TEOREMA FONDAMENTALE DELL' ALGEBRA

Teo

Sia $p(x) \in \mathbb{C}[x]$ $\deg p(x) = n$. Allora $\deg(p) = \sum_{z \in p^{-1}(0)} \text{molt}(z)$

dim

Sia $B_r(0) \subseteq \mathbb{C}$ (palla chiusa) con $p^{-1}(0) \subseteq \mathring{B}_r(0)$ ($\mathring{B}_r(0)$ contiene tutti gli zeri di $p(x)$
rientro nelle hp di Hopf (sul ∂ non si annulla))

$\Rightarrow$ gli zeri di $p(x)$ sono isolati

$\overset{\text{Hopf}}{\Rightarrow}$ $p: B_r(0) \to \mathbb{C}$ $\deg\left(\frac{p}{|p|}\Big|_{\partial B_r(0)}\right) = \sum_{z \in p^{-1}(0)} \text{ind}(p, z)$

La tesi segue da:

1) se $p(z_0) = 0 \Rightarrow \text{ind}(p, z_0) = \text{molt}(z_0)$
2) $\deg\left(\frac{p}{|p|}\Big|_{\partial B_r(0)}\right) = \deg(p)$

dim 1

$p(z_0) = 0 \Rightarrow p(z) = (z - z_0)^e q(z)$ tc $q(z_0) \neq 0$ $\quad e = \text{molt}(z)$

$$\text{ind}(p, z_0) = \deg\left(\frac{p}{|p|}\Big|_{\partial B_\varepsilon(z_0)}\right)$$

sia $g: S^1 \longrightarrow \partial B_\varepsilon(z_0)$ g è diffeo che cons. l'orientazione
$\quad z \longmapsto z_0 + \varepsilon z$ $\quad \det(dg_z) = \varepsilon^n$ $\quad J = \begin{pmatrix} \varepsilon & \nabla \\ \triangle & \varepsilon \end{pmatrix}$

g è che cons l'orient.

$$\deg\left(\frac{p}{|p|}\Big|_{\partial B_\varepsilon(z_0)}\right) = \deg\left(\frac{p \circ g}{|p \circ g|} : S^1 \to S^1\right)$$

$$\frac{p(z_0 + \varepsilon z)}{|p(z_0 + \varepsilon z)|} = \frac{(\cancel{z_0} - \cancel{z_0} + \varepsilon z)^e q(z_0 + \varepsilon z)}{\cancel{\varepsilon^e} \underset{\underset{1}{\parallel}}{|z|^e} |q(z_0 + \varepsilon z)|} = \frac{z^e q(z_0 + \varepsilon z)}{|q(z_0 + \varepsilon z)|}$$

$$h_t(z) = \frac{z^e q(z_0 + t\varepsilon z)}{|q(z_0 + t\varepsilon z)|} \qquad t \in [0,1]$$

$$h_0(z) = z^e \underbrace{\frac{q(z_0)}{|q(z_0)|}}_{= e^{i\theta_0}} = z^e e^{i\theta_0}$$

$$h_1(z) = \frac{z^e q(z_0 + \varepsilon z)}{|q(z_0 + \varepsilon z)|} = \frac{p \circ g}{|p \circ g|}(z)$$

dato che $e^{i\theta_0} \sim id_{S_n}$ $\qquad p \circ g \sim z^e e^{i\theta_0} \sim z^e id_{S_n} \sim f^e$

Quindi $\deg\left(\frac{p\circ g}{|p\circ g|} : S^1 \to S^1\right) \overset{!}{=} \deg\left(f_e : \begin{array}{c} S^1 \to S^1 \\ z \mapsto z^e \end{array}\right) = e$

dim 2

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots\dots + a_0 \qquad a_n \neq 0$

$g(z) = p(z) - a_n z^n = a_{n-1} z^{n-1} + \dots + a_0$

$$\lim_{|z| \to +\infty} \left| \frac{g(z)}{z^n} \right| = 0$$

$p_t(z) = a_n z^n + (1-t)\, g(z) \qquad t \in [0,1]$

$\left.\begin{array}{l} p_0(z) = a_n z^n + g(z) = p(z) \\ p_1(z) = a_n z^n \end{array}\right\}$ $p_t(z)$ è omotopia tra $p(z)$ e $a_n z^n$

Per $r \gg 0 \Rightarrow \left| \frac{p_t(z)}{z^n}\Big|_{\partial B_r(0)} \right| \geq |a_n| - \underbrace{(1-t)\left|\frac{g(z)}{z^n}\right|}_{\downarrow\, 0} > 0$

$\Rightarrow p_t(z)\Big|_{\partial B_r(0)} \neq 0 \quad \forall\, t \in [0,1]$

$\Rightarrow \deg\left(\frac{p(z)}{|p(z)|}\Big|_{\partial B_r}\right) \overset{r \gg 0}{\underset{p_t \text{ omotopia}}{=}} \deg\left(\frac{a_n z^n}{|a_n z^n|}\Big|_{B_r(0)}\right) \underset{a_n = \rho e^{i\theta}}{=} \deg\left(\frac{\rho e^{i\theta} z^n}{|\rho e^{i\theta} z^n|}\Big|_{B_r(0)}\right) \underset{\rho > 0}{=} \deg\left(\frac{e^{i\theta} z^n}{|e^{i\theta} z^n|}\Big|_{\partial B_r(0)}\right) =$

$= i(e^{i\theta} z^n, 0) = \deg(e^{i\theta} z^n : S^1 \to S^1) \underset{e^{i\theta} \sim id_{S^n} \Rightarrow \deg e^{i\theta} = 1}{=} \deg(z^n : S^1 \to S^1) = n = \deg(p(z))$