A correction to a proof in "The Amalgamation Property for automorphisms of ordered abelian groups"

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Abstract

We correct the proof of [1, Proposition 2.8].

In [1] we proved that the category of ordered abelian groups with a distinguished automorphism has the Amalgamation Property. We did this by reducing first to the analogous problem for $\mathbb{Q}$-vector spaces (Remark 2.4), and from there to that for $\mathbb{R}$-vector spaces (Proposition 2.8).

We recently became aware that the argument we gave for Proposition 2.8 implicitly uses an assumption that need not hold in general. Namely, in the notation of our previous paper, one needs isomorphisms with Hahn sums $h_0\colon A_0 \to \coprod_i (A_0)_{i\in I_0}$ and $h_1\colon A_1 \to \coprod_{i\in I_1}(A_1)_i$, with $I_0 \subseteq I_1$ and $\forall i\ (A_0)_i \subseteq (A_1)_i$, that commute with the given embedding $A_0 \to A_1$ and the embedding $\coprod_{i\in I_0}(A_0)_i \to \coprod_{i\in I_1}(A_1)_i$ induced by the aforementioned inclusions.

Equivalently, one needs A_0 to admit a *valuation basis* (see e.g. [2, before Corollary 0.5]) for the Archimedean valuation *that extends to one of* A_1. Even if both A_0 and A_1 admit valuation bases, e.g. because they have countable dimension, it may be the case that no valuation basis of A_0 extends to one of A_1. An easy example may be produced by taking as A_1 an immediate extension of A_0, e.g. take as A_0 the Hahn sum $\coprod_{i\in\omega}\mathbb{Q}$ and as A_1 the $\mathbb{Q}$-vector space generated, inside the Hahn product $\mathrm{H}(\omega, (\mathbb{Q} : i \in \omega))$, by A_0 together with $\sum_{i\in\omega} t^i$.

Below, we deduce Proposition 2.8 from [1, Theorem 2.6] (instead of modifying its proof as in the original argument).

Proposition ([1, Proposition 2.8])**.** If the category of models of $\mathbb{R}-\mathsf{OVSA}$ with maps the $L_{\mathbb{R},\sigma}$-embeddings has the AP, then so does the category of models of $\mathbb{Q}-\mathsf{OVSA}$ with maps the $L_{\mathbb{Q},\sigma}$-embeddings.

Proof. Given an amalgamation problem $C \leftarrow A \to B$ of models of $\mathbb{Q}-\mathsf{OVSA}$, we produce an amalgamation problem $C' \leftarrow A' \to B'$ of models of $\mathbb{R}-\mathsf{OVSA}$, together $L_{\mathbb{Q},\sigma}$-embeddings $A \to A'$, $B \to B'$ and $C \to C'$ commuting with those in said amalgamation problems.

Take the reduct of $C \leftarrow A \to B$ to $L_{\mathbb{Q}}$; since $\mathbb{Q}-\mathsf{OVS}$ eliminates quantifiers, there is a solution D_0 of the resulting amalgamation problem. Let D_1 be a $|D_0|^+$-strongly homogeneous elementary extension of D_0, and identify the $L_{\mathbb{Q}}$-reducts of A, B, C with the corresponding substructures of D_1.

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Again by quantifier elimination, σ_B is a partial elementary map from D_1 to itself. By strong homogeneity, we extend σ_B to an automorphisms σ_{B_1} of D_1, and set $B_1 \coloneqq (D_1, \sigma_{B_1})$. By running the same argument with B replaced by C, we analogously obtain a structure $C_1 \coloneqq (D_1, \sigma_{C_1})$, on the same underlying ordered $\mathbb{Q}$-vector space D_1.

Apply [1, Theorem 2.6] to D_1, obtaining a real vector space D_2 extending D_1, together with extensions of every automorphism of D_1 to one of D_2. Let σ_{B_2} and σ_{C_2} be the resulting extensions of σ_{B_1} and σ_{C_1} respectively.

Observe that σ_{B_2} and σ_{C_2} both restrict to σ_A on A, hence they have the same restriction $\sigma_{A'}$ to the real vector space A' it generates. By construction, $\sigma_{A'}\left(\sum_i r_i a_i\right) = \sum_i r_i \sigma_A(a_i)$, hence $\sigma_{A'}(A') \subseteq A'$. Similarly, as $\sigma_{B_2}^{-1}$ restricts to an automorphism of A, and is real vector space automorphism, we have $\sigma_{B_2}^{-1}(A') \subseteq A'$. It follows that the restriction of $\sigma_{B_2}^{-1}$ to A' is the compositional inverse of $\sigma_{A'}$, which is therefore an automorphism of A'.

We may therefore conclude by considering $(A', \sigma_{A'})$, together with its inclusion in $B' \coloneqq (D_2, \sigma_{B_2})$ and $C' \coloneqq (D_2, \sigma_{C_2})$, and the embeddings $A \to A'$, $B \to B'$, $C \to C'$ given by the construction above. □

References

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