $\mathrm{Age}(M)$ is a Fraïssé class.

Conversely, if L is countable and K is a Fraïssé class, then there is an ultrahomogeneous M with $|M| \leq \aleph_0$ and $\mathrm{Age}(M) = K$. Such an M is unique up to isomorphism.

Definition 1.2.8. If K is a Fraïssé class, the M costructed in the second part of Fraïssé's Theorem is called the *Fraïssé limit* of K.

Lemma 1.2.9. Let $|L|, |C|, |D| \leq \aleph_0$. Assume $\mathrm{Age}(C) \subseteq \mathrm{Age}(D)$. If D is weakly homogeneous, then every embedding of a finitely generated substructure of C into D can be extended to an embedding $C \to D$.

Proof. Let $A \subseteq C$ be finitely generated and $f: A \to D$ an embedding. Write C as the union of a chain (that is, $A_n \subseteq A_{n+1}$) of finitely generated structures $\bigcup_{n<\omega} A_n$, where $A_0 = A$. We extend the embedding $f_0 := f$ inductively. Start with $f_n : A_n \to D$. To extend this to $f_{n+1} : A_{n+1} \to D$, by weak homogeneity all we need to do is to find a substructure of D isomorphic to A_{n+1}. Such a substructure exists because $\mathrm{Age}(C) \subseteq \mathrm{Age}(D)$. The union of the f_n is the required embedding. □

Remark 1.2.10. So, countable weakly homogeneous structures are *universal*, in the sense that in the assumptions above there is an embedding $C \to D$ (just start with the structure generated in C by $\emptyset$).

Non-Example 1.2.11. Even if $\mathrm{Age}(\mathbb{Z}, <)) = \mathrm{Age}(\mathbb{Q}, <)$, we cannot embed $(\mathbb{Q}, <)$ into $(\mathbb{Z}, <)$. In fact, the latter is not ultrahomogeneous.

Lemma 1.2.12.

1. Let $|L|, |C|, |D| \leq \aleph_0$. Suppose that $\mathrm{Age}(C) = \mathrm{Age}(D)$ and that both C, D are ultrahomogeneous. For every $A \subseteq C$, every embedding $A \to D$ extends to an isomorphism $C \to D$. In particular, $C \cong D$.

2. If $|L|, |M| \leq \aleph_0$ and M, then M is ultrahomogeneous if and only if it is weakly homogeneous.

Proof. The previous lemma was proven by going "only forth". If we use the same strategy and go back and forth, we prove the first part of this lemma: write C and D as unions of chains, say $C = \bigcup_n C_n$, $D = \bigcup_n D_n$, starting with $C_0 = A$ and, with the same arguments as above, ensure that $\operatorname{dom} f_{2n} \supseteq C_n$ and $\operatorname{im} f_{2n+1} \supseteq D_n$.

For the second part, left to right is by extending to an automorphism and then restricting, and right to left is a special case of the first part with $C = D = M$. □

Proof of Fraïssé's Theorem. For the first part, by Theorem 1.1.6 we only need to check the AP. Take an amalgamation problem $B_0 \leftarrow A \to B_1$. If these three structures were actual substructures of M, and the maps were just inclusions, then we could simply solve this problem by taking the embeddings of the B_i into $\langle B_0 \cup B_1 \rangle$. But weak homogeneity allows to turn every amalgamation problem in $\mathrm{Age}(M)$ into one of the nice form above! So the first part is done.

For the second part, we proved that Fraïssé limits of a Fraïssé class are unique in Lemma 1.2.12, so we are left to prove existence.

Claim 1.2.13. There is a chain $(D_i : i < \omega)$ of structures in K such that if $A \subseteq B \in K$ then, for every i and every embedding $f: A \to D_i$ there is $j > i$ and an embedding $g: B \to D_j$ extending f.

Proof of the Claim. This the AP plus some clever bookkeeping. Consider the pairs of structures $A \subseteq B \in K$. Define two such pairs to be isomorphic in the obvious way (hint: this involves a square commuting). List all isomorphism types of pairs in a countable set P. Choose any bijection $\pi\colon \omega^2 \to \omega$ with the property that $\pi(i,j) \geq i$.

Start with $D_0 \in K$ arbitrary. Inductively, suppose we have built D_k. List as $((f_{kj}, A_{kj}, B_{kj} \mid j < \omega))$ the triples given by a pair $(A, B) \in P$ and an embedding $f\colon A \to D_k$.

In other words, after we have set up the k-th piece of the chain, we add to our "list of tasks" all the "weak homogeneity problems" involving it. We then use the bijection π to know which of the problems we solve now, and the AP to actually solve it. That is, we use AP to find D_{k+1} with the property that if $k = \pi(i,j)$ then f_{ij} extends to an embedding $B_{ij} \to D_{k+1}$. $\square_{\text{CLAIM}}$

Let $M \coloneqq \bigcup_{i<\omega} D_i$. By construction and the HP we have $\operatorname{Age}(M) \subseteq K$. For the other inclusion, take $A \in K$. By the JEP, there is $B \in K$ where both A and D_0 embed. By the Claim, the identity $D_0 \to D_0$ extends to an embedding of B into some D_j, hence in M, so $K \subseteq \operatorname{Age}(M)$.

The Claim then gives us weak homogeneity, hence ultrahomogeneity by Lemma 1.2.12. $\square$

1.3 Examples

There's no shortage of examples of Fraïssé limits.

Example 1.3.1. The class of finite linear orders, in the language $\{<\}$, is a Fraïssé class. Its limit is $(\mathbb{Q}, <)$.

Example 1.3.2. The class of finite graphs, in the language $\{E\}$, is a Fraïssé class. We will talk about its limit at length shortly.

Example 1.3.3. The class of finite groups, in the language $\{\cdot, e, (-)^{-1}\}$, is Fraïssé. Its Fraïssé limit is known as *Philip Hall's universal locally finite group*. This is not only ultrahomogeneous, but in fact every partial automorphism can be extended to an *inner* automorphism.[3] In other words, any two isomorphic finite subgroups are conjugate, and it follows that this group is simple. A direct construction can be obtained by starting with your favourite finite group G_0 with at least 3 elements, then inductively using Cayley's Theorem to embed G_n into $G_{n+1} \coloneqq S_{|G_n|}$, then taking the direct limit of this system.

Exercise 1.3.4. Which of the following classes are Fraïssé?

1. Triangle-free graphs.
2. Graphs with no cycles.
3. Finite k-uniform hypergraphs, that is, k-ary relations $R(x_1, \dots, x_k)$ that are irreflexive, that is, $R(x_1, \dots, x_k)$ implies that all x_i are pairwise distinct, and symmetric, that is, if $R(x_1, \dots, x_k)$ holds and σ is a permutation of $\{1, \dots, k\}$ then $Rx_{\sigma(1)}, \dots, x_\sigma(k)$ also holds.

[3] Recall that an inner automorphism is one of the form $x \mapsto gxg^{-1}$.

4. Finite tournaments. A *tournament* is a directed graph with no loops such that for every $x \neq y$ exactly one of $E(x, y)$ and $E(y, x)$ holds.
5. Finite-dimensional vector spaces over a field K with $|K| \leq \aleph_0$.
6. Finitely generated abelian groups.
7. Finitely generated torsion-free abelian groups.
8. Finitely generated ordered abelian groups.
9. Finite fields.
10. Finite boolean algebras.

Exercise 1.3.5. For those of the classes above that are Fraïssé, how does the Fraïssé limit look like?

Let us look at an instance of the exercise above in depth.

1.4 Case study: the Ra(n)do(m) Graph

Definition 1.4.1. The *Random Graph*, or *Rado graph*, is the Fraïssé limit of the class of finite graphs: binary, symmetric, irreflexive relations, in the language $\{E\}$ of graphs.

Remark 1.4.2. The Random Graph (M, E) has the *Alice Restaurant Property*: whenever U, V are disjoint finite subsets of M, there is $a \in M$ with an edge to every point of U and to no point of V.

Proof. Let $B := \langle U \cup V \rangle$. Let C be the graph obtained by adding to B a vertex c with the desired properties. As $\mathrm{Age}(M)$ is the class of all finite graphs, we can use Lemma 1.2.12 to embed C into M. □

Remark 1.4.3. The Alice Restaurant Property is expressible by an infinite conjunction of first-order sentences in the language of graphs: for each n, write a sentence expressing the restriction of the property to $|U|, |V| \leq n$.

Example 1.4.4. Every graph with the Alice Restaurant Property has diameter exactly 2.

In fact, we can say much more.

Theorem 1.4.5. Every countable graph with the Alice Restaurant Property is isomorphic to the Random Graph. In particular, the theory of the Random Graph is complete.

Proof. This is a back-and-forth argument, just like the proof of Cantor's theorem, with the Alice Restaurant Property replacing being dense without endpoints. If it does not sound obvious that it works, then it is a good exercise to spell out the details.

Completeness is proven as in Corollary 1.2.4.[4] □

[4] By the way, this trick is an instance of *Vaught's test*: if T has no finite models and has a unique model of some cardinality $\kappa \geq |L| + \aleph_0$, then T is complete. The proof is an easy corollary of the Löwenheim–Skolem Theorem, and we essentially saw it in the proof of Corollary 1.2.4.

The Alice Restaurant Property is quite ubiquitous. As a witness to this, here are some constructions that return the Random Graph.

Exercise 1.4.6. Recall that V_ω is the set of *hereditarily finite sets*: those whose transitive closure is finite. Equivalently, start with $V_0 = \emptyset$, let $V_{n+1} := \mathscr{P}(V_n)$, let $V_\omega := \bigcup_n V_n$. View this as a directed graph, where $a\ E\ b$ iff $a \in b$. Symmetrise it, so now $a\ E\ b$ iff $a \in b$ or $b \in a$. The resulting graph is the Random Graph.[5]

Exercise 1.4.7. On the natural numbers, define $a\ E\ b$ iff, when b is written in base 2, the a-th digit from the right is a 1.[6] Symmetrise the relation. The resulting graph is the Random Graph.

Exercise 1.4.8. On the set of primes congruent to 1 modulo 4, set $p\ E\ q$ iff p is a square modulo q. This is once again the Random Graph.

And in case you were wondering why the Random Graph is called that way...

Exercise 1.4.9. For every pair of distinct natural numbers, flip a coin (which heads with fixed probability $0 < p < 1$), independently. Put an edge between those numbers if and only if the coin heads. With probability 1, the resulting graph is the Random Graph.

Hint. How likely is the Alice Restaurant Property to fail? □

Corollary 1.4.10 (0-1 law)**.** Let φ be first-order sentence in the language $\{E\}$ of graphs. Then φ is true in almost all finite graphs or false in almost all finite graphs, in the sense that

$$\lim_{n\to\infty} \frac{|\{\text{graphs on } \{1,\dots,n\} \text{ that satisfy } \varphi\}|}{|\{\text{graphs on } \{1,\dots,n\}\}|} \in \{0,1\}$$

Proof. As the theory T_{rg} of the Random Graph is complete, either $T_{\mathrm{rg}} \vdash \varphi$ or $T_{\mathrm{rg}} \vdash \neg\varphi$. Up to replacing φ by $\neg\varphi$, assume $T_{\mathrm{rg}} \vdash \varphi$. By the Compactness Theorem, φ follows form the axioms of graphs (i.e. that E is symmetric and irreflexive) plus finitely many instances of the Alice Restaurant Property (cf. Remark 1.4.3). We may assume that these instances say that the property holds for $|U|, |V| \leq n$. The sentence saying this is true in almost all finite graphs (in the sense above; if it's not clear why, try doing Exercise 1.4.9), hence so is φ. □

Remark 1.4.11. This applies to things that are expressible to *first-order* sentences. For example, the property of being connected is not expressible this way; this is a standard compactness exercise. But note that almost all finite graphs have diameter 2, by Corollary 1.4.10 and Example 1.4.4, and of course this implies that they are connected.

For more on the Random Graph, see [Cam97], on which this section is based.

[5] Funny things happen if one looks at this kind of construction in non-well-founded set theories. See [AHM23] (and forgive me for the shameless self-advertising).

[6] The least significant digit is the 0-th digit.

1.5 Quantifier elimination

How general are the things above? Is there anything special about graphs? Not really.

Theorem 1.5.1 (see [Hod93, Theorem 7.4.1])**.** If L is a finite language and K is a Fraïssé class of *uniformly locally finite structures*: that is, there is a function f such that any n-generated structure in K has size at most $f(n)$.[7] Then, the theory of the Fraïssé limit of K has a unique countable model.

Exercise 1.5.2. The class of finite groups is not uniformly locally finite. Show that there are at least[8] two nonisomorphic countable model of the theory of its Fraïssé limit.

Under the same assumptions, that theory has *quantifier elimination.*

Definition 1.5.3. A theory T has *quantifier elimination* iff, for every n and every formula $\varphi(x_1, \dots, x_n)$, there is a quantifier-free formula $\psi(x_1, \dots, x_n)$ such that

$$T \vdash \forall x_1, \dots, x_n \ (\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

Note that this includes the case $n = 0$, that is, sentences. To avoid pointless complications, we convene that the logic we are using has 0-ary relational symbols $\top$ and $\bot$ that are always interpreted as "true" and "false" respectively.

What's the point of q.e.? Well, as trivial as it may sound, quantifier-free formulas are easier to understand than formulas with quantifiers. Well, at least if the language is simple enough (cf. Footnote 5). But having a quantifier elimination result in a nice language allows us to understand *definable sets.*

Definition 1.5.4. A subset of a cartesian power of an L-structure M is *definable* iff it is the set of solutions of a formula. That is, $X \subseteq M^n$ is definable if and only if there is an L-formula $\varphi(x_1, \dots, x_n)$ such that

$$X = \{(a_1, \dots, a_n) \in M^n \mid M \vDash \varphi(a_1, \dots, a_n)\}$$

One similarly talks of sets definable with parameters from $A \subseteq M$, with the obvious meaning.

The existential quantifier corresponds to a projection, and projections are tricky enough that even Lebesgue made a mistake with them.

So, for example, quantifier elimination in DLO implies that every definable subset of $\mathbb{Q}^n$ is a finite boolean combination of conditions of the form $x_i = x_j$ and $x_\ell < x_k$. For instance, in dimension 2, definable sets, even with parameters, are just (finite) boolean combinations of vertical lines, horizontal lines, the diagonal, the above-diagonal, and vertical or horizontal half-planes.

How does one prove quantifier elimination? One way of proving it involves the AP. We said something above and will say something more later. Another way is the following; in practice, in many concrete cases both approaches boil down to verifying the same things.

[7] For example, this happens whenever the language has no function symbol.

[8] Fun fact: there is no complete theory in a countable language with *exactly* two countable models up to isomorphism.

Theorem 1.5.5. Let T be a theory with following property: for every $M_0, N_0 \models T$ there are $M \succeq M_0$ and $N \succeq N_0$ such that the family of all partial isomorphisms between M and N with finitely generated domain has the back-and-forth property. That is, if f is such a map, $a \in M$, and $b \in N$, there are a partial isomorphisms g, h extending f with $a \in \mathrm{dom}(g)$ and $b \in \mathrm{im}(h)$.

Then T has quantifier elimination.

These ideas can also be phrased in terms of the existence of winning strategies in certain games. Look for *Ehrenfeucht–Fraïssé games* and their variants.

Chapter 2

Existentially closed structures

Most of the material in this chapter is contained in [Hod93, Chapter 8].

2.1 Ordered abelian groups

Definition 2.1.1. An *ordered abelian group* is a structure $(G, +, 0, -, <)$ consisting of an abelian group plus a linear order such that $\forall x, y, z\ (x < y) \rightarrow (x + z) < (y + z)$.

Question 2.1.2. Is the class of finitely generated ordered abelian groups Fraïssé?

1. HP: sure.
2. JEP: as $\{0\}$ embeds in every ordered abelian group, it suffices to prove AP.
3. AP: this can be proven with a reasonable amount of effort. For a quite direct proof, see for example [Hil, Proposition 2.3].

Still, this class fails to be essentially countable.

Exercise 2.1.3. Show that there are uncountably many pairwise non-isomorphic 2-generated ordered abelian groups.

Still, having the AP does have its consequences. Let us drop the "finitely generated" assumption and change context.

2.2 Existentially closed structures

Definition 2.2.1.
1. Let $M \subseteq N$ be L-structures. We say that M is *existentially closed in N* iff: whenever $\varphi(x_0, \dots, x_n, y_0, \dots, y_m)$ is quantifier-free and $a_0, \dots, a_m \in M$, if $N \vDash \exists x_0, \dots, x_n\ \varphi(x_0, \dots, x_n, a_0, \dots, a_m)$ then $M \vDash \exists x_0, \dots, x_n\ \varphi(x_0, \dots, x_n, a_0, \dots, a_m)$.
2. If K is a class of L-structures, say closed under isomorphism[1] we say that $M \in K$ is *existentially closed in K* iff, whenever $M \subseteq N \in K$ then M is existentially closed in N.

[1] Otherwise, you need to start talking about embeddings.

The idea is: M is existentially closed in K if a solution to some quantifier-free formula, possibly with parameters, can be added in an extension, while staying in K, then there was already a solution in M.

Question 2.2.2. Do these things exist?

Sure:

Example 2.2.3. A field is existentially closed[2] if and only if it is algebraically closed.

Proof. To prove the nontrivial direction, one uses Rabinowitsch trick and some syntax manipulation to reduce to formulas stating that if a variety has a point in a larger field then it already has one, then applies (a form of) the Nullstellensatz. □

Example 2.2.4. A $\mathbb{Q}$-vector space is existentially closed if and only if it is nontrivial. Here the language is very important: the field is not part of the structure, but part of the language, and this makes being linearly independent not expressible.

Example 2.2.5. A torsion-free abelian group is existentially closed if and only if it nontrivial and divisible. That is, if and only if it is a nontrivial $\mathbb{Q}$-vector space.

Example 2.2.6. A graph is existentially closed if and only if it satisfies the Alice Restaurant Property. This is true almost by definition.

Well, in fact:

Example 2.2.7. Let $|L| \leq \aleph_0$ and fix a Fraïssé class J. Let K be the class of all structures (not necessarily countable) M with $\operatorname{Age}(M) \subseteq J$. Then the Fraïssé limit of J is existentially closed in K.

Question 2.2.8. Do *enough* of these things exist? Also, is there a general method of construction?

Of course:

Definition 2.2.9. A class of L-structures K is *inductive* iff it is closed under isomorphism and under unions of chains.

Exercise 2.2.10. Being closed under unions of chains is equivalent to the (apparently more general) property of being closed under inductive limits (that is, limits along an upward directed system).

Theorem 2.2.11. If K is inductive, then for every $A \in K$ there is an existentially closed $B \in K$ with $A \subseteq B$.

Proof. This is bookkeeping: enumerate things in a sensible way, keep adding solutions, take unions at limit stages; being inductive ensures the last thing can be done. □

[2] I.e., it is existentially closed in the class of fields. This abuse of terminology is quite standard and we will use it.

If K is the class of models of a first-order theory, there is a very nice way to check whether it is inductive.

Theorem 2.2.12 (Chang–Łos–Suzsko)**.** Let T be an L-theory. Then $\mathrm{Mod}(T)$ is inductive if and only if T has a $\forall\exists$-axiomatisation.[3]

$\forall\exists$-axiomatisable theories are called *inductive* for the reason above.

2.3 Q.e. via the AP

Definition 2.3.1. Given T, write $T_\forall$ for the set of its universal consequences. That is, those sentences φ where all quantifiers are $\forall$ and are all in the beginning of the formula, and such that $T \vdash \varphi$.

Here is another standard fact:

Theorem 2.3.2. Models of $T_\forall$ are the same as substructures of models of T.

Theorem 2.3.3. Let T be an inductive theory. Assume that:

1. The class of existentially closed models of T is *elementary*, that is, of the form $\mathrm{Mod}(T')$, and[4]
2. $\mathrm{Mod}(T_\forall)$ has the AP.

Then T' has quantifier elimination.

These matters are very sensitive to the language. This only makes sense, as these notions are related to embeddings, and what counts as an embedding does depend on the language. And so does quantifier elimination![5]

2.4 Q.e. in fields

Definition 2.4.1. The *language of rings* is $L_{\mathrm{ring}} := \{+, 0, -, \cdot, 1\}$. The theory ACF of *algebraically closed fields* is the L_{ring}-theory with these axioms.

1. Axioms of fields.
2. For every $n \geq 2$, an axiom saying "every monic polynomial of degree n has a root".

We already said that, for a field, being algebraically closed is the same as being existentially closed. Substructures of fields in L_{ring}, that is, models of $T_\forall$, are the same as integral domains, and these can be amalgamated. Therefore, Theorem 2.3.3 gives us:

[3] All that counts is alternations, but multiple consecutive $\forall$ or $\exists$ are allowed. For example, $\forall x\, \forall y\, \exists z\, \exists w\, \varphi(x, y, z, w)$, with $\varphi(x, y, z, w)$ quantifier-free, counts as a $\forall\exists$-sentence.

[4] This assumption can be relaxed if one is happy to work in a different logic. An example of this is [DM24]. This is also another example of shameless self-advertising.

[5] In fact, every L-theory has an *expansion by definitions*, called its *morleyisation*, that is $\forall\exists$ axiomatised and has quantifier elimination. This theory is in a larger language, say L' (hence, there's fewer embeddings), but every L'-formula is equivalent modulo it to some L-formula.

Theorem 2.4.2. ACF has quantifier elimination in L_{ring}.

Definition 2.4.3. Recall that, if $K \vDash \mathsf{ACF}$ a set $X \subseteq K^n$ is

1. *Zariski closed* iff it is the set of zeroes of a family of polynomials;

2. *constructible* iff it is a finite boolean combination of Zariski closed sets.

With this terminology, and keeping in mind that an existential quantifier corresponds to a coordinate projection, we get:

Corollary 2.4.4 (Chevalley–Tarski)**.** If $K \vDash \mathsf{ACF}$ and $X \subseteq K^{n+1}$ is constructible, then its projection to K^n is constructible.

This is false for Zariski closed sets: consider the formula $\exists x\ xy = 1$.

Is ACF complete? No, as it does not decide whether $1+1=0$. But choosing the characteristic is the only obstruction.

Theorem 2.4.5. The completions of ACF are obtained by specifying the characteristic.

Proof. Fix p either a prime or 0. The only nontrivial thing to prove is that ACF_p, obtained from ACF by saying that the characteristic is p (for 0, use infinitely may axioms: $1+1 \neq 0$, $1+1+1 \neq 0 \dots$), is complete. If you know about transcendence bases, you can prove this by Vaught's test.

Instead, let us use quantifier elimination. Given $K, L \vDash \mathsf{ACF}_p$, consider the (unique) embedding of the prime field F (either $\mathbb{Q}$ or $\mathbb{F}_p$) in them. Take a sentence φ. By quantifier elimination, φ is equivalent modulo ACF, hence modulo ACF_p, to a quantifier-free sentence ψ. But if $A \subseteq B$, and ψ is quantifier-free, then $A \vDash \psi \iff B \vDash \psi$. So we have[6]

$$K \vDash \varphi \iff K \vDash \psi \iff F \vDash \psi \iff K \vDash \psi \iff K \vDash \varphi$$

So every two models of ACF_p satisfy the same sentences. By the Completeness Theorem, ACF_p is complete. □

Elisabeth will tell you more about model theory of fields, and show you some very nice proofs exploiting what we have just seen.

Just to mention another instance of quantifier elimination in fields:

Definition 2.4.6. The *language of ordered rings* is $\{+, 0, -, \cdot, 1, <\}$. The theory RCF of *real closed fields* is axiomatised by:

1. Axioms of fields.

2. Addition is increasing, multiplication by positive numbers is increasing.

3. Every positive element has a square root.

4. Every polynomial of odd degree has a zero.

There's several other characterisations, see [Wik]

Theorem 2.4.7. RCF is complete, coincides with the theory of $\mathbb{R}$, and eliminates quantifiers in the language of ordered rings.

[6] Note that ψ will not in general be equivalent to φ in F.

Definition 2.4.8. If $K \vDash \mathsf{RCF}$,a set $X \subseteq K^n$ is *semialgebraic* iff it is a finite union of sets, each of which is the set of solutions of a finite system of polynomial equations and inequalities.

Corollary 2.4.9 (Tarski–Seidenberg)**.** The projection of a semialgebraic set is semialgebraic.

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