A NON TRIVIAL HOMOTOPY THAT SENDS $\mathbb{Q}$ TO $\mathbb{Q}$ EXCEPT FOR AT MOST ONE VALUE

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Abstract. In this note we present the construction of a non-trivial continuous function H from $[0,1] \times [0,1]$ to $\mathbb{R}$ such for each fixed $t \in [0,1]$ then $H_t(x) = H(x,t)$ maps $[0,1] \cap \mathbb{Q}$ to $\mathbb{Q}$ with at most one exception. In particular we can construct the function H making it an homotopy between the zero map ($H_0(x) = 0$ for all x) and the identity ($H_1(x) = x$ for all x) such that $H_t(x)$ increases both in x and in t, and such that $H_t(x)$ for fixed t is Lipschitz with Lipschitz constant 1.

A little modification of this argument also provides a non-trivial homotopy that at every time sends irrational numbers to irrational numbers, with at most one exception.

In the beginning of this note we also show a funny and very simple example of a non-trivial homotopy that sends rationals to to irrationals, with at most one exception.

In this note we consider parametric functions $H : [0,1] \times [0,1] \to \mathbb{R}$, that we will see as function on the first argument 'position', the second argument being a parameter 'time'. While we work with the closed interval $[0,1]$ for both position and time, all constructions can easily generalized to functions defined to all positions and times while preserving the requested properties (e.g. by a reflection argument).

When such a function is required to map rationals to rationals (or irrationals to irrationals) at every time t, it is immediate to see that $H_t(x) = H(x,t)$ must be constant while time varies for every x. But the problem is more complicated if we allow one exception to exist.

We will give an answer to a few variants of the following problem:

Problem 1. *Does there exist a function $H : [0,1] \times [0,1] \to \mathbb{R}$ such that for every t, the function H_t maps rationals to rationals, with at most one exception?*

Additionally to this formulation, we will also consider the irrationals-to-irrationals case and the rationals-to-irrationals case.

In facts, the only remaining case irrationals-to-rationals can easily seen to be trivial: in facts every non-constant function on the interval assumes an interval of irrational numbers, and since all-but-one irrationals must go to $\mathbb{Q}$, the values assumed on the rational numbers cannot provide such uncountable set of irational images. This, such a function $H_t(x)$ mush be constant in x for every t, and consequently it must also be constant in t.

1. Rationals to irrationals case

In this section we provide a very simple example of a function sending all-but-one rational numbers to irrationals number.

Just take (this example is automatically defined in $\mathbb{R} \times \mathbb{R}$)

$$H_t(x) = e^{t+x}.$$

In facts, suppose that for fixed t, and for some rational numbers x, y we have $H_t(x), H_t(y) \in \mathbb{Q}$. Since theese values are both non-zero, their ratio $H_t(x)/H_t(y)$

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must also be a rational number, i.e. we have that $e^{x-y} \in \mathbb{Q}$. But if $x - y = p/q$, we have taking the q-th power that also $e^p \in \mathbb{Q}$, and this is absurd because of the transcendence of e.

2. Rationals to rationals case

First, it is easy to see that instead of working with $\mathbb{Q}$ we can equally work with the diadic rational numbers $\mathcal{D}$, the rationals of the form $p/2^k$ for some $p, k \in \mathbb{Z}$ (or equivalently $\mathcal{D} = \mathbb{Z}[1/2]$, for algebraists). In facts we have:

Lemma 1. *It is possible to construct an homeomorphism $\mu : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{Q}$ is mapped to $\mathcal{D}$, and we can require additionally that $\mu([0,1]) = [0,1]$.*

Proof. Exercise for the reader. □

We will now construct an homotopy $H_t(x)$ that for every time that sends all-but-one the $x \in \mathcal{D} \cap [0,1]$ to $\mathcal{D}$, ad to get a function with the same property relatively to $\mathbb{Q}$ we can just take $\mu^{-1}(H_t(\mu(x)))$.

We can now state the following:

Proposition 1. *There exist function $H_t(x) : [0,1] \times [0,1] \to \mathbb{R}$ such that for every $t \in [0,1]$, then $H_t(x) \in \mathcal{D}$ for all $x \in \mathcal{D} \times [0,1]$, with at most one exception. The example is such that $H_0(x) = x$ and $H_1(x) = x$ for every $x \in [0,1]$, and $H_t(x)$ is a Lipschitz function of x with Lipschitz constant 1 for every fixed t.*

Proof. The idea of the contruction is based con the famous example of the 'Devil's staircase':

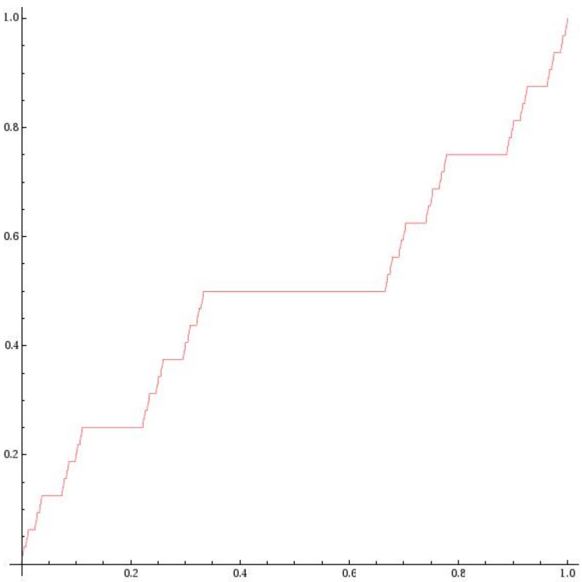

FIGURE 1. The 'Devil's staircase' function.

This function is a continuous function that maps the unit interval to $\mathcal{D}$, except for the points in a Cantor set.

More precisely, let $\sigma_n(x)$ be the sequence of functions inductively defined do $n \geq 0$ as

$$\sigma_0(x) = 1_{[1/3,1]}(x),$$
$$\sigma_i(x) = \sigma_{i-1}(3x) + \sigma_{i-1}(3x + 2/3) \qquad \text{for } i \geq 1.$$

Then $s(x)$ can be defined as the pointwise limit of the series

$$s(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} \sigma_i(x).$$

We can consider the sets

$$\begin{aligned} E_1 &= [0,1], \\ E_{1/2} &= [0,1/3] \cup [2/3,1], \\ E_{1/4} &= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1], \\ &\dots \\ E_{1/2^i} &= \frac{1}{3} E_{1/2^{i-1}} \cup (\frac{2}{3} + \frac{1}{3} E_{1/2^{i-1}}), \qquad \text{for } i \geq 1. \end{aligned}$$

It is easy to check that outside of $E_{1/2^i}$, for $i \geq 0$, the function $s(x)$ takes values in $\frac{1}{2^i}\mathbb{Z}$. If we call C the Cantor set defined as

$$C = \bigcap_{i=0}^{\infty} E_{1/2^i},$$

then outside of C the function $s(x)$ takes values in $\mathcal{D}$.

Note also that given $s(x)$ the set $E_{1/2^i}$ can be defined as the closure of the set

$$s^{-1}(\mathbb{R} \setminus \frac{1}{2^i}\mathbb{Z}),$$

and it is easy to check that this set is a union of intervals outside of which $s(x)$ takes values in the set

$$0, \frac{1}{2^i}, \frac{2}{2^i}, \frac{3}{2^i}, \dots, 1$$

and that at the extremals of each interval the value of the function differs of precisely $1/2^i$.

Let's remark this little but important observarion, i.e. that $E_{1/2^i}$ is the union of disjoit intervals, and that in any of them, $[a,b]$ say, $s(x)$ moves from $k/2^i$ to $(k+1)/2^i$ for some integer k, and in $[a,b]$ it takes values that are all in the interval $[k/2^i, (k+1)/2^i]$ (this is obvious considering that $s(x)$ is monotonically increasing).

In the following we will work with functions different from $s(x)$, but that share with this function the existence of unions of intervals with the same properties as the $E_{1/2^i}$.

The construction now proceeds inductively in the following way: put $s_0(x) = 0$ and $s_1(x) = s(x)$. After having defines $s_u(x)$ and $s_v(x)$ with $u, v \in \frac{1}{2^i}\mathbb{Z}$ and such that $|u - v| = 1/2^i$ we will be able to define $s_{(u+v)/2}(x)$, and we will end with a function $s_u(x)$ for each $u \in \mathcal{D}$.

Supposing that we defined the function $s_u(x)$ for a diadic rational $u \in \mathcal{D}$, for $i \geq 0$ we define $E_{1/2^i}(s_u)$ to be

$$E_{1/2^i}(s_u) = \overline{s_u^{-1}(\mathbb{R} \setminus \frac{1}{2^i}\mathbb{Z})},$$

similarly to what could be done for $s(x)$.

At every stage in the construction we will grant the following properties:

(1) $E_{1/2^i}(s_u)$ is a disjount union of closed intervals such that in each interval the function $s_u(x)$ grows monotically from a number of the form $k/2^i$ to one of the form $(k+1)/2^i$ for some integer k;
(2) if u, v are elements of $\frac{1}{2^i}\mathbb{Z}$ that differ of $1/2^i$ for some i, i.e. are of the form $k/2^i, (k+1)/2^i$ for some non-negative integers k, i, then the sets $E_{1/2^i}(s_u)$ and $E_{1/2^i}(s_v)$ have empty intersection;

(3) for u, v defined in the same way, the following estimate holds:

$$\|s_u - s_v\|_\infty \leq \frac{1}{2^i};$$

(4) have additionally that $s_u(t) \leq s_v(t)$ for each t if $u < v$.

Note that out definition of the pair of functions $s_0(t)$ and $s_1(t)$ satisfies all the properties 1,2,3,4.

Now lets define $s_{(u+v)/2}(t)$, supposing that we have already defined $s_u(t), s_v(t)$ satisfying all above properties. Let $u < v$ be $k/2^i, (k+1)/2^i$ for some non-negative integers k, i.

Since $E_{1/2^i}(s_u)$ and $E_{1/2^i}(s_v)$ are disjoint by property 2, the complementary of the union of these sets is a disjoint union of open intervals. In each of these open intervals, (a, b) say, s_u and s_v are constant and the values they assume are either equal, or differ precisely by $1/2^i$ (thanks to property 3).

Let $[a_\ell, b_\ell]$ for $\ell = 1, \ldots, m$ be the finite set of disjoint closed intervals that are the union of $E_{1/2^{i+1}}(s_u)$ and $E_{1/2^{i+1}}(s_v)$. In each of these sets $[a_\ell, b_\ell]$ precisely one of s_u or s_v grows of $1/2^{i+1}$, and in each of these intervals we will define $s_{(u+v)/2}$ to be constant chosing its value in the following way:

- $s_u(a_\ell) = s_v(a_\ell)$ and $s_v(a_\ell)$ grows of $1/2^{i+1}$, we take the value equal to $s_v(a_\ell)$.
- $s_u(a_\ell) + 1/2^{i+1} = s_v(a_\ell)$ and $s_v(a_\ell)$ grows of one more $1/2^{i+1}$, we take the value $s_v(a_\ell)$.
- $s_u(a_\ell) + 1/2^i = s_v(a_\ell)$ and $s_u(a_\ell)$ grows of $1/2^{i+1}$, we take the value equal to $s_u(b_\ell)$.
- $s_u(a_\ell) + 1/2^{i+1} = s_v(a_\ell)$ and $s_u(a_\ell)$ grows of $1/2^{i+1}$, we take the value equal to $s_u(b_\ell)$.

In this way we have defined $s_{(u+v)/2}$ with a constant value in all the intervals $[a_\ell, b_\ell]$, and note that in each of these intervals the pairs of functions $s_u, s_{(u+v)/2}$ and $s_{(u+v)/2}, s_v$ satisfy the required properties 3, 4.

Now, we can connect the values of $s_{(u+v)/2}$ in b_ℓ and $a_{\ell+1}$ making it constant in a neighborhood of $b_\ell, a_{\ell+1}$, and connecting the two values with a small 'Devil's staircase'. Since all the closed intervals that make up $E_{1/2^{i+1}}(s_u)$ and $E_{1/2^{i+1}}(s_u)$ are 'jumped over' defining $s_{(u+v)/2}$ to be constant with value in $\frac{1}{2^{i+1}}\mathbb{Z}$ in an open neighborhood, it is trivial to verify that the pairs $s_u, s_{(u+v)/2}$ and $s_{(u+v)/2}, s_v$ satisfy property 2. And of course 1 is also satisfied.

To proceed with the construction, note that there is no harm in the above definition of $s_{(u+v)/2}$ if we additionally require it to be locally constant in some neighborhood of a closed set K with empty interior, when defining it in the interval $(b_\ell, a_{\ell+1})$ we just have to find a closed interval contained in $(b_\ell, a_{\ell+1})$ and in the complementary of K, and put there the small 'Devil's staicase'.

Now let's call $K(s_u)$ the set

$$K(s_u) = \bigcap_{i=0}^{\infty} E_{1/2^i}(s_u),$$

and note that outside of this closed set with empty interior, an s_u as constructed above always takes values in $\mathcal{D}$. If we constructed s_u making it locally constant on a neighborhood of the closed set K with empty interior, the we have $K \cap K(s_u) = \emptyset$.

Now let's enumerate the elements of $\mathcal{D} \cap [0, 1]$ in the following way:

$$0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \ldots,$$

and let d_r be the r-th element of $\mathcal{D} \cap [0, 1]$ in this ordering, for $r \geq 1$.

We already defined s_0, s_1, supposing now that we have defined s_{d_r} for $r = 1, \ldots, p-1$, we can define s_{d_p} taking it locally constant in a neighborhood of

$$\bigcup_{r=1}^{p} K(s_{d_r}),$$

which is a closed set with empty interior.

In this way we can inductively define s_u for each $u \in \mathcal{D}$, and note that if we put $H_t(x) = s_x(t)$ this function is Lipschitz with Lipschitz constant 1 for fixed t by property 4, and it is continuous in t for each fixed x. Consequently it is continuous as function from $(\mathcal{D} \cap [0,1]) \times [0,1]$ to $\mathbb{R}$, and can be extended to a function from $[0,1] \times [0,1]$ to $\mathbb{R}$.

Note that but construction $K(s_u) \cap K(s_v) = \emptyset$ for each distinct $u, v \in \mathcal{D} \cap [0,1]$, an these are precisely the values of t where respectively s_u and s_v may take values outside of $\mathcal{D}$. Consequently for each t no two s_u, s_v can both take values outside of $\mathcal{D}$, and this means that $H_t(x)$ for fixed t will map $\mathcal{D} \cap [0,1]$ to $\mathcal{D}$ with at most one exception.

We also have that $H_0(x) = 0$ and $H_0(x) = x$ for each $x \in [0,1]$, and the proof is complete. □

Making the union of the $K(s_u)$ defined above over all $u \in \mathcal{D} \cap [0,1]$, we have the following

Corollary 1. *The set of t such that $H_t(x)$ is not mapping* all $x \in \mathcal{D} \cap [0,1]$ *to* $\mathcal{D} \cap [0,1]$ *is a rare set, in the sense of Baire's theorem.*

Proof. Trivial. □

We also have the following:

Corollary 2. *There exist a function $J : [0,1] \times [0,1] \to \mathbb{R}$ such that for fixed t the function $J_t(x)$ maps all the elements outside of $\mathcal{D}$ outside of $\mathcal{D}$, with at most one exception.*

Of course we can say the same with $\mathbb{Q}$ instead of $\mathcal{D}$ operating as explained above.

Proof. The $H_t(x)$ that we constructed above is weakily monotonic in x, and Lipschitz with Lipschitz constant 1. Consequently, the function $\tilde{H}_t(x) = H_t(x) + x$ is strictly monotonic, its inverse is Lipschitz with Lipschitz constant 1, and for each t maps bijectively the interval $[0,1]$ to an interval $[0, \alpha]$ for some $\alpha \geq 1$.

Furthermore, $\tilde{H}_t$ is againg mapping $\mathcal{D} \cap [0,1]$ to $\mathcal{D}$ with at most one exception. Now for each t we can define J_t to be the inverse of $\tilde{H}_t$, it is defined on the interval $[0,1]$ and continuous. To each $x \notin \mathcal{D}$ mapped to $\mathcal{D}$ correspond to an $x \in \mathcal{D} \cap [0,1]$ mapped outside of $\mathcal{D}$ by $\tilde{H}_t$, so the can be at most one exception, and we are done. □

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