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**The Geometry of Eulerian Posets:
cd-index and KLS Theory of Posets**

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Contents

1	Posets and the Combinatorics of Polytopes	1
1.1	Posets	1
1.1.1	Incidence Algebra	3
1.1.2	Weakly Ranked, Ranked and Graded Posets	5
1.2	Polytopes	8
1.2.1	Operations on Polytopes	11
1.3	f -Vector, h -Vector and g -Vector	13
1.3.1	Shellability and Simpliciality	16
1.4	Flag f -Vector and Flag h -Vector	19
2	Eulerian Posets	23
2.1	Geometric and Topological Motivation	25
2.2	Generalized Dehn–Sommerville Equations	28
2.3	The cd -Index	32
2.3.1	The cd -Index as a Newtonian Coalgebra Morphism	37
3	Toric h-Polynomial	43
3.1	Relation to the Flag f -Vector and Flag h -Vector	46
3.1.1	Fine’s Algorithm	46
3.1.2	Toric h -Polynomial from the cd -Index	50
4	KLS Theory of (Eulerian) Posets	61
4.1	KLS Theory of Weakly Ranked Posets	61
4.1.1	The Z -Function	66
4.1.2	The Chow Function	67
4.1.3	The Augmented Chow Function	71
4.2	Eulerian KLS Theory	73
4.2.1	Eulerian Z -Function and the Interval Poset	74
4.2.2	Eulerian Chow Function and the Order Complex	78
4.2.3	Eulerian Augmented Chow Function and the Stellanahedral Transform	80
4.2.4	Relation to the cd -Index	90
	Bibliography	101

Introduction

Eulerian posets form a special class of partially ordered sets which are often encountered in combinatorics. Relevant examples of Eulerian posets are the face lattices of polytopes, the posets of cells of regular CW-spheres, intervals in the Bruhat order on finite Coxeter groups, and the lattices of regions of oriented matroids. They provide a natural setting for generalizing many results that are valid for polytopes. This thesis investigates the theory of Eulerian posets and its connection with polytopes, with particular attention to the framework of the Kazhdan–Lusztig–Stanley (KLS) theory of posets.

This thesis is divided into four chapters. In Chapter 1, we introduce the fundamental notions of poset and polytope theory. In Chapter 2, we define Eulerian posets and develop some of their properties. We explore the geometric and topological settings in which they arise, present various generalizations of the Dehn–Sommerville equations, and then construct the cd -index. In Chapter 3, we explain how to generalize the construction of the h -vector of simplicial polytopes into the toric h -vector of Eulerian posets, which, in the case of face lattices of polytopes, encodes the intersection-cohomology Betti numbers of the associated toric varieties. We then describe how to compute the toric h -vector from the cd -index by interpreting it as a Newtonian coalgebra map.

In Chapter 4, we introduce the KLS theory of posets and some of its invariants: the Z -polynomial, the Chow polynomial and the augmented Chow polynomial. We proceed by studying these invariants in the Eulerian case, interpreting KLS polynomials in terms of toric g -vectors, Z -polynomials in terms of interval posets, and Chow polynomials in terms of order complexes. We present results from joint work with Luis Ferroni and Tommaso Faustini in which we introduce the stellahedral transform of posets and polytopes, providing a combinatorial and geometrical interpretation of Eulerian augmented Chow polynomials. In the last part of the chapter, we cover a gap in the literature by expressing the previous invariants in terms of the cd -index, again using its interpretation as a Newtonian coalgebra map.

1 Posets and the Combinatorics of Polytopes

In this chapter, we introduce the basic notions concerning partially ordered sets, polytopes and the relationship between them. The combinatorial study of polytopes relies on the partially ordered sets that arise from the faces of a polytope, creating a fruitful interaction between poset theory and polytope theory. This interaction both motivates new questions in combinatorics and provides useful combinatorial tools to obtain deeper results in discrete geometry.

In Section 1.1, we provide the key definitions and concepts regarding partially ordered sets (also known as posets). In Section 1.2 we introduce the basics of polytope theory and of the face lattice. In Section 1.3 we discuss the f -vector, the h -vector and g -vector of polytopes, give an interpretation of the h -vector in terms of shellability in the simplicial case and prove the Dehn–Sommerville Equations. In Section 1.4 we conclude the chapter by extending the notions of the f -vector and h -vector to those of the flag f -vector and flag h -vector for finite graded bounded posets.

During the chapter, we will assume familiarity with the fundamentals of CW-complexes and simplicial complexes.

§1.1 Posets

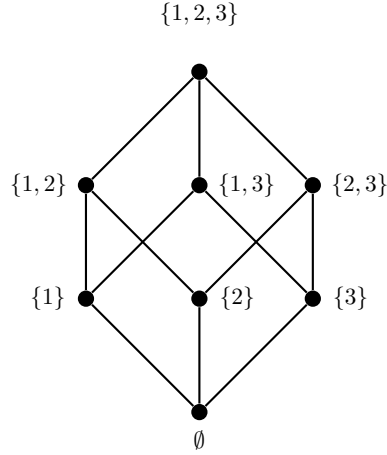
We start the section by providing some introductory definitions. For a more exhaustive introduction to poset theory, one may refer to [47].

Definition 1.1. A *partially ordered set* (also called *poset*, for brevity) is a pair $(P, \leq)$ where P is a set and $\leq$ is a binary relation on P that satisfies the following properties:

1. (reflexivity) for every $x \in P$, $x \leq x$;
2. (antisymmetry) for every $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$;
3. (transitivity) for every $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

By abuse of notation, we may denote P as a poset when we want to refer to the pair $(P, \leq)$, if $\leq$ is clear from the context. We will also write $y \geq x$ to mean $x \leq y$. Similarly, we write $x < y$ if $x \leq y$ and $x \neq y$. We say that two elements of a poset $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$, and we say that they are *incomparable* otherwise. Furthermore, given $x, y \in P$ we say that y *covers* x (denoted by $x < y$) if there is no $z \in P$ such that $x < z < y$. The *Hasse diagram* of a finite poset P is a graph whose nodes are the elements of P and whose edges correspond to the covering relations $x < y$ and are drawn by placing y at a strictly higher vertical coordinate than x .

Example 1.2. Let $[n]$ denote the set $\{1, 2, \dots, n\}$. The *Boolean poset* on $[n]$ is the poset $B_n = (2^{[n]}, \subseteq)$ of the subsets of $[n]$, ordered by inclusion. The figure below is the Hasse diagram of B_3 .


 Figure 1.1: The Hasse diagram of B_3 .

We say that Q is an *induced subposet* of P if Q is a subset of P and for every $x, y \in Q$ we have $x \leq_Q y$ if and only if $x \leq_P y$. A common type of subposet is that of *closed intervals*. For $x \leq y$ in P , the closed interval $[x, y]$ is the subposet of P defined as $\{z \in P \mid x \leq z \leq y\}$. We say that a poset is *locally finite* if all of its closed intervals are finite. An induced subposet $Q \subseteq P$ is called a *chain* if every pair of distinct elements of Q are comparable, while it is called an *antichain* if every pair of distinct elements of Q are incomparable. A chain C of P is *saturated* if there does not exist an element $z \in P \setminus C$ such that $x < z < y$ for some $x, y \in C$ and such that $C \cup \{z\}$ is a chain. A chain C of P has *length* $\ell(C)$ where $\ell(C) = |C| - 1$. A poset P has *length* $\ell(P)$ where $\ell(P) = \sup\{\ell(C) \mid C \text{ chain in } P\}$. We also denote the length of a closed interval $[s, t]$ of P as $\ell(s, t) := \ell([s, t])$. If a poset P has a minimum [resp. maximum] element, we denote it by the symbol $\hat{0}$ [resp. $\hat{1}$]. If P is a poset with $\hat{0}$ and $\hat{1}$, we say that it is *bounded*. For example, B_n is a bounded poset with $\hat{0} = \emptyset$ and $\hat{1} = [n]$. Given two posets P and Q , we say that they are *isomorphic* if there exists a bijection $f : P \rightarrow Q$ such that both f and f^{-1} are order-preserving.

Given one or more posets, there are many ways in which we can combine them to obtain another poset. Let us introduce some of them.

Definition 1.3. Let P and Q be two posets. Then

1. the *dual* of P is the poset P^* defined on the same set of P such that for every $x, y \in P$ we have $x \leq_{P^*} y$ if and only if $y \leq_P x$;
2. the *direct product* of P and Q is the poset $P \times Q$ defined on the Cartesian product of the set of P and Q where $(x, y) \leq_{P \times Q} (x', y')$ if and only if $x \leq_P x'$ and $y \leq_Q y'$ for every $x, x' \in P$ and $y, y' \in Q$;
3. the *ordinal sum* of P and Q is the poset $P \oplus Q$ defined on the disjoint union $P \sqcup Q$ where $x \leq_{P \oplus Q} y$ if either $x \in P$ and $y \in Q$, $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$;
4. if P has a maximum $\hat{1}_P$ and Q has a minimum $\hat{0}_Q$, the *join* of P and Q is the poset $(P \setminus \{\hat{1}_P\}) \oplus (Q \setminus \{\hat{0}_Q\})$.

We notice that if P and Q have length respectively n and m , then

1. the dual poset P^* has length n ;

2. the direct product $P \times Q$ has length $n + m$;
3. the ordinal sum $P \oplus Q$ has length $n + m + 1$;
4. the join $P * Q$ has length $n + m - 1$.

Observe also that the direct product is commutative, meaning that $P \times Q$ is always isomorphic to $Q \times P$. On the contrary, the ordinal sum is not commutative.

§1.1.1 Incidence Algebra

Let P be a locally finite poset. We can define the incidence algebra as follows.

Definition 1.4. The *incidence algebra* $\mathcal{I}(P)$ of P is the $\mathbb{Q}$ -algebra of functions $f : \text{Int } P \rightarrow \mathbb{Q}[x]$, where $\text{Int } P$ is the set of all closed intervals of P . The multiplication in $\mathcal{I}(P)$ is defined by the convolution

$$(f \cdot g)([x, y]) = \sum_{x \leq z \leq y} f([x, z])g([z, y]),$$

for every $f, g \in \mathcal{I}(P)$ and $[x, y] \in \text{Int } P$.

If $f \in \mathcal{I}(P)$ and $x \leq y$, by abuse of notation we will denote by $f(x, y)$ the evaluation $f([x, y])$. Endowed with this multiplication, $\mathcal{I}(P)$ is an associative algebra with identity element

$$\delta_P(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y. \end{cases}$$

An element of the incidence algebra is invertible if and only if it satisfies one of the equivalent conditions of the following theorem.

Proposition 1.5 ([47, Proposition 3.6.2]). *Let P be a locally finite poset and $f \in \mathcal{I}(P)$. The following conditions are equivalent*

1. f has a left inverse;
2. f has a right inverse;
3. f has a two-sided inverse;
4. $f(x, x) \neq 0$ for all $x \in P$.

One particular element of the incidence algebra of P is

$$\zeta_P(x, y) = 1, \quad \text{for all } x \leq y \text{ in } P.$$

This element is sometimes called the *zeta function* of P (for example, in [47]). We will not use this name to avoid confusion with the Z -function defined in Chapter 4. It is interesting to observe that it satisfies some enumerative properties. Given a finite poset P , let $Z(P, \bullet) : \mathbb{Z}_{\geq 2} \rightarrow \mathbb{N}$ be the function defined by

$$Z(P, n) := \#\{\text{multichains } t_1 \leq t_2 \leq \cdots \leq t_{n-1} \text{ in } P\}.$$

It can be described by a polynomial which is closely related to ζ_P .

Proposition 1.6. *Let P be a finite poset. Then*

1. $Z(P, n)$ is a polynomial function of degree $\ell(P)$, where

$$\ell(P) := \max\{\ell(C) \mid C \text{ is a chain in } P\},$$

and in particular, given $b_i := \#\{\text{chains } t_1 < t_2 < \cdots < t_{i-1} \text{ of } P\}$, it is equal to

$$Z(P, n) = \sum_{i=2}^{\ell(P)+2} b_i \binom{n-2}{i-2};$$

2. If P is bounded, then

$$Z(P, n) = \zeta^n(\hat{0}, \hat{1}).$$

Proof.

1. The number of multichains of length $n-1$ whose distinct elements are $t_1 < \cdots < t_{i-1}$ is the multiset coefficient $\left(\binom{i-1}{n-1-(i-1)}\right) = \binom{n-2}{i-2}$. From this observation, the formula follows directly.
2. The number of multichains $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$ in P is the same as the number of multichains $\hat{0} = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq t_n = \hat{1}$ in P starting with $\hat{0}$ and ending with $\hat{1}$, hence

$$\begin{aligned} \zeta^n(\hat{0}, \hat{1}) &= \sum_{\hat{0} \leq t_1 \leq \cdots \leq t_{n-1} \leq \hat{1}} \zeta(\hat{0}, t_1) \zeta(t_1, t_2) \cdots \zeta(t_{n-2}, t_{n-1}) \zeta(t_{n-1}, \hat{1}) \\ &= \#\{\text{multichains } \hat{0} \leq t_1 \leq \cdots \leq t_{n-1} \leq \hat{1} \text{ of } P\} = Z(P, n). \quad \square \end{aligned}$$

As a consequence of the proposition, we can also deduce the following enumerative results.

Corollary 1.7 ([47, Chapter 3.6]). *Let P be a finite poset. Then*

1. *for every natural number $k \in \mathbb{N}$ and every pair of elements $s < t$ in P , we have*

$$(\zeta - \delta)^k(s, t) = \#\{\text{chains } s = s_0 < s_1 < \cdots < s_k = t \text{ in } P\};$$

2. *for every pair of elements $s < t$ in P , we have*

$$\begin{aligned} (2\delta - \zeta)^{-1}(s, t) &= (\delta - (\zeta - \delta))^{-1}(s, t) \\ &= \#\{\text{chains } s = s_0 < s_1 < \cdots < s_k = t \text{ in } P \text{ for some } k \in \mathbb{N}\}. \end{aligned}$$

Another relevant element of the incidence algebra is the *Möbius function*. It is the function $\mu_P \in \mathcal{I}(P)$ defined as

$$\mu_P(x, x) = 1, \quad \mu_P(x, y) = - \sum_{x \leq z < y} \mu_P(x, z)$$

for all $x \leq y$ in P . It is equivalently obtained as the multiplicative inverse $\mu_P = \zeta_P^{-1}$, which is well defined by Proposition 1.5. As a consequence of this, the Möbius function satisfies the following property.

Proposition 1.8 (Möbius inversion formula). *Let P be a finite poset. Given any two functions $f, g : P \rightarrow \mathbb{Q}$, we have*

$$g(t) = \sum_{s \leq t} f(s), \quad \text{for all } t \in P,$$

if and only if

$$f(t) = \sum_{s \leq t} g(s) \mu_P(s, t), \quad \text{for all } t \in P.$$

Similarly, we also have that

$$g(s) = \sum_{t \geq s} f(t), \quad \text{for all } s \in P,$$

if and only if

$$f(s) = \sum_{t \geq s} \mu_P(s, t) g(t), \quad \text{for all } s \in P.$$

We can also give an enumerative result for the Möbius function.

Theorem 1.9 (Philip Hall). *Let P be a finite poset, and let $\widehat{P} := P \sqcup \{\hat{0}, \hat{1}\}$. Let $c_i := \#\{\text{chains } \hat{0} = t_0 < t_1 < \cdots < t_i = \hat{1} \text{ in } \widehat{P}\}$, we have*

$$\mu_{\widehat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots.$$

Proof. Let ℓ be the length of $\widehat{P}$. Note that, for any element of the incidence algebra $f \in \mathcal{I}(\widehat{P})$ and any $u \leq v$ in $\widehat{P}$, we have

$$\begin{aligned} & (\delta - f)(\delta + f + f^2 + \cdots + f^\ell)(u, v) \\ &= (\delta - f^{\ell+1})(u, v). \end{aligned}$$

If we set $f = \zeta - \delta$, then the formula above implies that $(\delta - f)^{-1} = \delta + f + \cdots + f^\ell$ since $(\zeta - \delta)^{\ell+1} = 0$ by Corollary 1.7. Hence,

$$\begin{aligned} \mu_{\widehat{P}}(\hat{0}, \hat{1}) &= \zeta^{-1}(\hat{0}, \hat{1}) \\ &= (\delta - (-f))^{-1}(\hat{0}, \hat{1}) \\ &= \sum_{i=0}^{\ell} (-1)^i f^i(\hat{0}, \hat{1}) \\ &= \sum_{i \geq 0} (-1)^i c_i. \end{aligned}$$

□

§1.1.2 Weakly Ranked, Ranked and Graded Posets

Many examples of posets that arise from geometry satisfy stricter conditions and have a richer structure. In the context of hyperplane arrangements, matroids, CW-complexes and Whitney stratified spaces, one could encounter the types of posets described by this subsection. These definitions may slightly differ across authors, so we now fix the terminology to avoid ambiguity.

Definition 1.10. Let P be a locally finite poset. A *weak rank function* on P is a map $\rho : \text{Int}(P) \rightarrow \mathbb{Z}_{\geq 0}$ such that

- for $s < t$, $\rho_{st} > 0$;

- for $s \leq w \leq t$, $\rho_{st} = \rho_{sw} + \rho_{wt}$,

where ρ_{st} denotes the value $\rho([s, t])$. If for every $s \leq w$ and $w \leq t$ in P we have $\rho_{st} = \rho_{sw} + 1$, then ρ is a *rank function* on P . We say that (P, ρ) is a *weakly ranked poset* [resp. *ranked poset*] whenever ρ is a weak rank function [resp. rank function] on P .

If P is a weakly ranked poset with a minimum element $\hat{0}$, we denote by $\rho(s)$ the value $\rho_{\hat{0}s}$ for any $s \in P$.

Remark 1.11. Let (P, ρ) be a weakly ranked poset. Any induced subposet $Q \subseteq P$ is weakly ranked by ρ , when restricted to elements of Q .

Definition 1.12. A poset P is *graded of rank n* if all its maximal chains have length n .

If P is graded, we denote its rank by $\rho(P)$. Notice that if P is graded, then every closed interval of P is also graded. Furthermore, a graded poset P is always ranked by the function

$$\rho_{st} = \rho([s, t]).$$

Unless stated otherwise, we will always endow P with the rank function above. We also denote by $\rho(s)$ the value ρ_{us} for a minimal element $u \in P$ such that $u \leq s$. Note that the value is independent of the choice of u . This defines a function $\rho : P \rightarrow \mathbb{N}$.

Remark 1.13. A bounded ranked poset (P, ρ) is a graded poset of rank $\rho_{\hat{0}\hat{1}}$.

Example 1.14. Let Γ be a finite CW-complex of dimension $n - 1$. The set

$$\hat{F}(\Gamma) := \{\text{closure of } s \mid s \text{ is a non-empty open cell of } \Gamma\}$$

ordered by inclusion is called the *face poset* of Γ . The *augmented face poset* of Γ is defined as $F(\Gamma) = \hat{F}(\Gamma) \sqcup \{\hat{0}, \hat{1}\}$. In general, $F(\Gamma)$ is weakly ranked by

$$\rho_{st} = \dim t - \dim s,$$

where we identify $\hat{0} = \emptyset$ and $\hat{1} = \Gamma$ and we also assume $\dim \emptyset = -1$ and $\dim \Gamma = n$. If Γ is regular, the face poset $\hat{F}(\Gamma)$ is ranked by the same function. If Γ is also pure, both $\hat{F}(\Gamma)$ and $F(\Gamma)$ are graded respectively of rank $n - 1$ and $n + 1$.

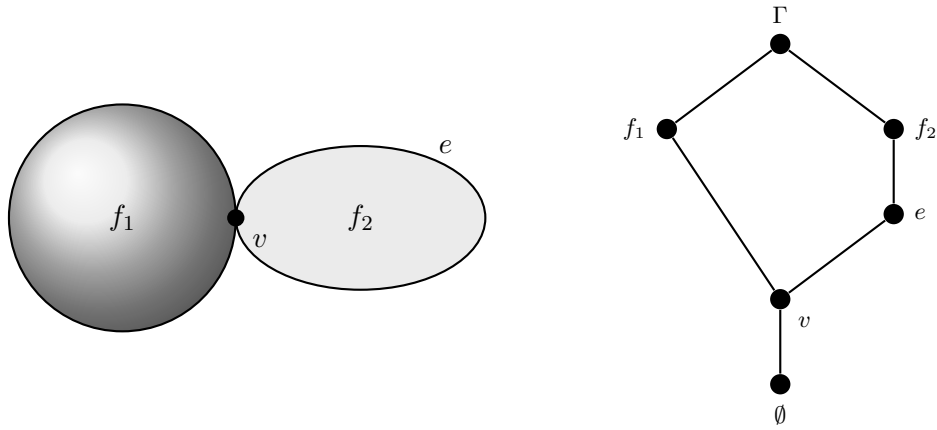


Figure 1.2: A non-regular CW-complex and its augmented face poset.

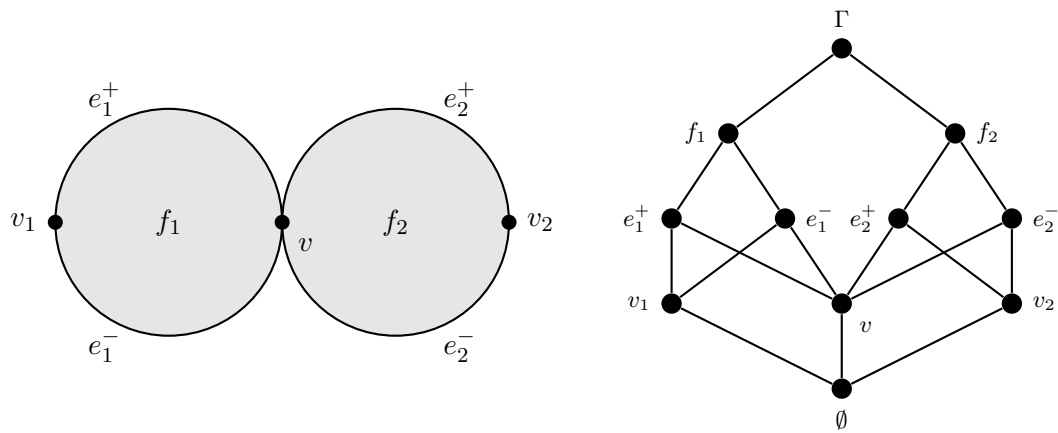


Figure 1.3: A pure regular CW-complex and its augmented face poset.

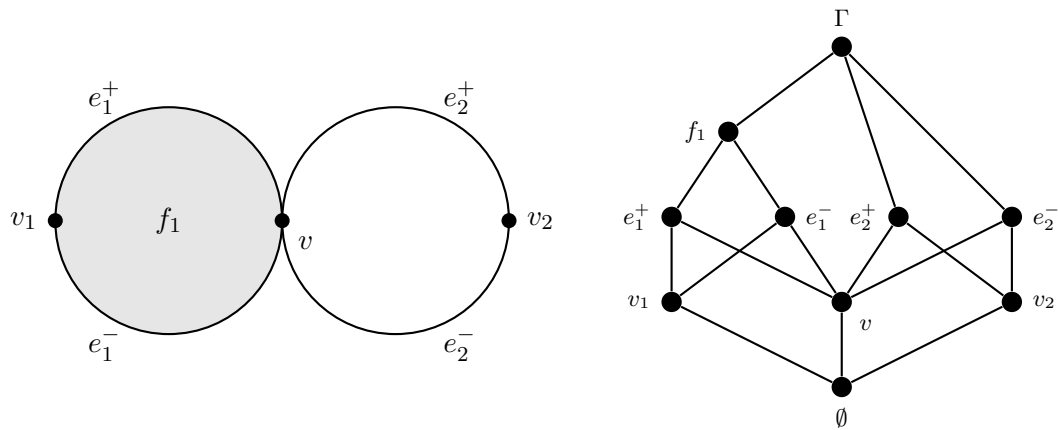


Figure 1.4: A non-pure regular CW-complex and its augmented face poset.

§1.2 Polytopes

Polytopes are among the main objects studied in this thesis. For this reason, it is important to recall some definitions and properties regarding them. For a more comprehensive discussion, see [53].

Definition 1.15. Let V be a finite set in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. A *polytope* $\mathcal{P}$ is the convex hull of V in $\mathbb{R}^n$. The *dimension* of a polytope $\mathcal{P}$ is the dimension of its affine span.

A *face* of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a subset $Q \subseteq \mathcal{P}$ such that there exists a linear form $\alpha \in (\mathbb{R}^n)^*$ and a real number $r \in \mathbb{R}$ such that $Q = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid \alpha(x) = r\}$ and $\mathcal{P}$ is contained in $\{x \in \mathbb{R}^n \mid \alpha(x) \geq r\}$. The *dimension* of a face of $\mathcal{P}$ is the dimension of its affine span. A face of $\mathcal{P}$ is called a *facet* if it has dimension $\dim \mathcal{P} - 1$ and it is called a *vertex* if it has dimension 0.

Notice that, by definition, both the empty set and the polytope $\mathcal{P}$ itself are faces of $\mathcal{P}$. The faces of a polytope satisfy the following properties.

Proposition 1.16 ([53, Proposition 2.3]). *Let $\mathcal{P}$ be a polytope and $F \subseteq \mathcal{P}$ one of its faces. Then*

1. F is a polytope;
2. every intersection of faces of $\mathcal{P}$ is a face of $\mathcal{P}$;
3. the faces of F are all the faces of $\mathcal{P}$ that are contained in F .

The combinatorial nature of polytopes mainly arises from the structure of their faces. In fact, every polytope has a natural poset structure induced by them.

Definition 1.17. Given a polytope $\mathcal{P}$, the *face lattice* of $\mathcal{P}$ is the poset

$$\mathcal{F}(\mathcal{P}) := (\{\text{faces of } \mathcal{P}\}, \subseteq).$$

Both $\emptyset$ and $\mathcal{P}$ itself are faces of the polytope, hence $\mathcal{F}(\mathcal{P})$ is a bounded poset. Notice that the boundary $\partial\mathcal{P}$ of a polytope is a regular CW-complex whose k -cells are the relative interiors of the k -dimensional faces of $\mathcal{P}$. For this reason, when we consider $\partial\mathcal{P}$ with its complex structure, we may refer to it as the *boundary complex* of $\mathcal{P}$. Since the closure of a k -cell equals the corresponding k -dimensional face, by identifying $\mathcal{P} \in \mathcal{F}(\mathcal{P})$ with $\partial\mathcal{P} \in \mathcal{F}(\partial\mathcal{P})$ we have the identity

$$\mathcal{F}(\mathcal{P}) = \mathcal{F}(\partial\mathcal{P}).$$

In particular, the face lattice $\mathcal{F}(\mathcal{P})$ is a graded poset of rank $\dim(\mathcal{P}) + 1$. It is also possible to prove that it is actually a lattice [53, Theorem 2.7 (i)].

Many properties of interest depend only on the structure of the face lattice, hence we introduce the following definition.

Definition 1.18. Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic as posets.

If we are also interested in geometric aspects of our polytope, we may use this other definition.

Definition 1.19. Two polytopes $\mathcal{P} \subseteq \mathbb{R}^n$ and $\mathcal{Q} \subseteq \mathbb{R}^m$ are *affinely isomorphic* if there exists an affine map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose restriction to $\mathcal{P}$ is a bijection $f|_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{Q}$.

Note that being affinely isomorphic is a stronger notion than being combinatorially equivalent.

Example 1.20. The face lattice of an n -dimensional simplex Δ_n is isomorphic to B_{n+1} , the boolean poset on $[n+1]$.

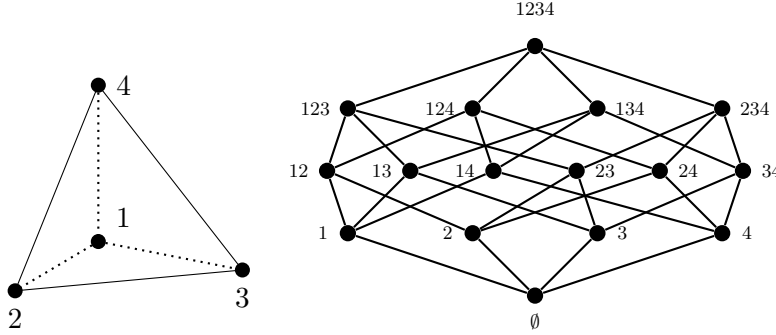


Figure 1.5: Face lattice of a 3-simplex: $\mathcal{F}(\Delta_3) \cong B_4$

Polytopes can be assembled into complexes as follows.

Definition 1.21. A *polytopal complex* $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^d$ such that

1. the complex $\mathcal{C}$ is non-empty, in particular $\emptyset \in \mathcal{C}$;
2. if $\mathcal{P} \in \mathcal{C}$, then all the faces of $\mathcal{P}$ are in $\mathcal{C}$;
3. if $\mathcal{P}, \mathcal{Q} \in \mathcal{C}$, then $\mathcal{P} \cap \mathcal{Q}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.

The *dimension* of the complex is the maximum dimension of a polytope in $\mathcal{C}$.

Given a polytopal complex $\mathcal{C}$, we call *face poset* of $\mathcal{C}$ the poset $\hat{F}(\mathcal{C}) = (\mathcal{C}, \subseteq)$. We also call *face lattice* of $\mathcal{C}$ the poset $F(\mathcal{C}) = \hat{F}(\mathcal{C}) \sqcup \{\hat{1}\}$ where we identify $\hat{1}$ with the complex $\mathcal{C}$ itself.

The notion of polytopal complex can be seen as a special case of the notion of regular CW-complex. In particular, the boundary complex of a polytope can always be seen as a polytopal complex. If the elements of a polytopal complex are all simplices, it is also a simplicial complex.

Since simplices are among the most fundamental polytopes, it is also natural to consider *simplicial* polytopes. A polytope $\mathcal{P}$ is said to be simplicial if all of its proper faces are simplices. This class can be characterized by a property of the poset $\mathcal{F}(\mathcal{P}) \setminus \{\hat{1}\}$, given in the following definition.

Definition 1.22. A poset P with minimum $\hat{0}$ is *simplicial* if, for any $t \in P$, the interval $[\hat{0}, t]$ is isomorphic to a boolean poset.

One way to generate simplicial posets from a poset P is the following.

Definition 1.23. Let P be a poset. The *order complex* of P is the CW-complex $\Delta(P)$ whose vertices are the elements of P and whose faces correspond to the chains of P .

The face poset of an order complex $\Delta(P)$ is always a simplicial poset. The geometric meaning of the order complex operation is given by the following observation: if Γ is a polytopal complex, one has

$$\hat{F}(\Delta(\hat{F}(\Gamma))) \cong \hat{F}(\text{sd}(\Gamma)), \quad (1.1)$$

where $\text{sd}(\Gamma)$ is the barycentric subdivision of the complex Γ . In particular, by taking the order complex of $\mathcal{F}(\mathcal{P}) \setminus \{\hat{0}, \hat{1}\}$ for some polytope $\mathcal{P}$, one obtains an abstract simplicial complex isomorphic to $\text{sd}(\partial\mathcal{P})$, which is the boundary complex of a simplicial polytope. A similar result to (1.1) also holds for all finite regular CW-complexes, up to properly defining barycentric subdivisions in this context [36, Section 10.3.5].

As we will also see later in this thesis, the setting of simplicial polytopes has been an active area of research where many strong results have been proven; some relevant examples include the Dehn–Sommerville equations (Theorem 1.46), the g -theorem [52, 10], the Upper Bound Theorem [38], the Lower Bound Theorem [3, 2].

Another significant class of polytopes is that of *simple* polytopes. A d -dimensional polytope is said to be simple if each of its vertices is contained in exactly d facets. A significant example of simple polytope is the n -dimension cube, defined as $[0; 1]^n \subseteq \mathbb{R}^n$. We now consider a more elaborate example.

Example 1.24. The *permutahedron* $\Pi_n \subseteq \mathbb{R}^n$ of order n is the $(n - 1)$ -dimensional polytope given by the convex hull of the vectors

$$\left\{ \begin{pmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(n) \end{pmatrix} \in \mathbb{R}^n : \sigma \in S_n \right\}.$$

Note that its vertices are in bijection with the permutations of $[n]$. Its face lattice is isomorphic to the poset of ordered set partitions of $[n]$, with the partial order defined by refinement [11, Proposition 1.4]. In particular, every vertex associated to the permutation σ is contained in the $n - 1$ facets associated to the ordered pairs $(\{\sigma(1), \dots, \sigma(k)\}, \{\sigma(k + 1), \dots, \sigma(n)\})$ for each $k = 1, \dots, n - 1$. It follows that Π_n is a simple polytope.

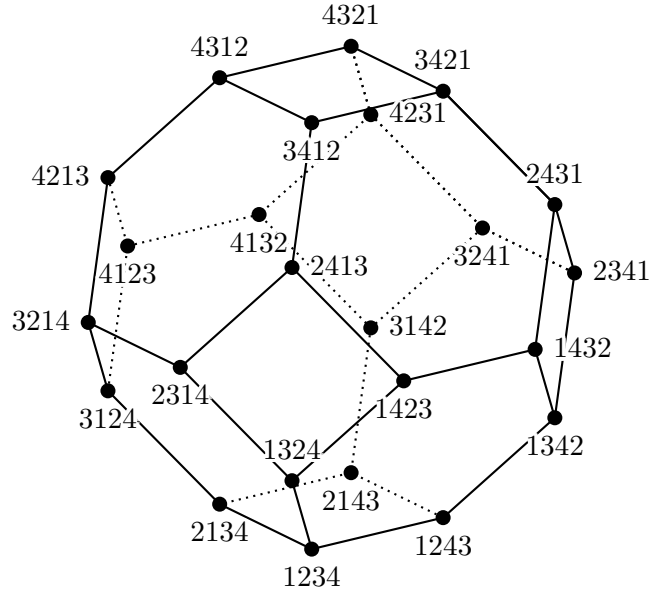


Figure 1.6: The permutahedron of order 4, projected in $\mathbb{R}^3$

Example 1.25 ([24]). The *stellahedron* $\text{St}_n \subseteq \mathbb{R}^n$ of order n is an n -dimensional simple polytope described as

$$\text{St}_n = \{u \in \mathbb{R}_{\geq 0}^n \mid \text{there exists } v \in \Pi_n \text{ such that } v - u \in \mathbb{R}_{\geq 0}^n\},$$

where $\mathbb{R}_{\geq 0}^n$ is the nonnegative orthant of $\mathbb{R}^n$.

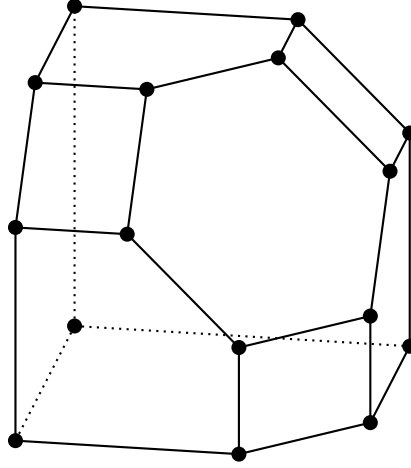


Figure 1.7: The Stellahedron of order 3.

§1.2.1 Operations on Polytopes

Definition 1.26. Let $\mathcal{P} \subseteq \mathbb{R}^n$ and $\mathcal{Q} \subseteq \mathbb{R}^m$ be two polytopes. Then, the *join* of $\mathcal{P}$ and $\mathcal{Q}$ is

$$\mathcal{P} * \mathcal{Q} := \text{conv} \left((\mathcal{P}, \mathbf{0}, 0), (\mathbf{0}, \mathcal{Q}, 1) \right) \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R},$$

and the *direct product* of $\mathcal{P}$ and $\mathcal{Q}$ is

$$\mathcal{P} \times \mathcal{Q} := \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p} \in \mathcal{P}, \mathbf{q} \in \mathcal{Q}\}.$$

Proposition 1.27. Let $\mathcal{P}$ and $\mathcal{Q}$ be polytopes. Then

$$\mathcal{F}(\mathcal{P} * \mathcal{Q}) \cong \mathcal{F}(\mathcal{P}) \times \mathcal{F}(\mathcal{Q});$$

$$\mathcal{F}(\mathcal{P} \times \mathcal{Q}) \cong (\mathcal{F}(\mathcal{P}) \setminus \{\hat{0}_{\mathcal{P}}\}) \times (\mathcal{F}(\mathcal{Q}) \setminus \{\hat{0}_{\mathcal{Q}}\}) \sqcup \{\hat{0}\}.$$

Proof. Consider the map

$$\begin{aligned} \phi : \mathcal{F}(\mathcal{P}) \times \mathcal{F}(\mathcal{Q}) &\rightarrow \mathcal{F}(\mathcal{P} * \mathcal{Q}) \\ (F, G) &\mapsto F * G. \end{aligned}$$

If ϕ is well-defined and bijective, then the first statement is true since ϕ and ϕ^{-1} would be order-preserving by construction. Consider a face $H \in \mathcal{F}(\mathcal{P} * \mathcal{Q})$. If H is empty, then $H = \emptyset * \emptyset$. Otherwise, take a linear functional $\ell : \mathbb{R}^{n+m+1} \rightarrow \mathbb{R}$ such that $H = \{\mathbf{v} \in \mathcal{P} * \mathcal{Q} \mid \ell(\mathbf{v}) = h\}$ for some $h \in \mathbb{R}$ and $\mathcal{P} * \mathcal{Q} \subseteq \{\ell(\mathbf{v}) \geq h\}$. By definition, every point $\mathbf{v} \in \mathcal{P} * \mathcal{Q}$ can be written as $(1-t)(\mathbf{x}, \mathbf{0}, 0) + t(\mathbf{0}, \mathbf{y}, 1) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ for some $t \in [0; 1]$. If we define two linear functionals $\alpha(\mathbf{x}) = \ell(\mathbf{x}, \mathbf{0}, 0)$ on $\mathbb{R}^n$ and $\beta(\mathbf{y}) = \ell(\mathbf{0}, \mathbf{y}, 0)$ on $\mathbb{R}^m$ and set $\gamma := \ell(\mathbf{0}, \mathbf{0}, 1)$, we may write

$$\ell(\mathbf{v}) = (1-t) \cdot \ell(\mathbf{x}, \mathbf{0}, 0) + t \cdot \ell(\mathbf{0}, \mathbf{y}, 1) = (1-t)\alpha(\mathbf{x}) + t(\beta(\mathbf{y}) + \gamma)$$

for every $\mathbf{v} \in \mathcal{P} * \mathcal{Q}$. Since $H \neq \emptyset$, this implies that

$$h = \min_{\mathbf{v} \in \mathcal{P} * \mathcal{Q}} \ell(\mathbf{v}) = \min \left\{ \min_{\mathbf{x} \in \mathcal{P}} \alpha(\mathbf{x}), \min_{\mathbf{y} \in \mathcal{Q}} \beta(\mathbf{y}) + \gamma \right\}.$$

Then, α and β are linear functionals such that $\mathcal{P} \subseteq \{\alpha(\mathbf{v}) \geq h\}$ and $\mathcal{Q} \subseteq \{\beta(\mathbf{v}) \geq h-\gamma\}$, so they both define a face of, respectively, $\mathcal{P}$ and $\mathcal{Q}$. This implies that every face of $\mathcal{P} * \mathcal{Q}$ is of the form $F * G$ for some $F \in \mathcal{F}(\mathcal{P})$ and $G \in \mathcal{F}(\mathcal{Q})$. Now, take two faces $F \in \mathcal{F}(\mathcal{P})$ and $G \in \mathcal{F}(\mathcal{Q})$ defined as

$$F = \mathcal{P} \cap \{\alpha(\mathbf{x}) = a\}, \quad G = \mathcal{Q} \cap \{\beta(\mathbf{y}) = b\}$$

for some linear functionals $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\beta : \mathbb{R}^m \rightarrow \mathbb{R}$, and some values $a, b \in \mathbb{R}$ such that $\mathcal{P} \subseteq \{\alpha(\mathbf{v}) \geq a\}$ and $\mathcal{Q} \subseteq \{\beta(\mathbf{v}) \geq b\}$. We may construct a linear functional defining the face $F * G \in \mathcal{F}(\mathcal{P} * \mathcal{Q})$ at the value a as follows

$$\ell(\mathbf{x}, \mathbf{y}, r) = \alpha(\mathbf{x}) + \beta(\mathbf{y}) + (a - b)r.$$

This proves that ϕ is well defined and surjective. The map is also injective since $\phi(F, G) \cap \mathcal{P} = F$ and $\phi(F, G) \cap \mathcal{Q} = G$, therefore we have proved the first statement.

Similarly, let us consider the map

$$\begin{aligned} \psi : (\mathcal{F}(\mathcal{P}) \setminus \{\hat{0}_{\mathcal{P}}\}) \times (\mathcal{F}(\mathcal{Q}) \setminus \{\hat{0}_{\mathcal{Q}}\}) \sqcup \{\hat{0}\} &\rightarrow \mathcal{F}(\mathcal{P} \times \mathcal{Q}) \\ (F, G) &\mapsto F \times G \\ \hat{0} &\mapsto \hat{0}. \end{aligned}$$

As we have done before, we only need to prove that it is well-defined and bijective. Consider a non-empty face $H \in \mathcal{F}(\mathcal{P} \times \mathcal{Q})$ and take a linear functional $\ell : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ such that $H = \{\mathbf{v} \in \mathcal{P} \times \mathcal{Q} \mid \ell(\mathbf{v}) = h\}$ for some $h \in \mathbb{R}$ and $\mathcal{P} \times \mathcal{Q} \subseteq \{\ell(\mathbf{v}) \geq h\}$. Given $\mathbf{v} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, we define two linear functionals $\alpha(\mathbf{x}) = \ell(\mathbf{x}, \mathbf{0})$ on $\mathbb{R}^n$ and $\beta(\mathbf{y}) = \ell(\mathbf{0}, \mathbf{y})$ on $\mathbb{R}^m$ that define respectively the non-empty faces $F \in \mathcal{F}(\mathcal{P})$ and $G \in \mathcal{F}(\mathcal{Q})$ at the values $a := \min_{\mathbf{x} \in \mathcal{P}} \alpha(\mathbf{x})$ and $b := \min_{\mathbf{y} \in \mathcal{Q}} \beta(\mathbf{y})$. We rewrite

$$\ell(\mathbf{x}, \mathbf{y}) = \ell(\mathbf{x}, \mathbf{0}) + \ell(\mathbf{0}, \mathbf{y}) = \alpha(\mathbf{x}) + \beta(\mathbf{y}), \tag{1.2}$$

which implies

$$h = \min_{\mathbf{v} \in \mathcal{P} \times \mathcal{Q}} \ell(\mathbf{v}) = a + b.$$

We can observe then that $H = \{(\mathbf{x}, \mathbf{y}) \in \mathcal{P} \times \mathcal{Q} \mid \alpha(\mathbf{x}) = a, \beta(\mathbf{y}) = b\} = F \times G$. We can prove similarly to the first statement that for every non-empty faces $F \in \mathcal{F}(\mathcal{P})$ and $G \in \mathcal{F}(\mathcal{Q})$ we can construct a linear functional defining the face $F \times G \in \mathcal{F}(\mathcal{P} \times \mathcal{Q})$. This implies that ψ is well defined and surjective. To prove that it is injective, we only need to observe that the projection of $\psi(F, G)$ onto $\mathbb{R}^n$ is F and the projection onto $\mathbb{R}^m$ is G . $\square$

Two common constructions on polytopes are the pyramid and the prism. Both of them can be described in terms of the previous operations.

Definition 1.28. Let $\mathcal{P}$ be a polytope. The *pyramid* over $\mathcal{P}$ is the polytope

$$\text{Pyr}(\mathcal{P}) := \mathcal{P} * \{0\}$$

and the *prism* over $\mathcal{P}$ is

$$\text{Prism}(\mathcal{P}) := \mathcal{P} \times \Delta_1.$$

Remark 1.29. The pyramid over $\mathcal{P}$ can be equivalently described as the convex hull of $\mathcal{P}$ with a point $\mathbf{v}$ (called the *apex* of the pyramid) which is not contained in the affine span of $\mathcal{P}$. In fact, any polytope of this type is affinely isomorphic to $\text{Pyr}(\mathcal{P})$.

In particular, it follows from Proposition 1.27 that they have the following face lattices.

Corollary 1.30. *Let $\mathcal{P}$ be a polytope. Then*

$$\begin{aligned}\mathcal{F}(\text{Pyr}(\mathcal{P})) &\cong \mathcal{F}(\mathcal{P}) \times B_1; \\ \mathcal{F}(\text{Prism}(\mathcal{P})) &\cong (\mathcal{F}(\mathcal{P}) \setminus \{\emptyset\}) \times (B_2 \setminus \{\hat{0}_{B_2}\}) \sqcup \{\hat{0}\}.\end{aligned}$$

For this reason, given an Eulerian poset P , we also define

$$\text{Pyr}(P) := P \times B_1, \quad \text{Prism}(P) := (P \setminus \{\hat{0}_P\}) \times (B_2 \setminus \{\hat{0}_{B_2}\}) \sqcup \{\hat{0}\},$$

so that we have

$$\mathcal{F}(\text{Pyr}(\mathcal{P})) \cong \text{Pyr}(\mathcal{F}(\mathcal{P})), \quad \mathcal{F}(\text{Prism}(\mathcal{P})) \cong \text{Prism}(\mathcal{F}(\mathcal{P})).$$

Another fundamental operation is that of the polar dual.

Definition 1.31. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope containing the origin $\mathbf{0} \in \mathbb{R}^n$ in its interior. The *polar dual* of $\mathcal{P}$ is the set

$$\mathcal{P}^\circ = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{y} \in \mathcal{P}\}.$$

This operation is an involution, since we have $(\mathcal{P}^\circ)^\circ = \mathcal{P}$. As we see in the next proposition, this construction is closely related to the dual operation on posets.

Proposition 1.32 ([53, Theorem 2.11, Corollary 2.14]). *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope containing the origin $\mathbf{0} \in \mathbb{R}^n$ in its interior. The polar dual $\mathcal{P}^\circ$ is a polytope which satisfies*

$$\mathcal{F}(\mathcal{P}^\circ) \cong \mathcal{F}(\mathcal{P})^*.$$

Remark 1.33. Every closed interval $[G, Q]$ of $\mathcal{F}(\mathcal{P})$ is the face lattice of a polytope. Translate Q such that it contains $\mathbf{0}$ in its relative interior and take the polar dual Q° in its affine span. Consider the face $\widehat{G} \in \mathcal{F}(Q^\circ)$ that corresponds to G in $\mathcal{F}(Q)$. If we take again the polar dual of $\widehat{G}$ (in its affine span, after an appropriate translation that brings $\mathbf{0}$ in the relative interior of $\widehat{G}$), we obtain the desired polytope: we have constructed a polytope of dimension $\dim(Q) - \dim(G) - 1$ whose face lattice is $[G, Q]$.

§1.3 f -Vector, h -Vector and g -Vector

One of the most natural objects that can be associated to a polytope is its f -vector. In some relevant cases, the information one might seek to extract from a polytope may become more visible after applying a linear transformation to the f -vector, such as those defining the h -vector and the g -vector.

Definition 1.34. Let $\mathcal{C}$ be an $(n-1)$ -dimensional polytopal complex. We define the *f -vector* of $\mathcal{C}$ as the vector $(f_{-1}, \dots, f_{n-1})$ where

$$\forall i = -1, 0, \dots, n-1: \quad f_i := \#\{\mathcal{Q} \in \mathcal{C} \mid \mathcal{Q} \text{ is an } i\text{-dimensional polytope in } \mathcal{C}\}.$$

where we set $f_{-1} := 1$, as the empty set is the only (-1) -dimensional polytope. We define the h -vector of $\mathcal{C}$ as the vector $(h_0, \dots, h_n)$ where

$$\sum_{i=0}^n h_i t^{n-i} = \sum_{i=0}^n f_{i-1} (t-1)^{n-i}.$$

We define the g -vector of $\mathcal{C}$ as the vector $(g_0, \dots, g_{\lfloor n/2 \rfloor})$ where

$$g_0 := 1, \quad \forall i = 1, \dots, \lfloor n/2 \rfloor : \quad g_i := h_i - h_{i-1}.$$

We call f -polynomial, h -polynomial and g -polynomial of a polytopal complex $\mathcal{C}$ the polynomials

$$f_{\mathcal{C}}(t) = \sum_{i=0}^n f_{i-1} t^{n-i}, \quad h_{\mathcal{C}}(t) = \sum_{i=0}^n h_i t^i, \quad g_{\mathcal{C}}(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} g_i t^i$$

where $(f_{-1}, \dots, f_{n-1})$, $(h_0, \dots, h_n)$ and $(g_0, \dots, g_{\lfloor n/2 \rfloor})$ are respectively the f -vector, h -vector and g -vector of $\mathcal{C}$.

Analogously, we give the following definition for polytopes.

Definition 1.35. Let $\mathcal{P}$ be a polytope. We define the f -vector, the h -vector and the g -vector of $\mathcal{P}$ as those of the polytopal complex $\partial\mathcal{P}$. We also call f -polynomial, h -polynomial and g -polynomial of $\mathcal{P}$ those of $\partial\mathcal{P}$.

We make a couple of observations on these definitions.

Remark 1.36. Given a $(n-1)$ -dimensional polytopal complex $\mathcal{C}$, the h -polynomial of $\mathcal{C}$ can be equivalently defined as

$$h_{\mathcal{C}}(x) := x^n f_{\mathcal{C}}(x^{-1} - 1).$$

Remark 1.37. Given an $(n-1)$ -dimensional polytopal complex $\mathcal{C}$, the h -vector $(h_0, \dots, h_n)$ may be equivalently defined by the equations

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{n-i}{k-i} f_{i-1} = \sum_{i=0}^k \binom{k-n-1}{k-i} f_{i-1}$$

for $k = 0, \dots, n$. Equivalently, we have

$$f_k = \sum_{i=0}^{k+1} \binom{n-i}{n-k-1} h_i = \sum_{i=0}^{k+1} \binom{n-i}{k+1-i} h_i.$$

Similar definitions can also be given for finite regular CW-complexes. Let us now look at some examples.

Example 1.38. Take the n -dimensional simplex Δ_n . Its f -polynomial is

$$f_{\Delta_n}(t) = \sum_{i=0}^n \binom{n+1}{i} t^{n-i} = \sum_{i=0}^n \binom{n+1}{i+1} t^i = \frac{(t+1)^{n+1} - 1}{t}$$

By the definition of the h -polynomial, we have

$$h_{\Delta_n}(t) = t^n f_{\Delta_n}(t^{-1} - 1) = \frac{t^{n+1} - 1}{t - 1} = 1 + t + \dots + t^n.$$

Consequently,

$$g_{\Delta_n}(t) = 1.$$

Example 1.39. Let $\mathcal{Q}_n = [0; 1]^n$ be the n -dimensional cube. We prove by induction on n that

$$f_{\mathcal{Q}_n}(t) = t^n + \frac{(2t+1)^n - 1}{t},$$

in particular $f_{n-1}(\mathcal{Q}_n) = 2n$ and $f_0(\mathcal{Q}_n) = 2^n$.

If $n = 1$, the formula is correct. For $n > 1$, consider the n -cube as the product $\mathcal{Q}_n = \mathcal{Q}_{n-1} \times [0; 1]$. Its k -dimensional faces are the k -dimensional faces of $\mathcal{Q}_{n-1}$ times $\{0\}$ or $\{1\}$, and the $(k-1)$ -dimensional faces of $\mathcal{Q}_{n-1}$ times the interval $[0; 1]$. Hence we have $f_k(\mathcal{Q}_n) = 2f_k(\mathcal{Q}_{n-1}) + f_{k-1}(\mathcal{Q}_{n-1})$ for every $1 \leq k \leq n-2$, while we have $f_0(\mathcal{Q}_n) = 2f_0(\mathcal{Q}_{n-1})$ and $f_{n-1}(\mathcal{Q}_n) = 2 + f_{n-2}(\mathcal{Q}_{n-1})$. In particular

$$\begin{aligned} f_{\mathcal{Q}_n}(t) &= \sum_{i=0}^n f_{i-1}(\mathcal{Q}_n) t^{n-i} \\ &= t^n + f_0(\mathcal{Q}_n) t^{n-1} + f_{n-1}(\mathcal{Q}_n) + \sum_{i=2}^{n-1} f_{i-1}(\mathcal{Q}_n) t^{n-i} \\ &= t^n + 2f_0(\mathcal{Q}_{n-1}) t^{n-1} + 2 + f_{n-2}(\mathcal{Q}_{n-1}) + 2t \sum_{i=2}^{n-1} f_{i-1}(\mathcal{Q}_{n-1}) t^{n-1-i} \\ &\quad + \sum_{i=2}^{n-1} f_{i-2}(\mathcal{Q}_{n-1}) t^{n-i} \\ &= t^n + 2^n t^{n-1} + 2n + 2t(f_{\mathcal{Q}_{n-1}}(t) - t^{n-1} - 2^{n-1} t^{n-2}) \\ &\quad + (f_{\mathcal{Q}_{n-1}}(t) - t^{n-1} - 2(n-1)) \\ &= (2t+1)f_{\mathcal{Q}_{n-1}}(t) - t^n - t^{n-1} + 2 \end{aligned}$$

which, as stated, equals

$$\begin{aligned} &(2t+1) \left(t^{n-1} + \frac{(2t+1)^{n-1} - 1}{t} \right) - t^n - t^{n-1} + 2 \\ &= (2t+1)t^{n-1} + \frac{(2t+1)^n - 2t - 1}{t} - t^n - t^{n-1} + 2 \\ &= t^n + \frac{(2t+1)^n - 1}{t}. \end{aligned}$$

Furthermore, we have

$$h_{\mathcal{Q}_n}(t) = t^n f_{\mathcal{Q}_n}(t^{-1} - 1) = t^n \left((t^{-1} - 1)^n + \frac{(2t^{-1} - 1)^n - 1}{t^{-1} - 1} \right) = (1-t)^n + t \frac{(2-t)^n - t^n}{1-t}.$$

For example, for $n = 3$

$$h_{\mathcal{Q}_3}(t) = 1 + 5t - t^2 + t^3$$

and

$$g_{\mathcal{Q}_3}(t) = 1 + 4t.$$

Note that the h -vector of Δ_n is palindromic and has non-negative coefficients. However, these properties do not always hold for the n -cubes. Although the h -vector and the g -vector are defined for any polytope, they are mostly used in the simplicial case: we will

se in Chapter 3 that there exist more appropriate definitions for non-simplicial polytopes. As we will see in Theorem 1.46, in the simplicial case the h -vector is palindromic, which implies that the g -vector contains the same information. The possible g -vectors of a simplicial polytope have been, in fact, characterized by the g -theorem ([52, 10]), one of the most celebrated results in polytope theory. One possible reason that explains why the simplicial case behaves better resides on the fact that the linear transformation applied to the f -vector seems to acquire a particular geometric meaning for simplicial polytopes, but not in general. We examine this phenomenon in the next subsection.

§1.3.1 Shellability and Simpliciality

The notion of shellability plays a key role in combinatorial topology. There are slightly different definitions depending on the author and the object that we are studying. The following definition comes from [53].

Definition 1.40. Let $\mathcal{C}$ be a pure k -dimensional polytopal complex. A *shelling* of $\mathcal{C}$ is an ordering $F_1, \dots, F_s$ of its maximal polytopes such that either $k = 0$ or it satisfies the following conditions:

1. the boundary complex ∂F_1 has a shelling;
2. for every $j = 2, \dots, s$ the facet F_j is such that

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right) = \bigcup_{i=1}^r G_i$$

for some shelling $G_1, \dots, G_t$ of ∂F_j and some $1 \leq r \leq t$.

A pure polytopal complex is *shellable* if it admits a shelling.

Example 1.41. Let Δ_n be the n -dimensional simplex. We may prove by induction on n that every ordering of its facets is a shelling. If $n = 0$, the statement is trivial. If $n > 0$, we notice that the intersection of a pair of facets of Δ_n is a $(n - 2)$ -dimensional simplex. Given any ordering $F_1, \dots, F_{n+1}$ of the facets of Δ_n , the intersection $F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right)$ must be the union of some $(n - 2)$ -dimensional faces of ∂F_j , which is a shelling by induction.

Remark 1.42. Given the boundary complex of a simplicial polytope, by the previous example we know that condition 1. in the definition of a shelling is redundant.

We now proceed to prove that every polytope is shellable. In order to obtain this result, we need a preliminary lemma.

Lemma 1.43. *If $F_1, \dots, F_s$ is a shelling of the boundary complex of a polytope $\mathcal{P}$, then so is the reverse sequence $F_s, \dots, F_1$.*

Proof. Fix F_j and consider one of its facets $G \subseteq F_j$. We know that $[G, \mathcal{P}]$ is the face lattice of a 1-dimensional polytope by Remark 1.33, hence G is contained in exactly one facet of $\mathcal{P}$ which is different from F_j . Hence, if

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right) = \bigcup_{i=1}^r G_i$$

for some shelling $G_1, \dots, G_t$ of ∂F_j and $r \leq t$, then we also have

$$F_j \cap \left(\bigcup_{i=j+1}^s F_i \right) = \bigcup_{i=r+1}^t G_i,$$

and by induction on the dimension of $\mathcal{P}$, the reversed sequence $G_t, \dots, G_1$ is a shelling of ∂F_j . $\square$

The exposition of the following theorem follows the proof of [53, Theorem 8.12].

Theorem 1.44. *The boundary complex of a polytope is shellable.*

Proof. We prove a stronger theorem by induction on the dimension of the polytope. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a d -dimensional polytope and $v \in \mathbb{R}^d$ be a point outside $\mathcal{P}$, which is in *general position* with respect to $\mathcal{P}$ (which means that it is not contained in the affine span of any facet of $\mathcal{P}$). We claim that there is a shelling of $\partial\mathcal{P}$ in which the facets that are *visible* from v come first. We say that a facet F of a polytope is visible from a point x if the interior of the polytope and the point x are on different sides of the hyperplane spanned by F .

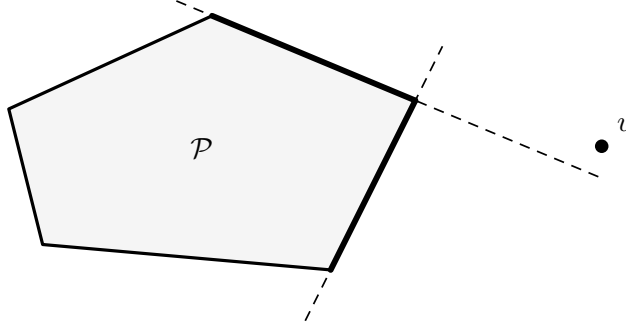


Figure 1.8: Visible facets from a point v .

Take a point $w \in \mathcal{P}$ in the interior of the polytope (or equivalently, a point $w \in \mathcal{P}$ which is general position with respect to $\mathcal{P}$) and construct a line L that passes through v and w . Without loss of generality we can assume that L is not parallel to any of the facets. This implies that it intersects all the hyperplanes generated by the facets. We can also assume that the intersections are all distinct. Since the line L intersects the polytope and both v and w are in general position with respect to $\mathcal{P}$, it intersects its boundary $\partial\mathcal{P}$ at two points lying in the relative interior of two distinct facets F and F' . Suppose without loss of generality that F is visible from v . The set $L \setminus \text{int}(\mathcal{P})$ is the union of two half-lines: the half-line L_1 which contains v and starts from F , and the half-line L_2 which starts from F' . We consider the following ordering of the facets of $\mathcal{P}$: we take the facets associated to the hyperplanes that intersect L_1 in the order in which the intersections are placed in the half-line by starting from $F_1 := F$, obtaining a sequence $F_1, \dots, F_s$; we repeat the same process with the half-line L_2 but we place the facets after F_s in the reverse ordering instead, obtaining at the end a sequence $F_1 = F, \dots, F_s, \dots, F_{s+s'} = F'$.

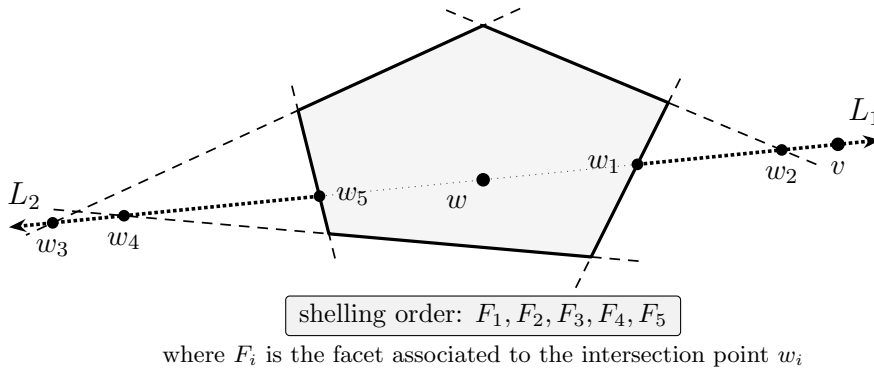


Figure 1.9: Shelling given by the line L passing through v and w .

Notice that this is a well-defined complete list of the facets of $\mathcal{P}$ by the assumptions made on L , and also that the facets visible from v come first. We now prove that it is in fact a shelling. Consider the intersection $\partial F_j \cap (\cup_{i=1}^{j-1} F_i)$. If $2 \leq j \leq s$, this is exactly the union of the facets of F_j which are visible from $L \cap \text{AffSpan}(F_j)$, and they form the start of a shelling of ∂F_j by the induction hypothesis. If $j > s$, the same argument holds by applying Lemma 1.43. $\square$

Given the previous result, we can provide the following interpretation of the h -vector of simplicial polytopes.

Proposition 1.45. *Let $\mathcal{P}$ be an n -dimensional simplicial polytope and let $F_1, \dots, F_s$ be a shelling of $\partial\mathcal{P}$. We treat the boundary complex as an abstract simplicial complex by identifying the faces with their vertex sets (hence, for example, the facets F_i 's are finite sets of cardinality n). Then, the h -vector $(h_0, \dots, h_n)$ of $\mathcal{P}$ satisfies*

$$h_i = \# \{j \in [s] : |R_j| = i\}$$

where $R_j := \{v \in F_j : F_j \setminus \{v\} \subseteq F_i \text{ for some } i \in [j-1]\}$.

Proof. Given the shelling $F_1, \dots, F_s$, we build the complex $\partial\mathcal{P}$ by adding at each step all the faces in the interval $[R_j, F_j]$. This gives us the following partition of $\partial\mathcal{P}$ as a set

$$\bigsqcup_{i=1}^s [R_i, F_i].$$

From this partition, we may calculate the f -vector of $\partial\mathcal{P}$ as

$$f_{k-1} = \sum_{j=1}^s \binom{n - |R_j|}{k - |R_j|} = \sum_{i=0}^k \# \{j \in [s] : |R_j| = i\} \cdot \binom{n-i}{k-i}, \quad \text{for all } k = 0, \dots, n.$$

In particular, using Remark 1.37, this determines the h -polynomial of $\mathcal{P}$ as

$$\sum_{i=0}^n \# \{j \in [s] : |R_j| = i\} \cdot x^i,$$

which concludes the proof. $\square$

The proof above works for all simplicial complexes Δ for which there exists a finite set of faces $R_1, \dots, R_s \in \Delta$ associated to some facets $F_1, \dots, F_s \in \Delta$ such that $\Delta = \bigsqcup_{i=1}^s [R_i, F_i]$. Simplicial complexes of this kind are called *partitionable*. A consequence of the proposition is that the coefficients of the h -vector of a simplicial polytope are non-negative. Since the boundary complex of an n -cube is not simplicial, the above theorem does not apply, and in fact the non-negative property fails. Using this theorem, we can also prove that the h -vector of a simplicial polytope is palindromic.

Theorem 1.46 (Dehn–Sommerville Equations). *Let $\mathcal{P}$ be an n -dimensional simplicial polytope. Then, the h -vector $(h_0, \dots, h_n)$ of $\mathcal{P}$ satisfies*

$$h_i = h_{n-i}, \quad \text{for } i = 0, 1, \dots, n.$$

Proof. Let $F_1, \dots, F_s$ be a shelling of $\partial\mathcal{P}$. For every $j \in [s]$, define the sets

$$R_j := \left\{ v \in F_j : F_j \setminus \{v\} \subseteq F_i \text{ for some } i \in [j-1] \right\}$$

and

$$R'_j := \left\{ v \in F_j : F_j \setminus \{v\} \subseteq F_{s+1-i} \text{ for some } i \in [s-j] \right\}.$$

By Lemma 1.43, we know that $F_s, \dots, F_1$ is also a shelling. Since Proposition 1.45 is independent of the chosen shelling, we have that

$$\# \{j \in [s] : |R_j| = k\} = \# \{j \in [s] : |R'_j| = k\}, \quad \text{for every } k = 0, \dots, n.$$

Notice that we have the identity $R'_i = F_i \setminus R_i$ for every $i \in [s]$. This implies that $|R'_i| = n - |R_i|$, hence

$$h_i = h_{n-i}, \quad \text{for } i = 0, 1, \dots, n. \quad \square$$

Remark 1.47. Let $\mathcal{P}$ be an n -dimensional simplicial polytope. Then, the Dehn–Sommerville Equations are equivalent to the identity

$$h_{\mathcal{P}}(x) = f_{\mathcal{P}}(x-1).$$

The equations may also be stated in terms of the f -vector as follows.

Theorem 1.48. *Let $\mathcal{P}$ be an n -dimensional simplicial polytope. Then, the f -vector $(f_{-1}, \dots, f_{n-1})$ of $\mathcal{P}$ satisfies the equations*

$$f_k = \sum_{i=k}^{n-1} (-1)^{n-i-1} \binom{i+1}{k+1} f_i, \quad \text{for } k = -1, 0, \dots, n-2.$$

§1.4 Flag f -Vector and Flag h -Vector

In the case of finite graded posets, we can generalize the constructions introduced in the previous section. One possible way is the following.

Definition 1.49. Let P be a finite graded poset of rank n . If $S \subseteq [0, n]$, we call

$$P_S := \{t \in P \mid \rho(t) \in S\}$$

the S -rank-selected subposet of P , where ρ is the rank function of P . We call the functions $\alpha_P : 2^{[0,n]} \rightarrow \mathbb{N}$ and $\beta_P : 2^{[0,n]} \rightarrow \mathbb{Z}$ defined by

$$\begin{aligned} \alpha_P(S) &:= \#\{C \subseteq P_S \mid C \text{ is a maximal chain of } P_S\}, \\ \beta_P(S) &:= \sum_{T \subseteq S} (-1)^{\#(S \setminus T)} \alpha_P(T) \end{aligned}$$

respectively the *flag f -vector* of P and the *flag h -vector* of P .

Note that, by Möbius inversion, these two functions also satisfy the following identity

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T).$$

The motivation behind the naming comes from the following observation.

Remark 1.50. Let P be a finite graded poset of rank n . For all $i = -1, \dots, n$ and

$j = 0, \dots, n+1$ we have

$$\begin{aligned}
 f_i(\Delta(P)) &= \sum_{\#S=i+1} \alpha_P(S), \\
 h_j(\Delta(P)) &= \sum_{k=0}^j (-1)^{j-k} \binom{n+1-k}{j-k} f_{k-1}(\Delta(P)) \\
 &= \sum_{k=0}^j (-1)^{j-k} \sum_{\#L=k} \binom{n+1-k}{j-k} \alpha_P(L) \\
 &= \sum_{k=0}^j (-1)^{j-k} \sum_{\#S=j} \sum_{\substack{T \subseteq S \\ \#T=j-k}} \alpha_P(S \setminus T) \\
 &= \sum_{\#S=j} \sum_{T \subseteq S} (-1)^{\#T} \alpha_P(S \setminus T) \\
 &= \sum_{\#S=j} \beta_P(S).
 \end{aligned}$$

We now state an important property of flag vectors.

Proposition 1.51. *Suppose P is a finite bounded graded poset of rank n . Then*

$$\begin{aligned}
 \forall S \subseteq [0, n] : \quad \alpha_P(S) &= \alpha_P(S \cap [1, n-1]), \\
 \forall S \not\subseteq [1, n-1] : \quad \beta_P(S) &= 0.
 \end{aligned}$$

Proof. The first identity follows from the bijection between the maximal chains in P_S and those in $P_S \setminus \{\hat{0}_P, \hat{1}_P\}$. The second identity is an easy consequence. Let $0 \in S$ (the case $n \in S$ is analogous), then

$$\begin{aligned}
 \beta_P(S) &= \sum_{T \subseteq S} (-1)^{\#(S \setminus T)} \alpha_P(T) \\
 &= \sum_{T \subseteq S \setminus \{0\}} \left((-1)^{\#(S \setminus T)} \alpha_P(T) + (-1)^{\#(S \setminus T)-1} \alpha_P(T \sqcup \{0\}) \right) \\
 &= \sum_{T \subseteq S \setminus \{0\}} \left((-1)^{\#(S \setminus T)} \alpha_P(T) - (-1)^{\#(S \setminus T)} \alpha_P(T) \right) = 0. \quad \square
 \end{aligned}$$

From now on, whenever we discuss finite bounded graded posets, if not specified otherwise we consider the flag vectors as functions with domain $2^{[1, n-1]}$. In fact, as this proposition suggests, this restriction does not lose any information coming from the poset.

Example 1.52. Let $\mathcal{P}$ be a cube. In the following table, we list all the values of the flag f -vector and of the flag h -vector of $\mathcal{F}(\mathcal{P})$. One can notice that, in this case, the flag h -vector satisfies the property

$$\beta_{\mathcal{F}(\mathcal{P})}(S) = \beta_{\mathcal{F}(\mathcal{P})}([1, n-1] \setminus S),$$

where $n = 4$ is the rank of the poset $\mathcal{F}(\mathcal{P})$. As we will see in Corollary 2.26, it is a property satisfied by every face lattice of a polytope.

S	$\alpha_{\mathcal{F}(\mathcal{P})}(S)$	$\beta_{\mathcal{F}(\mathcal{P})}(S)$
$\emptyset$	1	1
$\{1\}$	8	7
$\{2\}$	12	11
$\{3\}$	6	5
$\{1, 2\}$	24	5
$\{1, 3\}$	24	11
$\{2, 3\}$	24	7
$\{1, 2, 3\}$	48	1

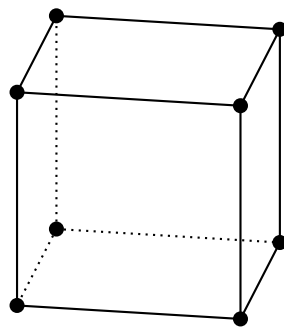


Figure 1.10: The flag f -vector and the flag h -vector of a cube.

2 Eulerian Posets

In this chapter, we discuss Eulerian posets and their properties. This class of posets arises naturally in many combinatorial, topological and geometric settings. The key example of Eulerian posets comes from polytopes, as the face lattice of a polytope is always Eulerian. Due to their many properties and generality, for some problems it is a natural setting in which to generalize the combinatorial study of polytopes. See [45] for a survey on Eulerian posets and [47] for a textbook introduction to the topic.

In the rest of the introduction of this chapter, we provide the definition and some basic properties of Eulerian posets. In Section 2.1 we explain how Eulerian posets arise from polytopes and regular CW-spheres. In Section 2.2 we discuss two different ways in which the Dehn–Sommerville Equations can be extended to Eulerian posets. In Section 2.3 we define the cd -index and its relation with the Generalized Dehn–Sommerville Equations and then we give an interpretation of the cd -index as a Newtonian coalgebra morphism. We conclude the chapter by providing a computational application.

We introduce the definition of an Eulerian poset.

Definition 2.1. A finite graded bounded poset P is *Eulerian* if

$$\forall s \leq t \in P : \quad \mu_P(s, t) = (-1)^{\rho(t) - \rho(s)}$$

where ρ is the rank function of P .

The property of being Eulerian admits the following equivalent combinatorial characterization.

Proposition 2.2. A finite graded bounded poset P is Eulerian if and only if all of its closed intervals $[s, t]$ where $s < t$ have the same amount of elements of odd rank as of even rank.

Proof. Let $[s, t]$ be a closed interval of P . The Möbius function is the only function in the incidence algebra of P such that

$$\mu(s, s) = 1, \quad \sum_{s \leq z \leq t} \mu(s, z) = 0,$$

for every $s < t$. The function $(-1)^{\ell(s, t)}$ always satisfies the first property. Furthermore,

$$\sum_{s \leq z \leq t} (-1)^{\ell(s, t)} = \sum_{\substack{s \leq z \leq t \\ \ell(s, z) \text{ is odd}}} 1 + \sum_{\substack{s \leq z \leq t \\ \ell(s, z) \text{ is even}}} -1$$

equals zero if and only if $[s, t]$ contains the same amount of elements of odd rank as of even rank. This proves the statement. $\square$

By definition, every closed interval of an Eulerian poset is always Eulerian. The property of being an Eulerian poset is also preserved by many fundamental operations, as we observe in the following proposition.

Proposition 2.3. Let P and Q be two locally finite posets. Then

1. the Möbius function of $P \times Q$ is

$$\mu_{P \times Q}((p_1, q_1), (p_2, q_2)) = \mu_P(p_1, p_2) \mu_Q(q_1, q_2),$$

for all $p_1 \leq p_2$ in P and $q_1 \leq q_2$ in Q ;

2. if P has a maximum $\hat{1}_P$ and Q has a minimum $\hat{0}_Q$, the Möbius function of $P * Q$ is

$$\mu_{P * Q}(p, q) = \begin{cases} \mu_P(p, q) & p, q \in P \\ \mu_Q(p, q) & p, q \in Q \\ -\mu_P(p, \hat{1}_P) \mu_Q(\hat{0}_Q, q) & p \in P, q \in Q \end{cases}$$

for all $p \leq q$ in $P * Q$;

3. the Möbius function of P^* is

$$\mu_{P^*}(q, p) = \mu_P(p, q),$$

for all $p \leq q$ in P .

In particular, the direct product, the join and the dual of Eulerian posets are always Eulerian.

Proof. 1. It can be proven by induction on the length of the interval. In fact

$$\begin{aligned} \mu_{P \times Q}((p_1, q_1), (p_2, q_2)) &= - \sum_{(p_1, q_1) \leq (p, q) < (p_2, q_2)} \mu_{P \times Q}((p_1, q_1), (p, q)) \\ &= - \sum_{(p_1, q_1) \leq (p, q) < (p_2, q_2)} \mu_P(p_1, p) \mu_Q(q_1, q) \\ &= - \sum_{p_1 \leq p < p_2} \mu_P(p_1, p) \sum_{q_1 \leq q < q_2} \mu_Q(q_1, q) \\ &\quad - \mu_P(p_1, p_2) \sum_{q_1 \leq q < q_2} \mu_Q(q_1, q) \\ &\quad - \mu_Q(q_1, q_2) \sum_{p_1 \leq p < p_2} \mu_P(p_1, p) \\ &= \mu_P(p_1, p_2) \mu_Q(q_1, q_2). \end{aligned}$$

2. If $p, q \in P \setminus \{\hat{1}_P\} \subseteq P * Q$ or $p, q \in Q \setminus \{\hat{0}_Q\} \subseteq P * Q$, then the statement is trivial.

Let $p \in P, q \in Q$. By induction on the length of the interval, we have that

$$\begin{aligned} \mu_{P * Q}(p, q) &= - \sum_{p \leq z < \hat{1}_P} \mu_{P * Q}(p, z) - \sum_{\hat{0}_Q < z < q} \mu_{P * Q}(p, z) \\ &= - \sum_{p \leq z < \hat{1}_P} \mu_P(p, z) + \mu_P(p, \hat{1}_P) \sum_{\hat{0}_Q < z < q} \mu_Q(\hat{0}_Q, z) \\ &= \mu_P(p, \hat{1}_P) + \mu_P(p, \hat{1}_P) (-\mu_Q(\hat{0}_Q, q) - 1) \\ &= -\mu_P(p, \hat{1}_P) \mu_Q(\hat{0}_Q, q). \end{aligned}$$

3. The identity follows from Philip Hall's Theorem. □

§2.1 Geometric and Topological Motivation

One of the main examples of an Eulerian poset is that of the face lattice of a polytope. It can be shown by a topological interpretation of Philip Hall's Theorem.

Theorem 2.4 (Philip Hall's Theorem, topological version). *Let P be a finite poset, then*

$$\mu_{P \sqcup \{\hat{0}, \hat{1}\}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P)),$$

where $\tilde{\chi}$ denotes the reduced Euler characteristic.

This alternative formulation of the theorem is a consequence of the fact that the Euler characteristic of a finite CW-complex Δ can be calculated as

$$\chi(\Delta) := \sum_{i \geq 0} (-1)^i \operatorname{rank} H_i(\Delta) = \sum_{c \text{ is a cell of } \Delta} (-1)^{\dim c}.$$

Hence, for a polytope $\mathcal{P}$ we have that the Möbius function of its face lattice is the reduced Euler characteristic

$$\mu_{\mathcal{F}(\mathcal{P})}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(\partial\mathcal{P})) = \tilde{\chi}(\partial\mathcal{P}).$$

The boundary of a polytope can be homeomorphically mapped onto a sphere, therefore

$$\mu_{\mathcal{F}(\mathcal{P})}(\hat{0}, \hat{1}) = (-1)^{\dim \mathcal{P} + 1}.$$

This result is equivalent to the *Euler–Poincaré formula*.

Theorem 2.5 (Euler–Poincaré Formula). *Let $\mathcal{P}$ be an n -dimensional polytope. Its f -vector $(f_{-1}, \dots, f_{n-1})$ satisfies the equation*

$$f_0 - f_1 + \dots + (-1)^{n-1} f_{n-1} = 1 - (-1)^n.$$

This is a classical result of discrete geometry. The first proof was provided by Poincaré [40, 39] by studying the Euler characteristics of manifolds. As Grünbaum observes [33], in 1995 Hadwiger's paper [34] renewed the interest in the problem of finding an elementary proof of the formula. For many years, all of the classical proofs that were proposed implicitly assumed that all polytopes are shellable. The first proof of shellability arrives only in 1971 [20] by Bruggesser and Mani, thus yielding the first valid combinatorial proof of the Euler–Poincaré formula. We now present a proof of the Euler–Poincaré formula that makes use of shellability.

Proof of Theorem 2.5. For any m -dimensional polytopal complex $\mathcal{C}$, we define the quantity

$$\delta(\mathcal{C}) := -f_{-1} + f_0 - f_1 + \dots + (-1)^m f_m,$$

where $(f_{-1}, f_0, \dots, f_m)$ is the f -vector of $\mathcal{C}$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two polytopal complexes such that $\mathcal{C}_1 \cup \mathcal{C}_2$ is also a polytopal complex. This happens if and only if for every pair of faces $F_1 \in F(\mathcal{C}_1)$ and $F_2 \in F(\mathcal{C}_2)$ we have that $F_1 \cap F_2 \in F(\mathcal{C}_1 \cap \mathcal{C}_2)$. In this case, δ satisfies the following additive property

$$\delta(\mathcal{C}_1) + \delta(\mathcal{C}_2) = \delta(\mathcal{C}_1 \cup \mathcal{C}_2) + \delta(\mathcal{C}_1 \cap \mathcal{C}_2).$$

Given a shelling $F_1, \dots, F_s$ of the boundary complex $\partial\mathcal{P}$, we prove by induction on the dimension n the formula

$$\delta(F_1 \cup \dots \cup F_j) = \begin{cases} 0 & j < s \\ (-1)^{n-1} & j = s \end{cases}$$

which implies the theorem.

If $n \leq 1$, the identity is trivial. If $n > 1$, we always have by induction that

$$\delta(F_i) = 0 \quad \text{for every } i = 1, \dots, s.$$

When $j < s$, the intersection $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is the first part of a shelling of ∂F_j that does not constitute the whole boundary complex, otherwise $F_s, \dots, F_1$ would not be a shelling of $\partial \mathcal{P}$, contradicting Lemma 1.43. By induction and additivity of δ , we have

$$\delta((F_1 \cup \dots \cup F_{j-1}) \cup F_j) = 0.$$

If $j = s$, we have $F_s \cap (\cup_{i=1}^{s-1} F_i) = \partial F_s$ instead, from which we conclude

$$\begin{aligned} \delta(\partial \mathcal{P}) &= \delta((F_1 \cup \dots \cup F_{s-1}) \cup F_s) \\ &= -\delta(\partial F_s) = (-1)^{n-1}. \end{aligned}$$

□

Using Remark 1.33, we can also calculate the Möbius function of $\mathcal{F}(\mathcal{P})$, thus obtaining

$$\mu_{\mathcal{F}(\mathcal{P})}(G, Q) = \mu_{\mathcal{F}(Q/G)}(\hat{0}, \hat{1}) = (-1)^{\dim Q - \dim G}.$$

This shows that the face lattice of a polytope is always Eulerian. This is a special case of a more general theorem.

Theorem 2.6. *Let Γ be a finite n -dimensional regular CW-complex. Suppose that its support $|\Gamma|$ is homeomorphic to a manifold without boundary whose Euler characteristic equals that of the n -sphere. Then the augmented face poset $F(\Gamma)$ is Eulerian.*

To prove this result, it is necessary to use some machinery regarding the homology of simplicial complexes. We first recall some definitions.

Definition 2.7. Let Δ be a finite simplicial complex and $G \in \Delta$. Then the *link* of G is the subcomplex

$$\text{lk } G := \{H \in \Delta : H \cap G = \emptyset, H \cup G \in \Delta\}.$$

The *star* of G is the subcomplex

$$\text{st } G := \{H \in \Delta : H \cup G \in \Delta\}.$$

Notice that the star of G is the join $G * \text{lk } G$. From the definition, it directly follows the lemma below.

Lemma 2.8. *Let P be a finite poset and $s < t$ be a pair of elements of P . Given any saturated chains $s_1 \triangleleft s_2 \triangleleft \dots \triangleleft s_i = s$ and $t = t_1 \triangleleft t_2 \triangleleft \dots \triangleleft t_j$ such that s_1 is minimal and t_j is maximal in P , we take the face of the order complex $G = \{s_1 \triangleleft \dots \triangleleft s_i \triangleleft t_1 \triangleleft \dots \triangleleft t_j\} \in \Delta(P)$. Then*

$$\Delta((s, t)) = \text{lk } G.$$

In particular, $\text{lk } G$ has dimension $\ell(s, t) - 2$.

We now prove an analogous result of Philip Hall's theorem for the link of a face.

Proposition 2.9 ([30, p. 223, Proposition 1]). *Let Δ be a finite simplicial complex such that its support $|\Delta|$ is homeomorphic to a manifold without boundary. If $G \in \Delta$ is a non-empty face, then $\text{lk } G$ has the same homology groups as a $\dim(\text{lk } G)$ -sphere. In particular, we have*

$$\tilde{\chi}(\text{lk } G) = (-1)^{\dim(\text{lk } G)}.$$

Proof. Let $k = \dim G$. By the properties of the suspension of topological spaces Σ , we have for every $i \geq k + 1$ that

$$\begin{aligned}\widetilde{H}_{i-k-1}(\mathrm{lk} G) &= \widetilde{H}_{i-1}(\Sigma^k \mathrm{lk} G) \\ &= \widetilde{H}_{i-1}(S^{k-1} * \mathrm{lk} G) \\ &= \widetilde{H}_{i-1}((\partial G) * \mathrm{lk} G)\end{aligned}$$

where the last equality follows from the fact that the boundary of a k -simplex is homeomorphic to a $(k - 1)$ -sphere. Let x_0 be a point in the relative interior of G , we can conclude

$$\begin{aligned}\widetilde{H}_{i-1}((\partial G) * \mathrm{lk} G) &= H_i(G * \mathrm{lk} G, (\partial G) * \mathrm{lk} G) \\ &= H_i(G * \mathrm{lk} G, (G \setminus \{x_0\}) * \mathrm{lk} G) \\ &= H_i(\Delta, \Delta \setminus \{x_0\}) \\ &= \begin{cases} \mathbb{Z} & i = \dim \Delta \\ 0 & i \neq \dim \Delta \end{cases}\end{aligned}$$

where the first equality follows from the relative long exact sequence, since $G * \mathrm{lk} G = \mathrm{st} G$ is contractible, the second equality follows by homotopy invariance, the third equality follows from the excision theorem and the last equality follows from the fact that $|\Delta|$ is a manifold without boundary. $\square$

Hence, given a finite regular CW-complex Γ whose support is a manifold without boundary and a pair of elements $s < t$ in $\hat{F}(\Gamma)$ such that $(s, t) \neq (\hat{0}, \hat{1})$, then

$$\mu_{F(\Gamma)}(s, t) = \widetilde{\chi}(\Delta((s, t))) = \widetilde{\chi}(\mathrm{lk} G) = (-1)^{\ell(s, t)}.$$

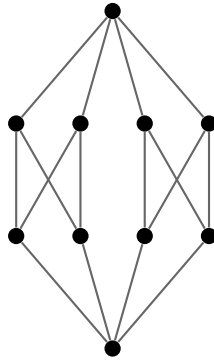
We can summarize the discussion above with the following theorem, which specializes to Theorem 2.6 .

Theorem 2.10. *Let Γ be a finite regular CW-complex. If $|\Gamma|$ is a manifold without boundary, then*

$$\mu_{F(\Gamma)}(s, t) = \begin{cases} \widetilde{\chi}(|\Gamma|) & (s, t) = (\hat{0}, \hat{1}) \\ (-1)^{\ell(s, t)} & (s, t) \neq (\hat{0}, \hat{1}). \end{cases}$$

It is relevant to note that not every Eulerian poset arises as the augmented face poset of a regular CW-complex.

Example 2.11. Consider the Eulerian poset P described by the Hasse diagram below:



Let C_2 denote the chain of two elements. The product $P \times C_2$ is an Eulerian poset by Proposition 2.3; we now prove that it is not the augmented face poset of a regular CW-complex. Let $t = (\hat{1}_P, \hat{0}_{C_2}) \in P \times C_2$ and suppose that $P \times C_2$ is the augmented face poset of a regular CW-complex Δ . By regularity, the boundary of the cell t is homeomorphic to S^1 . In particular, this implies that the order complex of the open interval $((\hat{0}_P, \hat{0}_{C_2}); t) \subseteq P \times C_2$ would also have to be homeomorphic to S^1 . However, it is homeomorphic to $S^1 \sqcup S^1$, which is a contradiction.

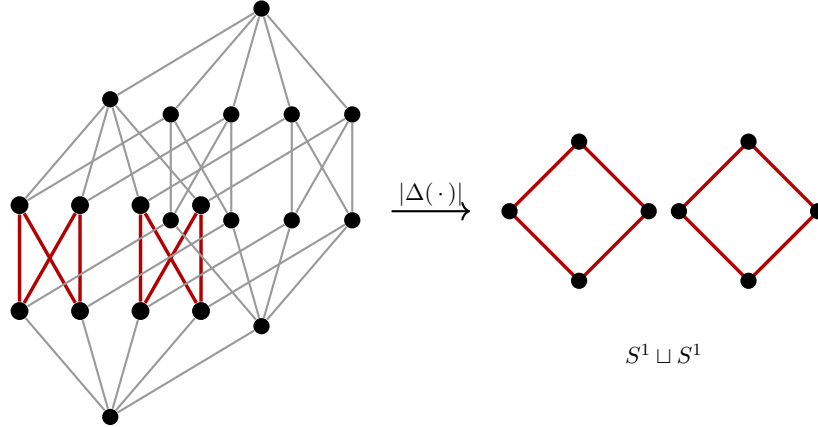


Figure 2.1: Hasse diagram of $P \times C_2$, with the open interval $((\hat{0}_P, \hat{0}_{C_2}), (\hat{1}_P, \hat{0}_{C_2}))$ highlighted in red (left), together with its geometric realization (right).

§2.2 Generalized Dehn–Sommerville Equations

Eulerian posets satisfy plenty combinatorial properties known for polytopes. One informative example is given by the different generalizations of the Dehn–Sommerville Equations. In this section, we discuss two ways in which we can generalize them. The first generalization comes from Stanley [46], while the second generalization has been discovered by Bayer and Billera in [7].

Given a finite bounded poset P , we have proved in Proposition 1.6 the existence of a polynomial $Z(P, t) \in \mathbb{Q}[t]$ such that

$$Z(P, n) = \zeta^n(\hat{0}, \hat{1}) = \#\{\text{multichains } t_1 \leq t_2 \leq \dots \leq t_{n-1} \text{ in } P\}, \quad \text{for every } n \in \mathbb{Z}_{>0}.$$

We introduce some properties of the polynomial. Let P and Q be two finite bounded posets. By Proposition 2.3, we have that $Z(P \times Q, n)$ factorizes as

$$Z(P \times Q, n) = \mu_{P \times Q}^{-n}(\hat{0}, \hat{1}) = \mu_P^{-n}(\hat{0}, \hat{1}) \mu_Q^{-n}(\hat{0}, \hat{1}) = Z(P, n) Z(Q, n). \quad (2.1)$$

There are some notable cases for which we can explicitly calculate the polynomial.

Example 2.12. Let $P = C_n$ be the chain of n elements. The multichains $t_1 \leq \dots \leq t_{m-1}$ in P are in bijection with the $(m-1)$ -multisets of elements of $[n]$, hence $Z(P, m) = \binom{n+m-1}{m-1}$.

Example 2.13. Let $P = B_n$ be the Boolean poset on $[n]$. Since B_n is isomorphic to $(C_2)^n$, by equation 2.1 we have $Z(P, m) = Z(C_2, m)^n = \binom{m-1}{m-1}^n = \binom{m}{m-1}^n = m^n$.

In the Eulerian case, it also satisfies a reciprocity property.

Proposition 2.14. *Let P be an Eulerian poset of rank n . Then*

$$Z(P, -m) = (-1)^n Z(P, m).$$

Proof. By construction, we have

$$\begin{aligned}
 Z(P, -m) &= \mu^m(\hat{0}, \hat{1}) \\
 &= \sum_{\hat{0} \leq t_1 \leq \dots \leq t_{m-1} \leq \hat{1}} \mu(\hat{0}, t_1) \mu(t_1, t_2) \cdots \mu(t_{m-2}, t_{m-1}) \mu(t_{m-1}, \hat{1}) \\
 &= \sum_{\hat{0} \leq t_1 \leq \dots \leq t_{m-1} \leq \hat{1}} (-1)^{\rho(t_1) - \rho(\hat{0})} (-1)^{\rho(t_2) - \rho(t_1)} \cdots (-1)^{\rho(\hat{1}) - \rho(t_{m-1})} \\
 &= \sum_{\hat{0} \leq t_1 \leq \dots \leq t_{m-1} \leq \hat{1}} (-1)^n = (-1)^n Z(P, m). \quad \square
 \end{aligned}$$

For simplicial posets, the following additional identity holds. Notice that every simplicial poset is ranked.

Proposition 2.15. *Let P be a simplicial poset with a minimum element $\hat{0}$, and let $f_i := \#\{x \in P : \rho(x) = i + 1\}$. Then*

$$Z(P, m) = \sum_{i=0}^{\ell(P)} f_{i-1} (m-1)^i.$$

Proof. Let $t \in P$ and denote by $Z_t(P, m)$ the number of multichains $t_1 \leq \dots \leq t_{m-2} \leq t_{m-1} = t$ in P . Since P is simplicial, we have $[\hat{0}, t_{m-1}] \cong B_i$ for some $i \in \mathbb{N}$, which implies $Z_t(P, m) = (m-1)^i$. We conclude

$$Z(P, m) = \sum_{t \in P} Z_t(P, m) = \sum_{i \geq 0} f_{i-1} (m-1)^i. \quad \square$$

If we combine Proposition 2.14 and Proposition 2.15, we get our first generalization of the Dehn–Sommerville Equations.

Theorem 2.16. *Let P be an Eulerian poset of rank n and let $P \setminus \{\hat{1}\}$ be simplicial. Given $f_i := \#\{x \in P : \rho(x) = i + 1\}$, then*

$$\sum_{i=0}^{n-1} f_{i-1} (m-1)^i = \sum_{i=0}^{n-1} (-1)^{n-1-i} f_{i-1} m^i.$$

Proof. Set $P' := P \setminus \{\hat{1}\}$. Since there are $Z(P, m)$ multichains with m elements in P that contain $\hat{1}$, we have

$$Z(P', m+1) = Z(P, m+1) - Z(P, m) =: \Delta Z(P, m),$$

where Δ denotes the difference operator. Since P' is simplicial, we can apply Proposition 2.15 and obtain

$$\Delta Z(P, m) = \sum_{i=0}^{n-1} f_{i-1} m^i.$$

On the other hand, since P is Eulerian we can also apply Proposition 2.14 and get $Z(P, -m) = (-1)^n Z(P, m)$, so

$$\begin{aligned}
 \Delta Z(P, -m) &= Z(P, -m+1) - Z(P, -m) \\
 &= (-1)^{n-1} (Z(P, m) - Z(P, m-1)) = (-1)^{n-1} \Delta Z(P, m-1).
 \end{aligned}$$

Combining it with the previous identity, we conclude

$$\sum_{i=0}^{n-1} f_{i-1} (m-1)^i = \sum_{i=0}^{n-1} (-1)^{n-1-i} f_{i-1} m^i. \quad \square$$

Remark 2.17. Given an n -dimensional simplicial polytope $\mathcal{P}$, Theorem 2.16 translates to

$$\sum_{i=k}^{(n+1)-1} (-1)^{i-k} \binom{i}{k} f_{i-1} = (-1)^{(n+1)-1-k} f_{k-1},$$

or equivalently

$$\sum_{i=k}^{n-1} (-1)^{n-i-1} \binom{i+1}{k+1} f_i = f_k,$$

which is exactly the f -vector form of the Dehn–Sommerville Equations.

We also have another generalization of the Dehn–Sommerville Equations. These are what most authors refer to as the *Generalized Dehn–Sommerville Equations*.

Theorem 2.18. *Let P be an Eulerian poset of rank n , and $S \subseteq \{0, 1, \dots, n-1\}$. If $i, k \in S \cup \{-1, n\}$, $i < k-1$ and $S \cap [i+1, k-1] = \emptyset$, then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} \alpha_P(S \cup \{j\}) = \alpha_P(S) \left(1 - (-1)^{k-i-1}\right).$$

Proof. Let $x_{k_1} < \dots < x_{k_r}$ be a maximal chain in the S -rank-selected subposet P_S such that $\rho(x_{k_l}) = k_l$ for any $l = 1 \dots, r$. Let us denote $x := x_i$ and $y := x_k$. By the proof of Proposition 2.2, we have

$$(-1)^{k-i} = \mu_P(x, y) = \sum_{j=i}^{k-1} (-1)^{j-i-1} \alpha_{[x,y]}(\{j\}),$$

which implies

$$1 - (-1)^{k-i-1} = \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \alpha_{[x,y]}(\{j\}).$$

Summing over all the maximal chains C of P_S (and denoting by x_C and y_C their elements of rank i and k respectively) we get

$$\begin{aligned} \alpha_P(S) \left(1 - (-1)^{k-i-1}\right) &= \sum_{\substack{C \text{ maximal} \\ \text{chain of } P_S}} \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \alpha_{[x_C, y_C]}(\{j\}) \\ &= \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \sum_{\substack{C \text{ maximal} \\ \text{chain of } P_S}} \alpha_{[x_C, y_C]}(\{j\}) \\ &= \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \alpha_P(S \cup \{j\}). \end{aligned} \quad \square$$

Remark 2.19. Given an n -dimensional simplicial polytope $\mathcal{P}$, for any $h = -1, 0, \dots, n-1$ we set $S = \{h+1\}$, $i = h+1$ and $k = n+1$ for some for $h = -1, 0, \dots, n-2$. Then, Theorem 2.18 gives us

$$\sum_{j=i+1}^n (-1)^{j-i-1} \alpha_{\mathcal{F}(\mathcal{P})}(\{i, j\}) = f_h \left(1 - (-1)^{n-i-1}\right),$$

where $(f_{-1}, f_0, \dots, f_{n-1})$ denotes the f -vector of $\mathcal{P}$. Since $\mathcal{P}$ is simplicial, we also have

$$\alpha_{\mathcal{F}(\mathcal{P})}(\{i, j\}) = \binom{j}{i} f_{j-1} = \binom{j}{h+1} f_{j-1}$$

for every $j = i+1, \dots, n$. Hence, we rewrite the formula as

$$\begin{aligned} f_h (1 - (-1)^{n-i}) &= \sum_{j=h+2}^n (-1)^{j-i-1} \binom{j}{h+1} f_{j-1} \\ &= \sum_{j=h+1}^n (-1)^{j-i-1} \binom{j}{h+1} f_{j-1} + f_h \end{aligned}$$

and by subtracting from both sides f_h and then multiplying by $(-1)^{n-i-1}$, it follows that

$$\begin{aligned} f_h &= \sum_{j=h+1}^n (-1)^{n-j} \binom{j}{h+1} f_{j-1} \\ &= \sum_{j=h}^{n-1} (-1)^{n-j-1} \binom{j+1}{h+1} f_j \end{aligned}$$

which is exactly the f -vector form of the Dehn–Sommerville Equations.

Since the Generalized Dehn–Sommerville Equations impose non-trivial linear relations on the components of the flag f -vector, they make it possible to lower the upper bound of the dimension of the linear space spanned by the flag f -vectors of Eulerian posets.

Proposition 2.20. *The linear span of the flag f -vectors of bounded graded posets of rank n satisfying the Generalized Dehn–Sommerville Equations has dimension less than or equal to F_n , the n -th Fibonacci number.*

Proof. Let P be a bounded graded poset of rank n satisfying the Generalized Dehn–Sommerville Equations. Define the set

$$\Psi^{(n)} := \{S \subseteq [n-2] : S \text{ contains no two consecutive integers}\}.$$

If $T \in \mathcal{P}([n-1]) \setminus \Psi^{(n)}$, then there exists $k \in [2, n]$ such that $k-1, k \in T \cup \{n\}$. Set $S := T \setminus \{k-1\}$ and $i = \max\{j \in T \cup \{0\} : j < k-1\}$. By taking the Generalized Dehn–Sommerville Equations we get

$$\alpha_P(T) = \sum_{j=i+1}^{k-2} (-1)^{k-j} \alpha_P(S \cup \{j\}) + \alpha_P(S)(1 - (-1)^{k-i-1}).$$

Since the subscript of every term of the right-hand side is less than T in the lexicographic order, we can prove by induction that every coordinate of the flag f -vector $\alpha_P(T)$ can be expressed as a linear combination of terms of the form $\alpha_P(S)$ where $S \in \Psi^{(n)}$.

We notice that $\#\Psi^{(n)} = F_n$. Indeed, every element $S \in \Psi^{(n)}$ might contain $n-2$ or not: the ones that do not contain it are all elements of $\Psi^{(n-1)}$; the ones that do contain it are in bijection with $\Psi^{(n-2)}$ since they must not contain $n-3$. Thus, $\#\Psi^{(n)} = \#\Psi^{(n-1)} + \#\Psi^{(n-2)}$, and since $\#\Psi^{(1)} = 1$ and $\#\Psi^{(2)} = 1$ we have $\#\Psi^{(n)} = F_n$. This proves the proposition. $\square$

As we will show in Theorem 2.24, the upper bound of the proposition can be proved to be an equality.

§2.3 The cd -Index

As we want to comprehend the structure of a poset P , it is of interest to study the flag h -vector β_P . Inspired by the discussion on the Generalized Dehn–Sommerville Equations, we recognize that one way to do that is to find all the linear relations that it satisfies. One way to encode these relations is through the cd -index, introduced Bayer and Klapper in [9] after some observations made by Jonathan Fine in an unpublished 1985 manuscript about Morse theory for convex polytopes. See [6] for a recent survey on the topic. We start the discussion by introducing some definitions.

Definition 2.21. Let P be a graded bounded poset of rank n . Given a set $S \subseteq [n-1]$, the *characteristic monomial* of S is

$$u_S := e_1 e_2 \cdots e_{n-1}$$

where

$$e_i := \begin{cases} a & i \notin S, \\ b & i \in S. \end{cases}$$

We define the ab -index of P as the noncommutative polynomial $\Psi_P(a, b) \in \mathbb{Z}\langle a, b \rangle$ in the variables a and b

$$\Psi_P(a, b) := \sum_{S \subseteq [n-1]} \beta_P(S) u_S.$$

Notice that, by equation 1.4, we have

$$\Psi_P(a+b, b) = \sum_{S \subseteq [n-1]} \left(\beta_P(S) \sum_{S \subseteq T} u_T \right) = \sum_{S \subseteq [n-1]} \left(\sum_{T \subseteq S} \beta_P(T) \right) u_S = \sum_{S \subseteq [n-1]} \alpha(S) u_S. \quad (2.2)$$

Thus, we may interpret $\Psi_P(a, b)$ and $\Psi_P(a+b, b)$ as noncommutative generating functions respectively for the flag h -vector β_P and the flag f -vector α_P .

Remark 2.22. Since $\Psi_P(a+b, b)$ is a generating function for the flag f -vector, we may calculate the ab -index as follows. Given a chain $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ of P , we associate to it a weight $w(c) = z_1 \cdots z_{n-1}$ where

$$z_i = \begin{cases} b & i \in \{\rho(x_1), \dots, \rho(x_k)\}, \\ a-b & \text{otherwise.} \end{cases}$$

Then the ab -index is

$$\Psi_P(a, b) = \sum_{c \text{ chain of } P} w(c).$$

Let's calculate an example.

Example 2.23. Take the poset P defined as the face lattice of the pyramid with a square base.

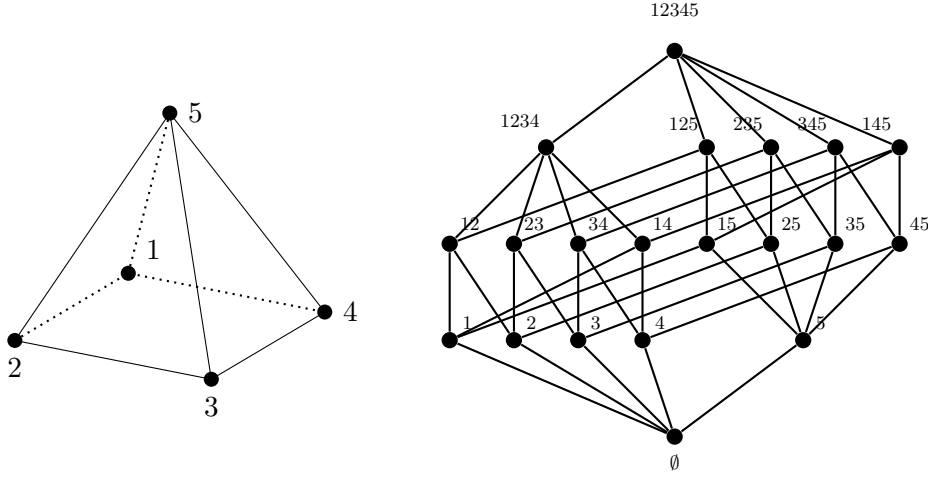


Figure 2.2: Face lattice of a pyramid with a square base

We may calculate

$$\Psi_P(a + b, b) = aaa + 5(baa + aab) + 8aba + 16(bba + bab + abb) + 32bbb$$

which implies

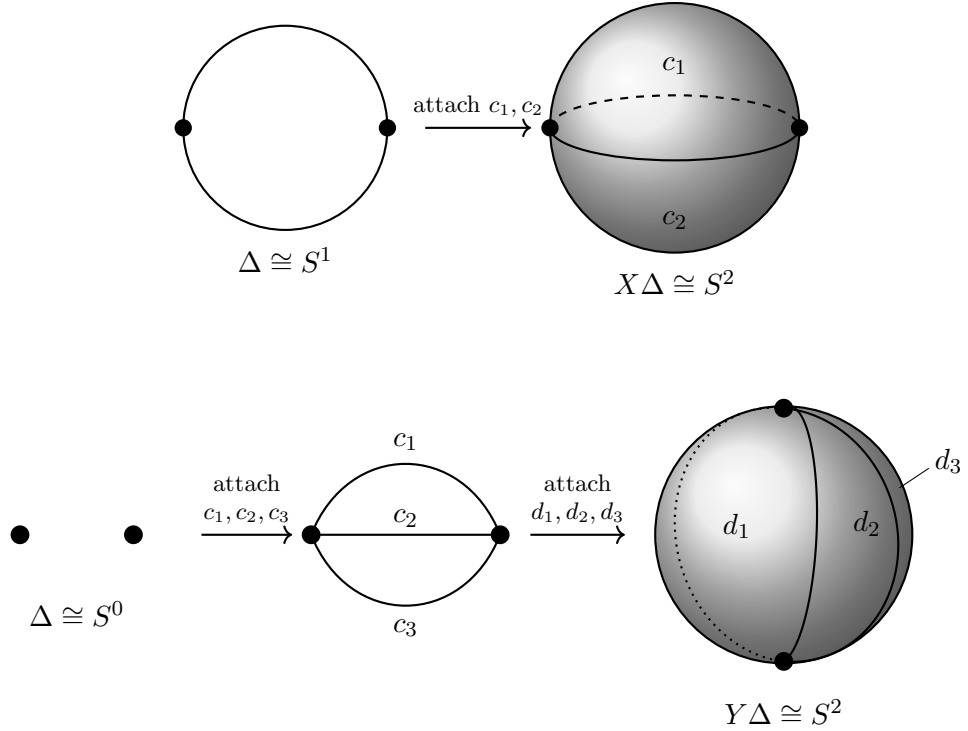
$$\Psi_P(a, b) = aaa + 4(baa + aab + bba + abb) + 7(aba + bab) + bbb.$$

Notice that the ab -index may be factorized as

$$\Psi_P(a, b) = (a + b)^3 + 3(a + b)(ab + ba) + 3(ab + ba)(a + b).$$

The factorization of the ab -index in the example is not an accident. In fact, as we are going to prove, every Eulerian poset admits a similar factorization of the ab -index. To prove that, we first need some specific constructions of Eulerian posets.

Given any regular CW-sphere Δ (i.e. a regular CW-complex whose support is homeomorphic to an n -sphere), we define two new regular CW-spheres, $X\Delta$ and $Y\Delta$, as follows. The complex $X\Delta$ is constructed by attaching to Δ two $(n + 1)$ -cells c_1 and c_2 with boundary Δ . The complex $Y\Delta$ is defined by attaching to Δ three $(n + 1)$ -cells c_1 , c_2 and c_3 with boundary Δ , and then attaching another three $(n + 2)$ -cells d_1 , d_2 and d_3 , each having a different pair of cells from c_1 , c_2 and c_3 in its boundary.


 Figure 2.3: Low-dimensional realizations of the operations X and Y .

By Remark 2.22, we can observe the following fact. Set $c := a + b$ and $d := ab + ba$. Let Δ be any regular CW-sphere, any chain of non-empty cells of Δ can be extended to a chain of non-empty cells of $X\Delta$ by adjoining either c_1 , c_2 or nothing, hence the ab -index of $F(X\Delta)$ is

$$\Psi_{F(X\Delta)}(a, b) = \Psi_{F(\Delta)}(a, b) \cdot (2b + (a - b)) = \Psi_{F(\Delta)}(a, b) \cdot c. \quad (2.3)$$

Similarly, one can observe

$$\Psi_{F(Y\Delta)}(a, b) = \Psi_{F(\Delta)}(a, b) \cdot (6b^2 + 3b(a - b) + 3(a - b)b + (a - b)^2) = \Psi_{F(\Delta)}(a, b) \cdot (c^2 + d). \quad (2.4)$$

Note that if Δ is the empty complex, then $\Psi_{F(\Delta)} = 1$. Consequently, every monomial in c and d is in the span of ab -indices of Eulerian posets. In fact, we have

$$\Psi_{F(X\Delta)} = \Psi_{F(\Delta)} \cdot c, \quad \Psi_{F(Y\Delta)} - \Psi_{F(X\Delta)} = \Psi_{F(\Delta)} \cdot d.$$

We now observe that there are no algebra relations between $a+b$ and $ab+ba$. Indeed, given any non-commutative polynomial $p(x, y) \in \mathbb{Q}\langle x, y \rangle$, the leading term of $p(a + b, ab + ba)$ in the graded lexicographic order induced by $a > b$ is given by the evaluation $p(a, ab)$. Consequently, if $p(a + b, ab + ba)$ equals 0, then $p(x, y)$ must also be 0. Thus, since $\deg c = 1$ and $\deg d = 2$, the degree k component of $\mathbb{Q}\langle c, d \rangle$ has dimension F_{k+1} , the $(k + 1)$ -th Fibonacci number. This implies that the dimension of the linear span of the ab -indices of Eulerian posets of rank n is at least F_n . Together with Proposition 2.20, we get the following theorem.

Theorem 2.24. *The linear span of the flag f -vectors of bounded graded posets of rank n satisfying the Generalized Dehn–Sommerville Equations has dimension F_n .*

Similarly, the linear span of the ab -indices of bounded graded posets satisfying the Generalized Dehn–Sommerville Equations is $\mathbb{Q}\langle c, d \rangle$.

This also shows that the Generalized Dehn–Sommerville Equations are exactly all the possible linear relations that we may find for Eulerian posets. We can use this result to compress our ab -index in a more compact invariant, whose existence is equivalent to satisfying the Generalized Dehn–Sommerville Equations. This follows easily from the second statement of Theorem 2.24.

Theorem 2.25. *Let P be a bounded graded poset of rank n . There exists a noncommutative polynomial $\Phi_P(c, d) \in \mathbb{Q}\langle c, d \rangle$ such that*

$$\Psi_P(a, b) = \Phi_P(a + b, ab + ba)$$

if and only if P satisfies the Generalized Dehn–Sommerville equations. In particular, Φ_P exists when P is Eulerian.

Hence, we have an optimal way to compress the ab -index up to linear relations. Some of these linear relations can now actually be recovered more naturally from this result. For example, since $\Psi_P(a, b) = \Psi_P(b, a)$ for any graded poset P satisfying the Generalized Dehn–Sommerville Equations, we have the following corollary.

Corollary 2.26. *Let P be a bounded graded poset of rank n , let $S \subseteq [n - 1]$ and set $\bar{S} := [n - 1] \setminus S$. If P satisfies the Generalized Dehn–Sommerville Equations, then $\beta_P(S) = \beta_P(\bar{S})$.*

One can also notice the following.

Remark 2.27. Given a poset P with a cd -index Φ_P , the coefficient of c^{n-1} in Φ_P is 1. In fact, the coefficient of a^{n-1} in the ab -index is also 1, and c^{n-1} is the only monomial in Φ_P which contributes to a^{n-1} . This implies that the affine span of the flag h -vectors of posets satisfying the Generalized Dehn–Sommerville Equations has dimension $F_{n-1} - 1$.

Example 2.28. Let P be the face lattice of a cube. We have already calculated in Example 1.52 its flag h -vector. In particular, its ab -index is

$$\Psi_P(a, b) = a^3 + 7ba^2 + 11aba + 5a^2b + 5b^2a + 11bab + 7ab^2 + b^3.$$

Its cd -index must be of the form

$$\Phi_P(c, d) = c^3 + \alpha \cdot cd + \beta \cdot dc$$

for some $\alpha, \beta \in \mathbb{Q}$. By expanding the cd -index, we get that the coefficient of ba^2 is $1 + \beta$ and the coefficient of b^2a is $1 + \alpha$, hence $\alpha = 4$ and $\beta = 6$. Therefore

$$\Phi_P(c, d) = c^3 + 4cd + 6dc.$$

Example 2.29. Let P be the poset defined as the join of n copies of B_2 . This is sometimes called the *butterfly poset* of rank $n + 1$.

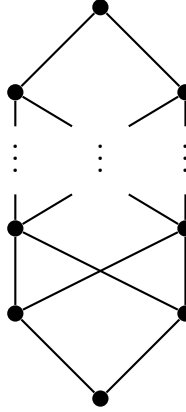


Figure 2.4: The butterfly poset.

For any $S \subseteq [n]$, we have

$$\alpha_P(S) = 2^{\#S}.$$

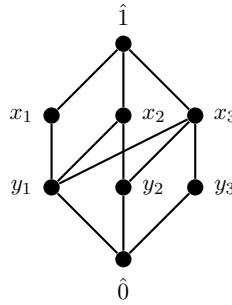
This implies that

$$\Psi_P(a+b, b) = \sum_{i=0}^n \sum_{\substack{\mathbf{w} \text{ is an } ab\text{-monomial} \\ \text{of degree } n \text{ and} \\ \text{with } i \text{ occurrences of } b}} 2^i \mathbf{w} = (a+2b)^n,$$

hence $\Psi_P(a, b) = (a+b)^n$ and

$$\Phi_P(c, d) = c^n.$$

Example 2.30. Let P be the poset represented by the following Hasse diagram.



Its ab -index is

$$\Psi_P(a, b) = a^2 + 2ab + 2ba + b^2 = (a+b)^2 + (ab+ba),$$

hence it has a cd -index which is

$$\Phi_P(c, d) = c^2 + d.$$

Notice that the interval $[y_1, \hat{1}]$ does not have a cd -index, since

$$\Psi_{[y_1, \hat{1}]}(a, b) = a + 2b.$$

This shows that if a poset satisfies the Generalized Dehn–Sommerville Equations, it is not guaranteed that every one of its intervals also satisfies them. This happens in contrast to

Eulerian posets, where the Eulerianity property is satisfied by all of its closed intervals. Margaret M. Bayer and Gábor Hetyei have actually proved that a poset is Eulerian if and only if every one of its interval satisfies the Generalized Dehn–Sommerville Equations [8, Corollary 1.3].

The cd -index behaves well under some specific operations on posets. The join is particularly relevant, as the cd -index is multiplicative with respect to it.

Proposition 2.31. *Let P and Q be two graded posets. Then*

1. *the ab -index of the join factorizes as $\Psi_{P*Q}(a, b) = \Psi_P(a, b)\Psi_Q(a, b)$;*
2. *given the linear involution*

$$\begin{aligned} \bullet^* : \mathbb{Q}\langle a, b \rangle &\rightarrow \mathbb{Q}\langle a, b \rangle \\ v_1 v_2 \dots v_n &\mapsto (v_1 v_2 \dots v_n)^* := v_n \dots v_2 v_1, \end{aligned}$$

the ab -index of the dual P^* is obtained as $\Psi_{P^*}(a, b) = \Psi_P(a, b)^*$.

Proof. 1. Consider the polynomial $\Psi_{P*Q}(a + b, b) = \sum_S \alpha_{P*Q}(S) u_S$. Maximal chains in $P * Q$ are the concatenation of a maximal chain in P and a maximal chain in Q , hence we get $\Psi_{P*Q}(a + b, b) = \Psi_P(a + b, b)\Psi_Q(a + b, b)$, which implies $\Psi_{P*Q}(a, b) = \Psi_P(a, b)\Psi_Q(a, b)$.

2. Let ρ be the rank function of P . Notice that the rank of an element x in P^* is $\rho(P) - \rho(x)$. Hence, by Remark 2.22, we have

$$\Psi_{P^*}(a, b) = \sum_{c \text{ chain of } P^*} w(c) = \sum_{c \text{ chain of } P} w(c)^*. \quad \square$$

Using Theorem 2.25, we can also state the following corollary.

Corollary 2.32. *Let P and Q be two graded posets. If they both satisfy the Generalized Dehn–Sommerville equations, then they are also satisfied by the join $P * Q$. Furthermore, its cd -index factorizes as $\Phi_{P*Q}(c, d) = \Phi_P(c, d)\Phi_Q(c, d)$.*

If P satisfies the Generalized Dehn–Sommerville equations, then also does the dual P^* .

§2.3.1 The cd -Index as a Newtonian Coalgebra Morphism

One important property of the ab -index and the cd -index is that they both can be interpreted as Newtonian coalgebra morphisms, as was shown by Ehrenborg and Readdy [23]. Let us introduce some definitions.

Definition 2.33. A *Newtonian coalgebra* is a vector space V over a field k of characteristic 0 that is equipped with an associative product

$$\mu : V \otimes V \rightarrow V$$

and a *coproduct*, defined as a map

$$\Delta : V \rightarrow V \otimes V.$$

The coproduct is required to be *coassociative*, meaning that

$$(\Delta \otimes \text{id}_V) \circ \Delta = (\text{id}_V \otimes \Delta) \circ \Delta.$$

Moreover, the product and the coproduct satisfy the *Newtonian condition*

$$\Delta \circ \mu = (\text{id}_V \otimes \mu) \circ (\Delta \otimes \text{id}_V) + (\mu \otimes \text{id}_V) \circ (\text{id}_V \otimes \Delta).$$

We will often use *Sweedler notation* for the coproduct. Since $\Delta(v)$ belongs to the tensor product $V \otimes V$, it is a finite sum of pure tensors. Sweedler notation expresses the sum as

$$\Delta(v) = \sum_v v_{(1)} \otimes v_{(2)}.$$

Here, $v_{(1)}$ and $v_{(2)}$ are not specific individual elements, but terms representing the first and second tensor factors across the entire sum.

Using Sweedler notation, the Newtonian condition can be rewritten more compactly as

$$\Delta(uv) = \sum_u u_{(1)} \otimes (u_{(2)}v) + \sum_v (uv_{(1)}) \otimes v_{(2)}.$$

The noncommutative polynomial algebra $k\langle a, b \rangle$ is a Newtonian coalgebra with the coproduct

$$\begin{aligned} \Delta : k\langle a, b \rangle &\rightarrow k\langle a, b \rangle \otimes k\langle a, b \rangle \\ v_1 v_2 \cdots v_n &\mapsto \sum_{i=1}^n v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n. \end{aligned}$$

The coproduct is not injective, but its kernel can be described explicitly.

Lemma 2.34. *If we restrict the coproduct Δ on the homogeneous ab -polynomials of degree n , we obtain a linear map*

$$\Delta : k\langle a, b \rangle_n \rightarrow \bigoplus_{i+j=n-1} k\langle a, b \rangle_i \otimes k\langle a, b \rangle_j$$

whose kernel is the linear span of $(a - b)^n$.

Proof. Let $p = \sum_{\mathbf{w}} \alpha_{\mathbf{w}} \mathbf{w}$ be an homogeneous ab -polynomial of degree n such that $\Delta(p) = 0$. Given any $\mathbf{u} \otimes \mathbf{v} \in k\langle a, b \rangle_i \otimes k\langle a, b \rangle_j$ for some i, j such that $i + j = n - 1$, it may be obtained by applying the coproduct to either $\mathbf{u} \cdot a \cdot \mathbf{v}$ or $\mathbf{u} \cdot b \cdot \mathbf{v}$, hence

$$\alpha_{\mathbf{u} \cdot a \cdot \mathbf{v}} + \alpha_{\mathbf{u} \cdot b \cdot \mathbf{v}} = 0.$$

Given an ab -monomial $\mathbf{w}$, let h be the number of b 's in the word $\mathbf{w}$. We have

$$\alpha_{\mathbf{w}} = (-1)^h \alpha_{a^n},$$

from which we conclude

$$p = \alpha_{a^n} \cdot \sum_{i=0}^n \sum_{\substack{\mathbf{w} \text{ is an } ab\text{-monomial} \\ \text{of degree } n \text{ and} \\ \text{with } i \text{ occurrences of } b}} (-1)^i \mathbf{w} = \alpha_{a^n} \cdot (a - b)^n. \quad \square$$

Since $(a - b)^2 = c^2 - 2d$, we also have the following corollary.

Corollary 2.35. *If we restrict the coproduct Δ on the homogeneous cd -polynomials of degree n , we obtain a linear map*

$$\Delta : k\langle c, d \rangle_n \rightarrow \bigoplus_{i+j=n-1} k\langle c, d \rangle_i \otimes k\langle c, d \rangle_j$$

which is injective when n is odd, and whose kernel is the linear span of $(c^2 - 2d)^{n/2}$ when n is even.

This also implies the following result, which may be used to reverse the operation on specific subsets of polynomials.

Corollary 2.36. *Let $p, q \in k\langle c, d \rangle_n$ be homogeneous cd-polynomials of degree n that have the same coefficient of c^n . If $\Delta(p) = \Delta(q)$, then $p = q$.*

Proof. If n is odd, the statement is trivial. If n is even, since $\Delta(p - q) = 0$ we have that the polynomial $p - q \in k\langle c, d \rangle_n$ is a multiple of $(c^2 - 2d)^{n/2}$. Since the coefficient of c^n in $p - q$ is 0, we have $p - q = 0$. $\square$

We may also give a Newtonian coalgebra structure to graded posets. Let $\mathcal{P}$ be the k -vector space spanned by the isomorphism classes of graded bounded posets of rank at least 1. We enrich the structure of $\mathcal{P}$ by defining a coproduct

$$\begin{aligned} \Delta : \mathcal{P} &\rightarrow \mathcal{P} \otimes \mathcal{P} \\ P &\mapsto \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} [\hat{0}, x] \otimes [x, \hat{1}] \end{aligned}$$

and taking the join as a product

$$\begin{aligned} * : \mathcal{P} \otimes \mathcal{P} &\rightarrow \mathcal{P} \\ P \otimes Q &\mapsto P * Q. \end{aligned}$$

It is an easy check to verify that $\mathcal{P}$ satisfies the Newtonian condition.

Proposition 2.37. *The vector space $\mathcal{P}$ with the product $*$ and the coproduct Δ is a Newtonian coalgebra.*

As anticipated, the ab -index can be seen as a map

$$\begin{aligned} \Psi : \mathcal{P} &\rightarrow k\langle a, b \rangle \\ P &\mapsto \Psi_P(a, b) \end{aligned}$$

which is compatible with the Newtonian coalgebra structure.

Proposition 2.38. *The linear map $\Psi : \mathcal{P} \rightarrow k\langle a, b \rangle$ is a Newtonian coalgebra map, i.e. it satisfies*

$$\Psi \circ \mu = \mu \circ (\Psi \otimes \Psi)$$

and

$$\Delta \circ \Psi = (\Psi \otimes \Psi) \circ \Delta.$$

Proof. The first identity is

$$\Psi(P * Q) = \Psi(P)\Psi(Q)$$

which was proved in Proposition 2.31. The second identity is

$$\Delta(\Psi(P)) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \otimes \Psi([x, \hat{1}]).$$

By Remark 2.22, we interpret the ab -index as the sum of the weights of all chains of P , in particular by linearity of Δ we have

$$\begin{aligned} \sum_{c \text{ chain of } P} \Delta(w(c)) &= \sum_{\hat{0} < x_1 < \dots < x_k < \hat{1}} \sum_{i=1}^k w_{[\hat{0}, x_i]}(\{\hat{0} < x_1 < \dots < x_i\}) \otimes w_{[x_i, \hat{1}]}(\{x_i < \dots < x_k < \hat{1}\}) \\ &= \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \left(\sum_{c_1 \text{ chain of } [\hat{0}, x]} w_{[\hat{0}, x]}(c_1) \right) \otimes \left(\sum_{c_2 \text{ chain of } [x, \hat{1}]} w_{[x, \hat{1}]}(c_2) \right) \end{aligned}$$

which proves the second identity. $\square$

Since the cd -index is obtained from the ab -index by a polynomial substitution, we get the following corollary.

Corollary 2.39. *Let $\mathcal{E} \subseteq \mathcal{P}$ be the k -vector space spanned by the isomorphism classes of Eulerian posets of rank at least 1. The linear map*

$$\begin{aligned} \Phi : \mathcal{E} &\rightarrow k\langle c, d \rangle \\ P &\mapsto \Phi_P(c, d) \end{aligned}$$

is a Newtonian coalgebra map.

Note that the corollary above would not work if we substituted Eulerian posets with all the posets satisfying the Generalized Dehn–Sommerville Equations. As we have seen in Example 2.30, it is not guaranteed that every interval of the poset has a cd -index. In fact, this only happens with Eulerian posets.

These results have a broad span of computational consequences related to the calculation of the cd -index for some families of Eulerian posets. We look at an example below. The formulas that we present have been found by Ehrenborg and Readdy [23, Proposition 4.2], we provide an alternative proof that avoids enumerating chains; instead, we explicitly make use of the Newtonian coalgebra structure.

Proposition 2.40. *Let $\mathcal{Q}$ be a polytope and set $P = \mathcal{F}(\mathcal{Q})$. Then*

$$\begin{aligned} \Phi_{\mathcal{F}(\text{Pyr}(\mathcal{Q}))} &= \frac{1}{2} \left[\Phi_P \cdot c + c \cdot \Phi_P + \sum_P \Phi_{P_{(1)}} \cdot d \cdot \Phi_{P_{(2)}} \right], \\ \Phi_{\mathcal{F}(\text{Prism}(\mathcal{Q}))} &= \Phi_P \cdot c + \sum_P \Phi_{P_{(1)}} \cdot d \cdot \Phi_{P_{(2)}}. \end{aligned}$$

Proof. Recall by Corollary 1.30 that

$$\mathcal{F}(\text{Pyr}(\mathcal{Q})) \cong P \times B_1, \quad \mathcal{F}(\text{Prism}(\mathcal{Q})) \cong (P \setminus \{\hat{0}_P\}) \times (B_2 \setminus \{\hat{0}_{B_2}\}) \sqcup \{\hat{0}\}.$$

Using Corollary 2.36, we know that it is sufficient to prove that

$$\Delta \Phi_{\mathcal{F}(\text{Pyr}(\mathcal{Q}))} = \Phi_{\Delta \mathcal{F}(\text{Pyr}(\mathcal{Q}))}, \quad \Delta \Phi_{\mathcal{F}(\text{Prism}(\mathcal{Q}))} = \Phi_{\Delta \mathcal{F}(\text{Prism}(\mathcal{Q}))}.$$

We first compute the coproduct of the posets

$$\begin{aligned}
 \Delta\mathcal{F}(\text{Pyr}(\mathcal{Q})) &= \Delta(P \times B_1) \\
 &= [(\hat{0}, \hat{0}), (\hat{1}, \hat{0})] \otimes [(\hat{1}, \hat{0}), (\hat{1}, \hat{1})] \\
 &\quad + [(\hat{0}, \hat{0}), (\hat{0}, \hat{1})] \otimes [(\hat{0}, \hat{1}), (\hat{1}, \hat{1})] \\
 &\quad + \sum_{\hat{0} < x < \hat{1}} [(\hat{0}, \hat{0}), (x, \hat{0})] \otimes [(x, \hat{0}), (\hat{1}, \hat{1})] \\
 &\quad + \sum_{\hat{0} < x < \hat{1}} [(\hat{0}, \hat{0}), (x, \hat{1})] \otimes [(x, \hat{1}), (\hat{1}, \hat{1})] \\
 &= P \otimes B_1 + B_1 \otimes P \\
 &\quad + \sum_P (P_{(1)} \otimes (P_{(2)} \times B_1) + (P_{(1)} \times B_1) \otimes P_{(2)}).
 \end{aligned}$$

Since the cd -index is a Newtonian coalgebra map, this implies

$$\Phi_{\Delta\mathcal{F}(\text{Pyr}(\mathcal{Q}))} = 2\Phi_P + \sum_P (\Phi_{P_{(1)}} \Phi_{\text{Pyr}(P_{(2)})} + \Phi_{\text{Pyr}(P_{(1)})} \Phi_{P_{(2)}}).$$

Similarly, we calculate

$$\Phi_{\Delta\mathcal{F}(\text{Prism}(\mathcal{Q}))} = 2\Phi_P + \sum_P (\Phi_{P_{(1)}} \Phi_{\text{Pyr}(P_{(2)})} + \Phi_{\text{Prism}(P_{(1)})} \Phi_{P_{(2)}}).$$

We may verify that these are exactly the coproducts of the formulas given in the statement. $\square$

3 Toric h -Polynomial

The construction of the toric h -polynomial was given for the first time by Richard Stanley in 1987 in the paper [48]. As we have seen in Section 1.3, the h -polynomial of a polytope $\mathcal{P}$ satisfies many appealing properties, but only when $\mathcal{P}$ is simplicial. The toric h -polynomial is an extension of the h -vector from the case of simplicial polytopes to the case of Eulerian posets. Despite its contrived and recursive definition, the toric h -polynomial preserves some of the most interesting properties of the usual h -polynomial of simplicial polytopes. In particular, we will prove that it preserves the palindromic property.

One of the most striking peculiarities of the toric h -polynomial is its algebro-geometric interpretation, in the case of rational polytopes. In this context, the recursive definition of the polynomial may be used to create a bridge with the theory of *intersection cohomology*, introduced by Goresky and MacPherson [32, 31] with the purpose of studying singular spaces. Whenever a space is not singular, in fact, it coincides with singular cohomology. The following theorem yields one of the main motivations for studying the toric h -polynomials.

Theorem 3.1. *Let $\mathcal{P}$ be a rational polytope of dimension n containing the origin in its interior. The toric h -vector $(h_0, \dots, h_n)$ of $F(\mathcal{P})$ satisfies the identity*

$$h_i = \dim \mathrm{IH}^{2i}(X_{\mathcal{P}}, \mathbb{C}),$$

where IH denotes intersection cohomology and $X_{\mathcal{P}}$ denotes the projective toric variety associated to the fan of cones over the proper faces of $\mathcal{P}$.

This fact was stated without proof in the introductory paper of Stanley, while citing independent unpublished works of other authors. The first proofs were actually published only 4 years later by Jan Denef and François Loeser [22], and by Karl-Heinz Fieseler [29].

In the context of rational polytopes, the properties of a combinatorial object like the toric h -vector can be translated into algebro-geometric properties of complex toric varieties, and vice versa. In particular, the palindromic property of the toric h -vector is implied by the Poincaré duality in intersection cohomology. Similarly, the Hard Lefschetz Theorem implies the unimodality of the toric h -vector.

In the rest of the introduction of this chapter, we define the toric h -vector and prove that it coincides with the usual h -vector in the case of simplicial polytopes. Then, we show that it is palindromic. In Section 3.1, we calculate the toric h -vector using other invariants of Eulerian posets, in particular the flag f -vector, the flag h -vector and the cd -index. The proof of the cd -index formula will significantly rely on the Newtonian coalgebra structure of the cd -index.

We now introduce the definition of the toric h - and g -polynomial of an Eulerian poset.

Definition 3.2. Let P be a graded bounded poset. Let $h(P, x) = \sum_i h_i(P)x^i$ and $g(P, x) = \sum_i g_i(P)x^i$ be polynomials in $\mathbb{Z}[x]$, recursively defined as follows. If $\rho(P) = 0$, then

$$h(P, x) = g(P, x) = 1.$$

Otherwise, if $\rho(P) = n + 1 > 0$, then h has degree n and

$$\begin{aligned} h(P, x) &= \sum_{\hat{0} \leq y < \hat{1}} g([\hat{0}, y], x)(x-1)^{n-\rho(y)} \\ g(P, x) &= h_0(P) + (h_1(P) - h_0(P))x + \cdots + (h_m(P) - h_{m-1}(P))x^m \end{aligned}$$

where $m = \lfloor n/2 \rfloor$ and ρ is the rank function of P . We call $h(P, x)$ and $g(P, x)$ respectively the *toric h -polynomial* of P and the *toric g -polynomial* of P . The sequences $(h_0(P), \dots, h_n(P))$ and $(g_0(P), \dots, g_m(P))$ are respectively called the *toric h -vector* and the *toric g -vector* of P .

In the literature, the toric h -polynomial is often denoted with the letter f . We will not use this notation to avoid ambiguity with a function that we will define in Chapter 4, the right KLS function.

The usual h -vector and g -vector are equal to their toric counterparts in the case of Boolean posets, or equivalently, in the case of face lattices of a simplex.

Example 3.3. Let B_n be the Boolean poset on $[n]$ for $n \geq 1$. We claim that

$$\begin{aligned} h(B_n, x) &= 1 + x + \cdots + x^{n-1}, \\ g(B_n, x) &= 1. \end{aligned}$$

By definition, we have $h(B_1, x) = 1$, $g(B_1, x) = 1$ and for $n+1 > 1$ we have the recursion

$$\begin{aligned} h(B_{n+1}, x) &= \sum_{\hat{0} \leq y < \hat{1}} g([\hat{0}, y], x)(x-1)^{n-\text{rank}([\hat{0}, y])} \\ &= \sum_{k=0}^n g(B_k, x) \binom{n+1}{k} (x-1)^{n-k}. \end{aligned}$$

Using an induction argument on n , we prove that

$$\begin{aligned} h(B_{n+1}, x) &= \sum_{k=0}^n \binom{n+1}{k} (x-1)^{n-k} \\ &= (x-1)^{-1} \sum_{k=0}^n \binom{n+1}{k} (x-1)^{n+1-k} \\ &= \frac{((x-1)+1)^{n+1} - 1}{x-1} \\ &= 1 + x + \cdots + x^n. \end{aligned}$$

Hence, we also have

$$g(B_{n+1}, x) = 1 + (1-1)x + \cdots + (1-1)x^{\lfloor n/2 \rfloor} = 1.$$

The definitions match whenever P is the face lattice of a simplicial polytope. This claim is proved by the following proposition.

Proposition 3.4. *Let P be an Eulerian poset of rank $n+1$ and $P \setminus \{\hat{1}\}$ be simplicial. Set $f_i := \#\{x \in P : \rho(x) = i+1\}$. We have*

$$h(P, x) = \sum_{i=0}^n f_{i-1} (x-1)^{n-i}.$$

Proof. Since $g(B_n, x) = 1$, by the simpliciality of $P \setminus \{\hat{1}\}$ we get

$$h(P, x) = \sum_{\hat{0} \leq y < \hat{1}} g([\hat{0}, y], x) (x-1)^{\text{rank}([y, \hat{1}]) - 1} = \sum_{\hat{0} \leq y < \hat{1}} (x-1)^{\text{rank}([y, \hat{1}]) - 1} = \sum_{i=0}^n f_{i-1} (x-1)^{n-1}.$$

□

We now prove that the palindromic property of the h -vector generalizes to Eulerian posets.

Theorem 3.5 ([47, Theorem 3.16.9]). *Let P be an Eulerian poset of rank $n + 1$. Then, for any $i = 0, \dots, n$ we have*

$$h_i(P) = h_{n-i}(P).$$

Proof. Let $h(P)$ and $g(P)$ denote $h(P, x)$ and $g(P, x)$, and let ρ be the rank function of P . Set $z := x - 1$, and for any $y \in P$ set $Q_y := [\hat{0}, y] \subseteq P$. By the definition of the toric h -polynomial, we have

$$\begin{aligned} g(P) + zh(P) &= g(P) + \sum_{\hat{0} \leq y < \hat{1}} g(Q_y) z^{n+1-\rho(y)} \\ &= \sum_{y \in P} g(Q_y) z^{\rho(\hat{1})-\rho(y)} \end{aligned}$$

hence

$$z^{-\rho(\hat{1})}(g(P) + zh(P)) = \sum_{y \in P} g(Q_y) z^{-\rho(y)}.$$

By Möbius inversion, we get

$$g(P) z^{-\rho(\hat{1})} = \sum_{y \in P} \mu_P(y, \hat{1}) (g(Q_y) + zh(Q_y)) z^{-\rho(y)}.$$

Using the Eulerianity of P , this implies

$$g(P) = \sum_{y \in P} (-z)^{\rho(\hat{1})-\rho(y)} (g(Q_y) + zh(Q_y)). \quad (3.1)$$

Given any $y \in P$ of rank $r + 1 \in \mathbb{N}$, let $h(Q_y) = a_0 + a_1x + \dots + a_rx^r$. Then, given $s := \lfloor r/2 \rfloor$, by the definition of toric g -vector we have

$$\begin{aligned} g(Q_y) + zh(Q_y) &= a_0 + (a_1 - a_0)x + \dots + (a_s - a_{s-1})x^s + (x-1)(a_0 + a_1x + \dots + a_rx^r) \\ &= (a_s - a_{s+1})x^{s+1} + \dots + (a_{r-1} - a_r)x^r + a_rx^{r+1}. \end{aligned}$$

By induction on $\rho(y)$, we may assume $a_i = a_{r-i}$ for any $r < n$ and $0 \leq i \leq r$, so

$$\begin{aligned} g(Q_y) + zh(Q_y) &= (a_{r-s} - a_{r-s-1})x^{s+1} + (a_{r-s-1} - a_{r-s-2})x^{s+2} \\ &\quad + \dots + (a_{r-r+1} - a_{r-r})x^r + a_rx^{r+1} \\ &= \begin{cases} (a_s - a_{s-1})x^{s+1} + (a_{s-1} - a_{s-2})x^{s+2} + \dots + a_0x^{r+1} & \text{even } r \\ (a_s - a_{s-1})x^{s+2} + (a_{s-1} - a_{s-2})x^{s+3} + \dots + a_0x^{r+1} & \text{odd } r \end{cases} \\ &= x^{\rho(y)} g(Q_y, 1/x). \end{aligned}$$

By subtracting $g(P) + zh(P)$ from both sides of (3.1), we obtain

$$-zh(P) = \sum_{\hat{0} \leq y < \hat{1}} x^{\rho(y)} g(Q_y, 1/x) (-z)^{\rho(\hat{1})-\rho(y)},$$

therefore

$$\begin{aligned}
 h(P) &= \sum_{\hat{0} \leq y < \hat{1}} (g(Q_y) + zh(Q_y)) (-z)^{\rho(\hat{1}) - \rho(y) - 1} \\
 &= \sum_{\hat{0} \leq y < \hat{1}} g(Q_y, 1/x) x^{\rho(y)} (1-x)^{\rho(\hat{1}) - \rho(y) - 1} \\
 &= x^n \sum_{\hat{0} \leq y < \hat{1}} g(Q_y, 1/x) \left(\frac{1-x}{x} \right)^{\rho(\hat{1}) - \rho(y) - 1} \\
 &= x^n \sum_{\hat{0} \leq y < \hat{1}} g(Q_y, 1/x) \left(\frac{1}{x} - 1 \right)^{\rho(\hat{1}) - \rho(y) - 1} \\
 &= x^n h(P, 1/x),
 \end{aligned}$$

which is equivalent to the statement. $\square$

§3.1 Relation to the Flag f -Vector and Flag h -Vector

The recursive definition of the toric h -vector does not display clearly all of its properties. Despite that, it has many relations to other statistics of an Eulerian poset that simplify its calculation.

§3.1.1 Fine's Algorithm

We start by calculating the toric h -polynomial using the flag f -vector data. Jonathan Fine formulated an algorithm [4, Theorem 7.14] that does exactly that for any Eulerian poset.

Let P be an Eulerian poset of rank $n + 1$ and let $S \subseteq [n]$. A *valid S -diagram* is a vector of signs ± 1 of length n such that the sum of the terms up to position s is positive for each $s \in S$. Denote by $j_{S,i}$ the number of valid S -diagrams with i negative terms and $n - i$ positive terms. If $n = 0$, we let $j_{\emptyset,0} := 1$.

Theorem 3.6. *Let P be an Eulerian poset of rank $n + 1$. The toric h -polynomial is*

$$h(P, x) = \sum_{i=0}^n \left(\sum_{S \subseteq [n]} (-1)^{|S| + n - i} j_{S,i} \alpha_P(S) \right) x^i.$$

Proof. We prove it by induction on n .

If $n = 0$, the statement is trivial as there is only one Eulerian poset of rank 1, and both sides of the formula are equal to 1.

Let $n > 0$, suppose that the formula holds for any Eulerian poset of rank at most n . Set $j_{S,i}^m$ to be the number of valid S -diagrams of length m with i negative terms. By the

definition of the toric h -vector and the induction hypothesis,

$$\begin{aligned} h(P, x) &= (x-1)^n + \sum_{\hat{0} < y < \hat{1}} \sum_{i=0}^{\lfloor (\rho(y)-1)/2 \rfloor} g_i([\hat{0}, y]) x^i (x-1)^{n-\rho(y)} \\ &= (x+1)^n + \sum_{\hat{0} < y < \hat{1}} \sum_{i=0}^{\lfloor (\rho(y)-1)/2 \rfloor} x^i (x-1)^{n-\rho(y)} \\ &\quad \cdot \sum_{S \subseteq [\rho(y)-1]} (-1)^{|S|+\rho(y)-1-i} (j_{S,i}^{\rho(y)-1} + j_{S,i-1}^{\rho(y)-1}) \alpha_{[\hat{0}, y]}(S). \end{aligned}$$

For each nonempty set $S \subseteq [n]$, set $T := S \setminus \{\max S\}$ and notice that

$$\sum_{\substack{y \in P \\ \rho(y) = \max S}} \alpha_{[\hat{0}, y]}(T) = \alpha_P(S).$$

Using this identity, we may rearrange the sum and rewrite the toric h -vector of P as

$$\begin{aligned} h(P, x) &= (x-1)^n + \sum_{\emptyset \neq S \subseteq [n]} \sum_{i=0}^{\lfloor (\max S-1)/2 \rfloor} (-1)^{|T|+\max S-1-i} (j_{T,i}^{\max S-1} + j_{T,i-1}^{\max S-1}) \\ &\quad \cdot \alpha_{[\hat{0}, y]}(T) x^i (x-1)^{n-\max S}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} [x^k]h(P, x) &= (x-1)^{n-k} \\ &\quad + \sum_{\emptyset \neq S \subseteq [n]} (-1)^{|S|+n-k} \alpha_P(S) \sum_{i=0}^{\lfloor (\max S-1)/2 \rfloor} \binom{n-\max S}{k-i} (j_{T,i}^{\max S-1} + j_{T,i-1}^{\max S-1}) \\ &= \sum_{S \subseteq [n]} (-1)^{|S|+n-k} \alpha_P(S) \sum_{i=0}^{\min(k, \lfloor (\max S-1)/2 \rfloor)} \binom{n-\max S}{k-i} (j_{T,i}^{\max S-1} + j_{T,i-1}^{\max S-1}) \end{aligned}$$

Hence, to prove the theorem it is sufficient to show that, for $S \subseteq [n]$ and $T := S \setminus \{\max S\}$, we have the identity

$$(-1)^{|S|+n-k} j_{S,k} = \sum_{i=0}^{\min(k, \lfloor (\max S-1)/2 \rfloor)} (-1)^{|T|-1+n-k} \binom{n-\max S}{k-i} (j_{T,i}^{\max S-1} + j_{T,i-1}^{\max S-1}),$$

or equivalently

$$j_{S,k} = \sum_{i=0}^{\min(k, \lfloor (\max S-1)/2 \rfloor)} \binom{n-\max S}{k-i} (j_{T,i}^{\max S-1} + j_{T,i-1}^{\max S-1})$$

which comes from imposing that the coefficient of $x^k \alpha_P(S)$ are the same in both polynomials. This can be proved by a combinatorial argument. Any valid S -diagram with k negative terms may arise from a valid T -diagram of length $\max S - 1$ and i or $(i-1)$ negative terms for any $i \leq \min(k, \lfloor (\max S-1)/2 \rfloor)$ as follows. Start with the T -diagram and extend it in position $\max S$ with $+1$ if the valid T -diagram has $i-1$'s, and with -1 if the valid T -diagram has $(i-1)-1$'s. Fill the remaining $n - \max S$ positions of the diagram with $(k-i)-1$'s and $(n-k)+1$'s. Hence, the procedure constructs $\sum_{i=0}^{\min(k, \lfloor (\max S-1)/2 \rfloor)} \binom{n-\max S}{k-i} (j_{T,i}^{\max S-1} + j_{T,i-1}^{\max S-1})$ valid S -diagrams with k negative terms. Since the S -diagrams produced are all different, and every S -diagram with k negative terms may be produced in this way, we have proved the theorem. $\square$

The previous theorem may be converted in a formula which uses the flag h -vector of P .

Corollary 3.7. *Let P be an Eulerian poset of rank $n+1$. Given any sequence $\lambda \in \{\pm 1\}^n$, there is a unique maximal set $S(\lambda) \subseteq [n]$ for which λ is a valid $S(\lambda)$ -diagram. Let i_λ be the number of -1 's in λ . Given $S \subseteq [n]$ and $i \in [0, n]$, let $m_{S,i}$ be the number of valid S -diagrams λ for which $S(\lambda) = S$ and $i_\lambda = i$. The toric h -polynomial is*

$$\begin{aligned} h(P, x) &= \sum_{\lambda \in \{\pm 1\}^n} \left(\sum_{T \subseteq S(\lambda)} (-1)^{|T|+n-i_\lambda} \alpha_P(T) \right) x^{i_\lambda} \\ &= \sum_{\lambda \in \{\pm 1\}^n} (-1)^{|S(\lambda)|+n-i_\lambda} \beta_P(S(\lambda)) x^{i_\lambda} \\ &= \sum_{i=0}^n \left(\sum_{S \subseteq [n]} (-1)^{|S|+n-i} m_{S,i} \beta_P(S) \right) x^i. \end{aligned}$$

Example 3.8. Let P be the face lattice of a cube. Since P has rank 4, we consider the subsets $S \subseteq [3]$. All the valid S -diagrams are

$$\begin{aligned} (\emptyset) : & [+1, +1, +1], [-1, +1, +1], [+1, -1, +1], [+1, +1, -1], \\ & [-1, -1, +1], [-1, +1, -1], [+1, -1, -1], [-1, -1, -1], \\ (1) : & [+1, +1, +1], [+1, -1, +1], [+1, +1, -1], [+1, -1, -1], \\ (2) : & [+1, +1, +1], [+1, +1, -1], \\ (3) : & [+1, +1, +1], [-1, +1, +1], [+1, -1, +1], [+1, +1, -1], \\ (12) : & [+1, +1, +1], [+1, +1, -1], \\ (13) : & [+1, +1, +1], [+1, -1, +1], [+1, +1, -1], \\ (23) : & [+1, +1, +1], [+1, +1, -1], \\ (123) : & [+1, +1, +1], [+1, +1, -1]. \end{aligned}$$

Let $\alpha_S := \alpha_P(S)$. Fine's formula returns

$$\begin{aligned} h(P, x) &= (-\alpha_\emptyset + \alpha_1 + \alpha_2 + \alpha_3 - \alpha_{12} - \alpha_{13} - \alpha_{23} + \alpha_{123}) \\ &\quad + (3\alpha_\emptyset - 2\alpha_1 - \alpha_2 - 3\alpha_3 + \alpha_{12} + 2\alpha_{13} + \alpha_{23} - \alpha_{123})x \\ &\quad + (-3\alpha_\emptyset + \alpha_1)x^2 + \alpha_\emptyset x^3. \end{aligned}$$

Recall from Example 1.52 the values of the flag f -vector of P :

S	$\alpha_P(S)$
$\emptyset$	1
$\{1\}$	8
$\{2\}$	12
$\{3\}$	6
$\{1, 2\}$	24
$\{1, 3\}$	24
$\{2, 3\}$	24
$\{1, 2, 3\}$	48

Hence,

$$\begin{aligned} h(P, x) &= (-1 + 8 + 12 + 6 - 24 - 24 - 24 + 48) \\ &\quad + (3 - 16 - 12 - 18 + 24 + 48 + 24 - 48)x + (-3 + 8)x^2 + x^3 \\ &= 1 + 5x + 5x^2 + x^3. \end{aligned}$$

Thus, the toric h -vector of the face lattice of the cube is

$$(1, 5, 5, 1).$$

Example 3.9. Let $\mathcal{Q}$ be the triangular bypiramyd, which is the polar dual of the polytope $\Delta_2 \times [0, 1]$. The flag f -vector of P is given by the following table.

S	$\alpha_P(S)$
$\emptyset$	1
$\{1\}$	5
$\{2\}$	9
$\{3\}$	6
$\{1, 2\}$	18
$\{1, 3\}$	18
$\{2, 3\}$	18
$\{1, 2, 3\}$	36

Hence,

$$\begin{aligned} h(P, x) &= (-\alpha_\emptyset + \alpha_1 + \alpha_2 + \alpha_3 - \alpha_{12} - \alpha_{13} - \alpha_{23} + \alpha_{123}) \\ &\quad + (3\alpha_\emptyset - 2\alpha_1 - \alpha_2 - 3\alpha_3 + \alpha_{12} + 2\alpha_{13} + \alpha_{23} - \alpha_{123})x \\ &\quad + (-3\alpha_\emptyset + \alpha_1)x^2 + \alpha_\emptyset x^3 \\ &= (-1 + 5 + 9 + 6 - 18 - 18 - 18 + 36) \\ &\quad + (3 - 10 - 9 - 18 + 18 + 36 + 18 - 36)x + (-3 + 5)x^2 + x^3 \\ &= 1 + 2x + 2x^2 + x^3. \end{aligned}$$

This agrees with the algebro-geometric interpretation of the toric h -vector. In fact, the variety associated with $\mathcal{Q}$ is $X_{\mathcal{Q}} = \mathbb{P}^2 \times \mathbb{P}^1$. Since

$$H^i(\mathbb{P}^2) = \begin{cases} \mathbb{C} & i = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H^i(\mathbb{P}^1) = \begin{cases} \mathbb{C} & i = 0, 2 \\ 0 & \text{otherwise} \end{cases},$$

by applying Künneth's Theorem, we get

$$H^i(X_{\mathcal{Q}}) = \begin{cases} \mathbb{C}^2 & i = 2, 4 \\ \mathbb{C} & i = 0, 6 \\ 0 & \text{otherwise} \end{cases}$$

which confirms that the toric h -vector of P is

$$(1, 2, 2, 1).$$

§3.1.2 Toric h -Polynomial from the cd -Index

Another strategy to calculate the toric h -vector is the one discovered by Margaret Bayer and Richard Ehrenborg in [4]. It is in fact possible to extract the necessary information to calculate the vector directly from the ab -index, by using its property of being a morphism of Newtonian coalgebras. They first construct a map from $\mathbb{Q}\langle a, b \rangle$ to $\mathbb{Q}[x]$ that returns the toric h -vector whenever evaluated at the ab -index, and then they describe it with an explicit formula. We now discuss their proof.

Define a map of algebras

$$\begin{aligned} \kappa : \mathbb{Q}\langle a, b \rangle &\rightarrow \mathbb{Q}[x] \\ a &\mapsto x - 1, \\ b &\mapsto 0. \end{aligned}$$

This map is defined on all non-commutative polynomials in $\mathbb{Q}\langle a, b \rangle$, and when it is evaluated on the ab -index of a poset P , it returns

$$\kappa(\Psi_P(a, b)) = (x - 1)^{\text{rank}(P)-1}.$$

In fact, the ab -index of a graded poset P is a homogeneous polynomial $\Psi_P(a, b)$ of degree $\text{rank}(P) - 1$ whose only term not contained in the kernel of κ is $a^{\text{rank}(P)-1}$ which has coefficient 1 (see Remark 2.27).

We also define a pair of linear maps $h, g : \mathbb{Q}\langle a, b \rangle \rightarrow \mathbb{Q}[x]$ using the following recursion. Denote by $U_{\leq m}$, $U_{> m}$ and $U_{=m}$ the linear maps on $\mathbb{Q}[x]$ such that, given any $p(x) = \sum_{i \geq 0} a_i x^i$, they output

$$U_{\leq m}[p(x)] := \sum_{i=0}^m a_i x^i, \quad U_{> m}[p(x)] := \sum_{i>m} a_i x^i, \quad U_{=m}[p(x)] = a_m x^m.$$

For any monomial $\mathbf{v} \in \mathbb{Q}\langle a, b \rangle$ of degree n and $m := \lfloor n/2 \rfloor$, using Sweedler notation we define

$$\begin{aligned} h(\mathbf{v}) &:= \kappa(\mathbf{v}) + \sum_{\mathbf{v}} g(\mathbf{v}_{(1)}) \kappa(\mathbf{v}_{(2)}), \\ g(\mathbf{v}) &:= U_{\leq m}[(1 - x)h(\mathbf{v})]. \end{aligned}$$

Note that $h(1) = 1$ and $g(1) = 1$. As anticipated, this map actually calculates the toric h -vector.

Proposition 3.10. *Given a graded bounded poset P , we have*

$$\begin{aligned} h(P, x) &= h(\Psi(P)), \\ g(P, x) &= g(\Psi(P)). \end{aligned}$$

Proof. We prove it by induction on the rank of P . If $\text{rank}(P) = 1$, then $h(P, x) = g(P, x) = h(1) = g(1) = 1$. Let $\text{rank}(P) = n + 1$. Observe that

$$\begin{aligned} h(P, x) &= (x - 1)^n + \sum_{\hat{0} < y < \hat{1}} g([\hat{0}, y]) (x - 1)^{\text{rank}([y, \hat{1}]) - 1} \\ &= \kappa(\Psi(P)) + \sum_{\hat{0} < y < \hat{1}} g([\hat{0}, y]) \kappa(\Psi([y, \hat{1}])) \\ &= \kappa(\Psi(P)) + \sum_P g(\Psi(P_{(1)})) \kappa(\Psi(P_{(2)})) \end{aligned}$$

and since the ab -index is a Newtonian coalgebra morphism by Proposition 2.38, we verify the first identity

$$h(P, x) = \kappa(\Psi(P)) + \sum_{\Psi(P)} g((\Psi(P))_{(1)}) \kappa((\Psi(P))_{(2)}) = h(\Psi(P)).$$

We now verify the second identity

$$g(P, x) = U_{\leq m}[(1-x)h(P, x)] = U_{\leq m}[(1-x)h(\Psi(P))] = g(\Psi(P)). \quad \square$$

In order to find an explicit description of h and g , we now define some polynomials.

Definition 3.11. Set $p(n, k) := \binom{n}{k} - \binom{n}{k-1}$. Given a positive integer n ,

$$\begin{aligned} Q_n(x) &:= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k p(n-1, k) x^k, \\ R_n(x) &:= x^{n-1} Q_n(x^{-1}). \end{aligned}$$

For odd n , we also have

$$T_n(x) := (-1)^{(n-1)/2} p(n-1, (n-1)/2) x^{(n-1)/2}$$

while for even n we let $T_n(x) := 0$.

Remark 3.12. For even n , $p(n, n/2)$ is the $(n/2)$ -th Catalan number $C_{n/2}$. Hence,

$$T_{2k+1}(x) = C_k(-x)^k.$$

This implies that, given the ordinary generating function of the Catalan numbers

$$C(z) := \sum_{k \geq 0} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z},$$

the ordinary generating function of $\{T_k(x)\}_{k \in \mathbb{N}}$ is

$$T(z) := \sum_{k \geq 0} T_k(x) z^k = zC(-xz^2).$$

These polynomials satisfy some recursive properties that are particularly relevant for our purposes.

Proposition 3.13. For any positive integer n ,

$$\begin{aligned} Q_{n+1}(x) &= (1-x)Q_n(x) + xT_n(x), \\ R_{n+1}(x) &= (x-1)R_n(x) + T_n(x), \end{aligned}$$

where $T_0(x) := 0$.

Proof. Note that for any integers n and k we have

$$\begin{aligned} p(n+1, k) &= \binom{n+1}{k} - \binom{n+1}{k-1} \\ &= \binom{n}{k} + \binom{n}{k-1} - \binom{n}{k-1} - \binom{n}{k-2} \\ &= p(n, k) + p(n, k-1) \end{aligned} \tag{3.2}$$

and also that for odd n

$$\begin{aligned}
 p(n+1, \lfloor n/2 \rfloor + 1) &= p(n, \lfloor n/2 \rfloor + 1) + p(n, \lfloor n/2 \rfloor) \\
 &= \binom{n}{(n+1)/2} - \binom{n}{(n-1)/2} + \binom{n}{(n-1)/2} - \binom{n}{(n-3)/2} \\
 &= \binom{n}{(n+1)/2} - \binom{n}{(n-3)/2} \\
 &= \binom{n}{(n-1)/2} - \binom{n}{(n-3)/2} \\
 &= p(n, \lfloor n/2 \rfloor).
 \end{aligned} \tag{3.3}$$

For even n , we have

$$\begin{aligned}
 (1-x)Q_n(x) &= (1-x) \sum_{k=0}^{n/2-1} (-1)^k p(n-1, k) x^k \\
 &= \sum_{k=0}^{n/2-1} (-1)^k p(n-1, k) x^k + \sum_{k=0}^{n/2-1} (-1)^{k+1} p(n-1, k) x^{k+1} \\
 &= \sum_{k=0}^{n/2-1} (-1)^k p(n-1, k) x^k + \sum_{k=0}^{n/2} (-1)^k p(n-1, k-1) x^k \\
 &= \sum_{k=0}^{n/2-1} (-1)^k (p(n-1, k) + p(n-1, k-1)) x^k \\
 &\quad + (-1)^{n/2} p(n-1, \lfloor (n-1)/2 \rfloor) x^{n/2}
 \end{aligned}$$

which, by equation 3.2 and equation 3.3, is equal to

$$\begin{aligned}
 (1-x)Q_n(x) &= \sum_{k=0}^{n/2-1} (-1)^k p(n, k) x^k + (-1)^{n/2} p(n, n/2) x^{n/2} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k p(n, k) x^k \\
 &= Q_{n+1}(x).
 \end{aligned}$$

For odd n , we have

$$\begin{aligned}
 (1-x)Q_n(x) + xT_n(x) &= (1-x) \sum_{k=0}^{(n-1)/2} (-1)^k p(n-1, k) x^k \\
 &\quad + (-1)^{(n-1)/2} p(n-1, (n-1)/2) x^{(n+1)/2} \\
 &= \sum_{k=0}^{(n-1)/2} (-1)^k (p(n-1, k) + p(n-1, k-1)) x^k \\
 &\quad + (-1)^{(n-1)/2} p(n-1, (n-1)/2) x^{(n+1)/2} \\
 &\quad + (-1)^{(n+1)/2} p(n-1, (n-1)/2) x^{(n+1)/2} \\
 &= \sum_{k=0}^{(n-1)/2} (-1)^k (p(n-1, k) + p(n-1, k-1)) x^k
 \end{aligned}$$

which again by equation 3.2 is equal to

$$\begin{aligned} (1-x)Q_n(x) + xT_n(x) &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k p(n, k) x^k \\ &= Q_{n+1}(x). \end{aligned}$$

This proves our first identity. The second identity can be proved by expanding the definition of $R_n(x)$ and using the first identity

$$\begin{aligned} (x-1)R_n(x) + T_n(x) &= (x-1)x^{n-1}Q_n(x^{-1}) + T_n(x) \\ &= x^n \left(1 - \frac{1}{x}\right) Q_n(x^{-1}) + T_n(x) \\ &= x^n \left(Q_{n+1}(x^{-1}) - \frac{1}{x}T_n(x^{-1})\right) + T_n(x) \\ &= R_{n+1}(x) + T_n(x) - x^{n-1}T_n(x^{-1}). \end{aligned}$$

Note that for odd n we have,

$$\begin{aligned} x^{n-1}T_n(x^{-1}) &= x^{n-1}(-1)^{(n-1)/2} p(n-1, (n-1)/2) x^{-(n-1)/2} \\ &= (-1)^{(n-1)/2} p(n-1, (n-1)/2) x^{(n-1)/2} \\ &= T_n(x), \end{aligned}$$

and for even n

$$x^{n-1}T_n(x) = 0 = T_n(x).$$

Thus, we have proved the second identity. $\square$

Remark 3.14. Let $Q(z)$ and $R(z)$ be the ordinary generating functions respectively of $\{Q_k(x)\}_{k \geq 1}$ and $\{R_k(x)\}_{k \geq 1}$. By Proposition 3.13, we have

$$Q(z) - z = (z - xz)Q(z) + xzT(z), \quad R(z) - z = (xz - z)R(z) + zT(z).$$

Hence

$$Q(z) = \frac{z + xz^2C(-xz^2)}{1 - z + xz}, \quad R(z) = \frac{z + z^2C(-xz^2)}{1 + z - xz}.$$

Corollary 3.15. Given a positive integer n and $m := \lfloor n/2 \rfloor$,

$$Q_{n+1}(x) = U_{\leq m}[(1-x)Q_n(x)].$$

Proof. Notice that $U_{\leq m}[xT_n(x)] = 0$ for any positive integer n : if n is even then $T_n(x) = 0$, if n is odd $xT_n(x)$ is a monomial of degree greater than m . Then, since $Q_{n+1}(x)$ has degree m , by Proposition 3.13

$$\begin{aligned} Q_{n+1}(x) &= U_{\leq m}[Q_{n+1}(x)] \\ &= U_{\leq m}[(1-x)Q_n(x) + xT_n(x)] \\ &= U_{\leq m}[(1-x)Q_n(x)]. \end{aligned} \quad \square$$

We can now analyse the function $h : \mathbb{Q}\langle a, b \rangle \rightarrow \mathbb{Q}[x]$.

Lemma 3.16. *For every ab-monomial $\mathbf{v}$,*

$$\begin{aligned} h(\mathbf{v}a) &= (x-1)h(\mathbf{v}) + g(\mathbf{v}), \\ h(\mathbf{v}b) &= g(\mathbf{v}). \end{aligned}$$

Proof. By the Newtonian condition we have $\Delta(\mathbf{v}a) = \sum_{\mathbf{v}} \mathbf{v}_{(1)} \mathbf{v}_{(2)} a + \mathbf{v}$, so by expanding the definition of h we get

$$\begin{aligned} h(\mathbf{v}a) &= \kappa(\mathbf{v}a) + \sum_{\mathbf{v}} g(\mathbf{v}_{(1)}) \kappa(\mathbf{v}_{(2)}a) + g(\mathbf{v}) \kappa(1) \\ &= \kappa(\mathbf{v})(x-1) + \sum_{\mathbf{v}} g(\mathbf{v}_{(1)}) \kappa(\mathbf{v}_{(2)})(x-1) + g(\mathbf{v}) \\ &= h(\mathbf{v})(x-1) + g(\mathbf{v}). \end{aligned}$$

The second identity can be proved in the same way, using $\kappa(\mathbf{v}b) = 0$. $\square$

Notice that, for any ab-monomial $\mathbf{v}$ of degree $n \geq 1$ and $m := \lfloor (n+1)/2 \rfloor$,

$$g(\mathbf{v}b) = U_{\leq m}[(1-x)h(\mathbf{v}b)] = U_{\leq m}[(1-x)g(\mathbf{v})]. \quad (3.4)$$

In particular, this remind us the recursion formula for $Q_n(x)$ in Corollary 3.15. Since $g(1) = 1$, we have

$$g(b^n) = Q_{n+1}(x). \quad (3.5)$$

By the previous lemma, we also calculate

$$h(b^n) = Q_n(x). \quad (3.6)$$

Lemma 3.17. *For every ab-monomial $\mathbf{v}$ of degree n and $m := \lfloor n/2 \rfloor$,*

$$g(\mathbf{v}a) = \begin{cases} U_{=m+1}[(x-1)h(\mathbf{v})] & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Proof. By Lemma 3.16, we have

$$\begin{aligned} h(\mathbf{v}a) &= (x-1)h(\mathbf{v}) + g(\mathbf{v}) \\ &= (x-1)h(\mathbf{v}) + U_{\leq m}[(1-x)h(\mathbf{v})] \\ &= U_{>m}[(x-1)h(\mathbf{v})]. \end{aligned}$$

Hence

$$\begin{aligned} g(\mathbf{v}a) &= U_{\leq \lfloor (n+1)/2 \rfloor}[(1-x)h(\mathbf{v}a)] \\ &= U_{\leq \lfloor (n+1)/2 \rfloor}[(1-x)U_{>m}[(x-1)h(\mathbf{v})]]. \end{aligned}$$

If n is even, then $g(\mathbf{v}a) = U_{\leq m}[(1-x)U_{>m}[(x-1)h(\mathbf{v})]] = 0$. If n is odd, then $g(\mathbf{v}a) = U_{\leq m+1}[(1-x)U_{>m}[(x-1)h(\mathbf{v})]] = U_{=m+1}[(x-1)h(\mathbf{v})]$. $\square$

Observe that

$$h(a^n) = (x-1)h(a^{n-1}) + g(a^{n-1}), \quad g(a^n) = \begin{cases} U_{=\lfloor (n-1)/2 \rfloor + 1}[(x-1)h(a^{n-1})] & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

These formulas may remind us of the definition of $T_n(x)$ and the recursion for $R_n(x)$ in Proposition 3.13. We may prove by induction on n that

$$h(a^n) = R_{n+1}(x), \quad g(a^n) = T_{n+1}(x). \quad (3.7)$$

If $n = 0$, the statement is trivial since $h(1) = g(1) = R_1(x) = T_1(x) = 1$. If $n > 0$,

$$h(a^n) = (x-1)h(a^{n-1}) + g(a^{n-1}) = (x-1)R_n(x) + T_n(x) = R_{n+1}(x)$$

and for even n , by recalling equation 3.3

$$\begin{aligned} g(a^n) &= U_{=n/2}[(x-1)h(a^{n-1})] \\ &= U_{=n/2}[(x-1)R_n(x)] \\ &= U_{=n/2} \left[(x-1) \sum_{k=0}^{n/2-1} (-1)^k p(n-1, k) x^{n-1-k} \right] \\ &= (-1)^{n/2} p(n-1, n/2-1) x^{n/2} \\ &= (-1)^{n/2} p(n, n/2) x^{n/2} \\ &= T_{n+1}(x), \end{aligned}$$

while for odd n , we have

$$g(a^n) = 0 = T_{n+1}(x).$$

Another instance we can calculate is the case of monomials of the form $b^n a^k$, for some positive integers n and k . In particular, we have

$$h(b^n a^k) = xT_n(x)R_k(x), \quad g(b^n a^k) = xT_n(x)T_k(x). \quad (3.8)$$

We can prove them by induction on k . If $k = 1$, then

$$\begin{aligned} h(b^n a) &= \kappa(b^n a) + \sum_{i=0}^{n-1} \kappa(b^i) g(b^{n-1-i} a) + \kappa(b^n) \\ &= g(b^{n-1} a) \\ &= xT_{n-1}(x)T_1(x) \\ &= xT_{n-1}(x)R_1(x), \\ g(b^n a) &= U_{\leq \lfloor (n+1)/2 \rfloor}[(1-x)f(b^n a)] \\ &= xT_{n-1}(x)T_1(x). \end{aligned}$$

Otherwise,

$$\begin{aligned} h(b^n a^k) &= (x-1)h(b^n a^{k-1}) + g(b^n a^{k-1}) \\ &= (x-1)xT_n(x)R_{k-1}(x) + xT_n(x)T_{k-1}(x) \\ &= xT_n(x)((x-1)R_{k-1}(x) + T_{k-1}(x)) \\ &= xT_n(x)R_k(x). \end{aligned}$$

Set $s := n + k$. If s is odd, then at least one between k and n is also even and

$$g(b^n a^k) = 0 = xT_n(x)T_k(x),$$

while for even s

$$\begin{aligned} g(b^n a^k) &= U_{=s/2}[(x-1)h(b^n a^{k-1})] \\ &= U_{=s/2}[(x-1)xT_n(x)R_{k-1}(x)]. \end{aligned}$$

By assuming n to be odd (otherwise, it is trivially equal to 0), we have that also k must be odd. Hence, $T_{k-1}(x) = 0$ and $(x-1)R_{k-1}(x) = R_k(x)$. Therefore,

$$\begin{aligned} g(b^n a^k) &= U_{=s/2}[(x-1)xT_n(x)R_{k-1}(x)] \\ &= U_{=s/2}[xT_n(x)R_k(x)] \\ &= xT_n(x)U_{=(k-1)/2}[R_k(x)] \\ &= xT_n(x)T_k(x), \end{aligned}$$

where the last equality follows from the definitions of R_k and T_k .

To complete our understanding of h and g it is sufficient to study their behaviour when they are evaluated on an ab -monomial containing the factor ab .

Proposition 3.18. *For any two ab -monomials $\mathbf{u}$ and $\mathbf{v}$,*

$$\begin{aligned} h(\mathbf{u}ab\mathbf{v}) &= g(\mathbf{u}a)h(b\mathbf{v}), \\ g(\mathbf{u}ab\mathbf{v}) &= g(\mathbf{u}a)g(b\mathbf{v}). \end{aligned}$$

Proof. Let k be the degree of $\mathbf{v}$, n the degree of $\mathbf{u}$ and $m := \lfloor n/2 \rfloor$. Let's prove the statement by induction on k . Let $k = 0$, then when n is even

$$g(\mathbf{u}ab) = U_{\leq m+1}[(1-x)g(\mathbf{u}a)] = 0 = g(\mathbf{u}a)h(b),$$

while when n is odd,

$$\begin{aligned} g(\mathbf{u}ab) &= U_{\leq m+1}[(1-x)g(\mathbf{u}a)] \\ &= U_{\leq m+1}[(1-x)U_{=m+1}[(x-1)h(\mathbf{u})]] \\ &= U_{=m+1}[(x-1)h(\mathbf{u})] \\ &= g(\mathbf{u}a). \end{aligned}$$

Since $g(b) = h(b) = 1$, both identities follow

$$g(\mathbf{u}ab) = g(\mathbf{u}a) = g(\mathbf{u}a)h(b), \quad h(\mathbf{u}ab) = g(\mathbf{u}a) = g(\mathbf{u}a)h(b).$$

Let $k > 0$. We have

$$\Delta(\mathbf{u}ab\mathbf{v}) = \sum_{\mathbf{u}} \mathbf{u}_{(1)}\mathbf{u}_{(2)}ab\mathbf{v} + \mathbf{u}b\mathbf{v} + \mathbf{u}a\mathbf{v} + \sum_{\mathbf{v}} \mathbf{u}ab\mathbf{v}_{(1)}\mathbf{v}_{(2)}.$$

Hence, using $\kappa(b) = 0$ and the induction hypothesis,

$$\begin{aligned} h(\mathbf{u}ab\mathbf{v}) &= \kappa(\mathbf{u}ab\mathbf{v}) + \sum_{\mathbf{u}} g(\mathbf{u}_{(1)})\kappa(\mathbf{u}_{(2)}ab\mathbf{v}) \\ &\quad + g(\mathbf{u})\kappa(b\mathbf{v}) + g(\mathbf{u}a)\kappa(\mathbf{v}) + \sum_{\mathbf{v}} g(\mathbf{u}ab\mathbf{v}_{(1)})\kappa(\mathbf{v}_{(2)}) \\ &= g(\mathbf{u}a) \left(\kappa(\mathbf{v}) + \sum_{\mathbf{v}} g(b\mathbf{v}_{(1)})\kappa(\mathbf{v}_{(2)}) \right) \\ &= g(\mathbf{u}a) \left(\kappa(b\mathbf{v}) + g(1)\kappa(\mathbf{v}) + \sum_{\mathbf{v}} g(b\mathbf{v}_{(1)})\kappa(\mathbf{v}_{(2)}) \right) \\ &= g(\mathbf{u}a)h(b\mathbf{v}). \end{aligned}$$

This proves the first identity. Consider now the case when n is even. Then

$$\begin{aligned} g(\mathbf{uabv}) &= U_{\leq \lfloor (n+k+2)/2 \rfloor} [(1-x)h(\mathbf{uabv})] \\ &= U_{\leq \lfloor (n+k+2)/2 \rfloor} [(1-x)g(\mathbf{ua})h(\mathbf{bv})] \\ &= 0 = g(\mathbf{ua})g(\mathbf{bv}). \end{aligned}$$

Let $m := \lfloor n/2 \rfloor$. When n is odd,

$$\begin{aligned} g(\mathbf{uabv}) &= U_{\leq \lfloor (n+k+2)/2 \rfloor} [(1-x)g(\mathbf{ua})h(\mathbf{bv})] \\ &= U_{\leq \lfloor (n+k+2)/2 \rfloor} [U_{=(n+1)/2} [(x-1)h(\mathbf{u})](1-x)h(\mathbf{bv})] \\ &= U_{=(n+1)/2} [(x-1)h(\mathbf{u})] U_{\leq \lfloor (k+1)/2 \rfloor} [(1-x)h(\mathbf{bv})] \\ &= g(\mathbf{ua})g(\mathbf{bv}). \end{aligned}$$

This completes our proof. $\square$

One may also calculate the following results regarding the evaluations of these functions on cd -monomials. We skip the proofs as the reasoning is similar to the one that we have encountered above. The details are found in the original paper of Bayer and Ehrenborg.

Proposition 3.19. *Let $n \in \mathbb{N}$ be a natural number. Then*

$$g(c^n) = Q_{n+1}(x), \quad g(d^n) = x^n, \quad g(c^n d) = xT_{n+1}(x), \quad h(c^n) = xR_n(x) + Q_n(x).$$

Furthermore, for any cd -monomials $\mathbf{u}$ and $\mathbf{v}$, we have

$$g(\mathbf{udv}) = g(\mathbf{ud})g(\mathbf{v}), \quad h(\mathbf{udv}) = g(\mathbf{ud})h(\mathbf{v}).$$

Now, we may finally deduce a formula for the toric h -vector that uses only data from the ab -index. Let $S \subseteq [n]$ and consider the partition of $[n]$ given by maximal strings of S and $\bar{S}$, with blocks of the partition ordered by least element. Write the sequence of sizes of the blocks, but add 1 to the leftmost term if $1 \notin S$ and call the resulting sequence $\sigma(S) = \sigma_1, \sigma_2, \dots, \sigma_r$.

Using Proposition 3.18 and equations 3.5, 3.7, 3.6, and 3.8, we can prove the following theorem.

Theorem 3.20. *Let P be a graded poset with $\hat{0}$ and $\hat{1}$ of rank $n+1$ with ab -index $\Psi_P(a, b) = \sum_S h_S \mathbf{u}_S$. The toric h -vector is given by $h(P, x) = \sum_S h_S h(\mathbf{u}_S, x)$, where $h(\mathbf{u}_S, x)$ equals*

1.

$$x^{\lfloor (r-1)/2 \rfloor} Q_{\sigma_r}(x) \prod_{t=1}^{r-1} T_{\sigma_t}(x) \text{ if } n \in S,$$

2.

$$x^{\lfloor r/2 \rfloor} R_{\sigma_r}(x) \prod_{t=1}^{r-1} T_{\sigma_t}(x) \text{ if } n \notin S.$$

Proof. For brevity, we discuss only the case where $n \in S$: if $n \notin S$, we can apply a similar argument.

The ab -monomial $\mathbf{u}_S$ has the following form:

1. if $1 \in S$, then r is odd and $\mathbf{u}_S = b^{\sigma_1} a^{\sigma_2} b^{\sigma_3} \dots a^{\sigma_{r-1}} b^{\sigma_r}$;

2. if $1 \notin S$, then r is even and $\mathbf{u}_S = a^{\sigma_1-1}b^{\sigma_2}a^{\sigma_3} \dots a^{\sigma_{r-1}}b^{\sigma_r}$.

Let us prove the first case by induction on the length of the word $\mathbf{u}_S$. For $1 \in S$, we have already discussed the case where r is 1, so we can assume r is odd and $r \geq 3$. By applying Proposition 3.18 and equation 3.8,

$$\begin{aligned} h(\mathbf{u}_S) &= g(b^{\sigma_1}a^{\sigma_2})h(b^{\sigma_3}a^{\sigma_4} \dots a^{\sigma_{r-1}}b^{\sigma_r}) \\ &= xT_{\sigma_1}(x)T_{\sigma_2}(x)x^{\lfloor (r-3)/2 \rfloor}Q_{\sigma_r}(x) \prod_{t=3}^{r-1} T_{\sigma_t}(x) \\ &= x^{\lfloor (r-1)/2 \rfloor}Q_{\sigma_r}(x) \prod_{t=1}^{r-1} T_{\sigma_t}(x). \end{aligned}$$

For $1 \notin S$, given any $r \geq 2$, we can apply Proposition 3.18 and equation 3.7,

$$\begin{aligned} h(\mathbf{u}_S) &= g(a^{\sigma_1-1})h(b^{\sigma_2}a^{\sigma_3} \dots a^{\sigma_{r-1}}b^{\sigma_r}) \\ &= T_{\sigma_1}(x)x^{\lfloor (r-2)/2 \rfloor}Q_{\sigma_r}(x) \prod_{t=2}^{r-1} T_{\sigma_t}(x) \\ &= x^{\lfloor (r-1)/2 \rfloor}Q_{\sigma_r}(x) \prod_{t=1}^{r-1} T_{\sigma_t}(x). \end{aligned}$$

Thus, we have proved the identities. $\square$

In a similar way to the theorem above, we can prove the following theorem using Proposition 3.19.

Theorem 3.21. *Let P be an Eulerian poset of rank $n+1$ with cd-index $\Phi_P(c, d) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} \mathbf{w}$. The toric h -polynomial of P is given by $h(P, x) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} h(\mathbf{w}, x)$ where*

$$h(c^{k_1}dc^{k_2}d \dots dc^{k_r}dc^k, x) = x^r(xR_k(x) + Q_k(x)) \prod_{j=1}^r T_{k_j+1}(x)$$

where we let $xR_0(x) + Q_0(x) := 1$.

Similarly, writing the toric g -polynomial as $g(P, x) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} g(\mathbf{w}, x)$,

$$g(c^{k_1}dc^{k_2}d \dots dc^{k_r}dc^k, x) = x^r Q_{k+1}(x) \prod_{j=1}^r T_{k_j+1}(x).$$

Example 3.22. Let P be the butterfly poset of rank $n+1$. By Example 2.29, we know that its cd-index is c^n , hence

$$h_P(x) = xR_n(x) + Q_n(x), \quad g_P(x) = Q_{n+1}(x).$$

Example 3.23. Let P be the face lattice of a cube. By Example 2.28, we know that its cd-index is $c^3 + 4cd + 6dc$. This implies that

$$\begin{aligned} h_P(x) &= (xR_3(x) + Q_3(x)) + 4xT_2(x) + 6x(xR_1(x) + Q_1(x)) \\ &= (x^3 - x^2 - x + 1) + 0 + 6x(x + 1) \\ &= x^3 + 5x^2 + 5x + 1 \end{aligned}$$

and

$$\begin{aligned} g_P(x) &= Q_4(x) + 4xT_2(x) + 6xQ_2(x) \\ &= (-2x + 1) + 0 + 6x \\ &= 4x + 1. \end{aligned}$$

One important result of Margaret Bayer and Richard Ehrenborg is that we can produce an explicit formula for the toric h -vector based purely on the flag h -vector. This formula is equivalent to Theorem 3.20.

Given a set $S \subseteq [n]$, we say that S is *odd* if and only if in the corresponding sequence $\sigma(S) = \sigma_1, \sigma_2, \dots, \sigma_r$ all terms except σ_r are odd, and σ_r has the same parity as n if $n \in S$, and the opposite parity if $n \notin S$.

Theorem 3.24. *Let P be a graded poset with $\hat{0}$ and $\hat{1}$ of rank $n + 1$. Then*

1. *for $0 \leq i < n/2$, $h_i(P)$ equals*

$$\sum_{\substack{\text{odd } S, n \in S \\ \sigma_r \geq n-2i}} (-1)^{|S|+n-i} p\left(\sigma_r - 1, \frac{\sigma_r - n + 2i}{2}\right) \prod_{t=1}^{r-1} p\left(\sigma_t - 1, \frac{\sigma_t - 1}{2}\right) \beta_P(S),$$

2. *for $n/2 \leq i \leq n$, $h_i(P)$ equals*

$$\sum_{\substack{\text{odd } S, n \notin S \\ \sigma_r \geq 2i+1-n}} (-1)^{|S|+n-i} p\left(\sigma_r - 1, \frac{\sigma_r + n - 1 - 2i}{2}\right) \prod_{t=1}^{r-1} p\left(\sigma_t - 1, \frac{\sigma_t - 1}{2}\right) \beta_P(S).$$

Proof. Let's prove that this theorem is equivalent to Theorem 3.20. Set S be an odd set. We discuss only the case where $n \in S$, since we only require minor differences to prove the case when $n \notin S$.

Set $\alpha(n, S, i) := |S| + n - i - (n - \sigma_r)/2 + \lfloor (r-1)/2 \rfloor$. Consider the coefficient of $\beta_S(P)$ in the toric h -vector formula given in the statement of the theorem. It is equal to

$$\begin{aligned} & \sum_{i=(n-\sigma_r)/2}^{\lfloor (n-1)/2 \rfloor} (-1)^{|S|+n-i} p\left(\sigma_r - 1, \frac{\sigma_r - n + 2i}{2}\right) \prod_{t=1}^{r-1} p\left(\sigma_t - 1, \frac{\sigma_t - 1}{2}\right) x^i \\ &= \left[\prod_{t=1}^{r-1} (-1)^{\frac{\sigma_t-1}{2}} p\left(\sigma_t - 1, \frac{\sigma_t - 1}{2}\right) x^{\frac{\sigma_t-1}{2}} \right] x^{\lfloor \frac{r-1}{2} \rfloor} \\ & \quad \sum_{i=(n-\sigma_r)/2}^{\lfloor (n-1)/2 \rfloor} (-1)^{\alpha(n, S, i)} p\left(\sigma_r - 1, \frac{\sigma_r - n + 2i}{2}\right) x^{i - \frac{n-\sigma_r}{2}} \\ &= \left[\prod_{t=1}^{r-1} (-1)^{\frac{\sigma_t-1}{2}} p\left(\sigma_t - 1, \frac{\sigma_t - 1}{2}\right) x^{\frac{\sigma_t-1}{2}} \right] x^{\lfloor \frac{r-1}{2} \rfloor} \\ & \quad \sum_{k=0}^{\lfloor (\sigma_r-1)/2 \rfloor} (-1)^{|S|+n+\lfloor (r-1)/2 \rfloor+k} p(\sigma_r - 1, k) x^k. \end{aligned}$$

Since $|S| + n + \lfloor (r-1)/2 \rfloor$ is even, this is

$$\begin{aligned} & \left[\prod_{t=1}^{r-1} (-1)^{\frac{\sigma_t-1}{2}} p\left(\sigma_t - 1, \frac{\sigma_t - 1}{2}\right) x^{\frac{\sigma_t-1}{2}} \right] x^{\lfloor \frac{r-1}{2} \rfloor} \\ & \quad \cdot \sum_{k=0}^{\lfloor (\sigma_r-1)/2 \rfloor} (-1)^k p(\sigma_r - 1, k) x^k \\ &= x^{\lfloor (r-1)/2 \rfloor} Q_{\sigma_r}(x) \prod_{t=1}^{r-1} T_{\sigma_t}(x) \end{aligned}$$

which is the term $h(\mathbf{u}_S, x)$ given in Theorem 3.20. □

4 KLS Theory of (Eulerian) Posets

The Kazhdan–Lusztig–Stanley (KLS) Theory of posets is a unifying framework for studying some classes of polynomial invariants of posets that was introduced by Richard Stanley in his seminal paper of 1992 [51]. The aim of his work was to generalize within a single framework the theory of formal subdivisions of lower Eulerian posets, toric g -polynomials of Eulerian posets and Kazhdan–Lusztig polynomials of Coxeter systems. The theory was then developed in subsequent works by Francesco Brenti [18, 17], which further generalized it from locally graded posets to any weakly ranked poset. A comprehensive survey of the state of the art was later presented by Nicholas Proudfoot in [41]. One difference from the previous expositions is the introduction of a distinction between left KLS functions and right KLS functions. He also introduced the notion of Z -function, as a generalization of the Z -polynomial of a matroid [42]. A recent contribution to this theory was then given by Luis Ferroni, Jacob P. Matherne and Lorenzo Vecchi in a 2024 paper [27]. In their work, they developed the theory of Chow functions inside the framework of KLS theory, with the purpose of generalizing the Hilbert–Poincaré series of the Chow ring of a loopless matroid.

In Section 4.1, we present the basic definitions of KLS Theory by defining kernels and constructing the associated left and right KLS functions. Then, we define the Z -function, the Chow function and the left and right augmented Chow function associated to a kernel. In Section 4.2, we define the Eulerian kernel in the case of Eulerian posets, and then provide geometric and combinatorial interpretations to the KLS invariants that we have already presented. For the specific case of the augmented Chow function, we present joint work conducted with Luis Ferroni and Tommaso Faustini [25] in which we define a new operation of posets called the stellahedral transform, and then provide an interpretation to augmented Chow functions in terms of this operation. In the last part of the chapter, we present original results from another work [25] in which we calculate the Eulerian Z -polynomial and the Eulerian augmented Chow polynomials using data from the cd -index.

§4.1 KLS Theory of Weakly Ranked Posets

Let us start the chapter by introducing some definitions. Given a locally finite poset P endowed with a weak rank function ρ , we define the subalgebra $\mathcal{I}_\rho(P) \subseteq \mathcal{I}(P)$ as

$$\mathcal{I}_\rho(P) := \{a \in \mathcal{I}(P) \mid \deg a_{st}(x) \leq \rho_{st} \text{ for all } s \leq t \text{ in } P\}.$$

This subalgebra admits an involution

$$\begin{aligned} \bullet^{\text{rev}} : \mathcal{I}_\rho(P) &\rightarrow \mathcal{I}_\rho(P) \\ a &\mapsto a^{\text{rev}} \text{ such that } (a^{\text{rev}})_{st}(x) = x^{\rho_{st}} a_{st}(x^{-1}). \end{aligned}$$

Remark 4.1. The *rev* operation satisfies the following properties. Given $a, b \in \mathcal{I}_\rho(P)$, it holds

$$(ab)^{\text{rev}} = a^{\text{rev}} b^{\text{rev}}.$$

Moreover, if a is also invertible, then $a^{-1} \in \mathcal{I}_\rho(P)$ and

$$(a^{-1})^{\text{rev}} = (a^{\text{rev}})^{-1}.$$

Definition 4.2. Let (P, ρ) be a weakly ranked poset. An element $\kappa \in \mathcal{I}_\rho(P)$ is a (P, ρ) -kernel if

$$\forall s \in P : \kappa_{ss} = 1, \quad \kappa^{-1} = \kappa^{\text{rev}}.$$

We say that κ is *non-degenerate* if $\deg \kappa_{st} = \rho_{st}$ for every $s \leq t$ in P .

An important example of a (P, ρ) -kernel is the following.

Example 4.3. Let P be a locally finite poset weakly graded by ρ . Notice that the Möbius function μ is a member of the algebra $\mathcal{I}_\rho(P)$. We define the *characteristic function* of P as

$$\chi = \mu \zeta^{\text{rev}} = \zeta^{-1} \zeta^{\text{rev}}.$$

By Remark 4.1, we know that χ is a (P, ρ) -kernel

$$\chi^{\text{rev}} = (\zeta^{-1} \zeta^{\text{rev}})^{\text{rev}} = (\zeta^{-1})^{\text{rev}} \zeta = (\zeta^{-1} \zeta^{\text{rev}})^{-1} = \chi^{-1}.$$

The argument in the example above applies to any other function $\kappa \in \mathcal{I}_\rho(P)$ of the form $\kappa = a^{-1} a^{\text{rev}}$, where $a \in \mathcal{I}_\rho(P)$ is an invertible element. In particular, all (P, ρ) -kernels arise in this way.

Theorem 4.4. Let P be a locally finite poset with a weak rank function ρ . Given $\kappa \in \mathcal{I}(P)$ such that $\kappa_{ss}(x) = 1$ for all $s \in P$, the followings are equivalent

1. κ is a (P, ρ) -kernel;
2. there exists an invertible element $a \in \mathcal{I}_\rho(P)$ such that $\kappa = a^{-1} a^{\text{rev}}$.

Proof. We only need to prove the implication (1) $\Rightarrow$ (2). Given $s \leq t$ in P , we define

$$a_{st}(x) := \sum_{i \geq 0} \binom{-1/2}{i} (\kappa - \delta)_{st}^i(x).$$

Since $(\kappa - \delta)_{ss}(x) = 0$ for all $s \in P$, it is a finite sum, hence it is well-defined. Notice that

$$a_{st}^{\text{rev}}(x) = \sum_{i \geq 0} \binom{-1/2}{i} (\kappa^{\text{rev}} - \delta^{\text{rev}})_{st}^i(x) = \sum_{i \geq 0} \binom{-1/2}{i} (\kappa^{-1} - \delta)_{st}^i(x).$$

By the binomial theorem we have $a = \kappa^{-1/2}$, or explicitly $a^2 = \kappa^{-1}$. Similarly, it holds $a^{\text{rev}} = \kappa^{1/2} = a^{-1}$. Using Remark 4.1 we conclude

$$\kappa = a^{-2} = a^{-1} a^{\text{rev}}. \quad \square$$

We now prove a lemma which is needed in order to construct the KLS functions. It provides us with a set of equations that must be satisfied by some classes of functions in the incidence algebra. These equations are (as we will see later) a generalization of the Dehn–Sommerville equations.

Lemma 4.5. Let (P, ρ) be a finite weakly graded poset with $\hat{1}$ and $\kappa \in \mathcal{I}_\rho(P)$ a (P, ρ) -kernel. Fix an element $u \in P$. Given a function $f \in \mathcal{I}_\rho(P)$ such that for all $s \in [u, \hat{1})$ satisfies

$$(f\kappa)_{us}(x) = f_{us}^{\text{rev}}(x)$$

then the polynomial $g(x) := \sum_{u \leq s < \hat{1}} f_{us}(x) \kappa_{s\hat{1}}(x)$ is antisymmetric, i.e. it satisfies

$$-g(x) = \sum_{u \leq s < \hat{1}} f_{us}^{\text{rev}}(x) \kappa_{s\hat{1}}^{\text{rev}}(x).$$

Proof. We have

$$\begin{aligned} \sum_{u \leq s < \hat{1}} f_{us}^{\text{rev}}(x) \kappa_{s\hat{1}}^{\text{rev}}(x) &= \sum_{u \leq s < \hat{1}} (f\kappa)_{us}(x) \kappa_{s\hat{1}}^{\text{rev}}(x) \\ &= \sum_{u \leq v \leq s < \hat{1}} f_{uv}(x) \kappa_{vs}(x) \kappa_{s\hat{1}}^{-1}(x) \\ &= \sum_{u \leq v < \hat{1}} f_{uv}(x) \sum_{v \leq s < \hat{1}} \kappa_{vs}(x) \kappa_{s\hat{1}}^{-1}(x) \\ &= \sum_{u \leq v < \hat{1}} f_{uv}(x) \left(-\kappa_{v\hat{1}}(x) \kappa_{\hat{1}\hat{1}}^{-1}(x) \right) \\ &= - \sum_{u \leq v < \hat{1}} f_{uv}(x) \kappa_{v\hat{1}}(x) = -g(x). \end{aligned}$$

□

Remark 4.6. Lemma 4.5 is a generalization of the Dehn–Sommerville equations. Let P be an Eulerian poset of rank n such that $P \setminus \{\hat{1}\}$ is simplicial, and let ρ be the usual rank function on P . As we will see in Proposition 4.29, the function $\varepsilon \in \mathcal{I}_\rho(P)$ such that $\varepsilon_{st}(x) := (x-1)^{\rho(t)-\rho(s)}$ for $s \leq t$ is a (P, ρ) -kernel when P is Eulerian. Let $f = \zeta \in \mathcal{I}_\rho(P)$ and $u = \hat{0}$, then for all $s \neq \hat{1}$ in P we have

$$(\zeta\varepsilon)_{\hat{0}s}(x) = \sum_{\hat{0} \leq t \leq s} (x-1)^{\rho(s)-\rho(t)}$$

and since $[\hat{0}, s]$ is a boolean poset

$$\begin{aligned} (\zeta\varepsilon)_{\hat{0}s}(x) &= \sum_{\hat{0} \leq t \leq s} (x-1)^{\rho(s)-\rho(t)} \\ &= \sum_{k=0}^{\rho(s)} \binom{\rho(s)}{k} (x-1)^k \\ &= x^{\rho(s)} = \zeta_{\hat{0}s}^{\text{rev}}(x). \end{aligned}$$

Then, the polynomial $g(x)$ is

$$g(x) = \sum_{\hat{0} \leq t < \hat{1}} (x-1)^{n-\rho(t)} = \sum_{i=0}^{n-1} \alpha_P(\{i\}) \cdot (x-1)^{n-i}.$$

Given $h_P(x)$ the h -polynomial of P , we notice that

$$g(x) = (x-1)h_P(x).$$

Since we also have

$$\sum_{u \leq s < \hat{1}} f_{us}^{\text{rev}}(x) \kappa_{s\hat{1}}^{\text{rev}}(x) = -(x-1)x^{n-1}h_P(x^{-1}),$$

Lemma 4.5 translates as

$$-(x-1)h_P(x) = -(x-1)x^{n-1}h_P(x^{-1}),$$

hence

$$h_P(x) = x^{n-1}h_P(x^{-1}),$$

which is equivalent to Theorem 2.16, one of the possible generalizations of the Dehn–Sommerville Equations.

Proposition 4.7. *Let P be a locally finite poset with a weak rank function ρ . Given a kernel $\kappa \in \mathcal{I}_\rho(P)$, let $\mathcal{F}_{st}^\kappa(x)$ denote the space of all the functions $f \in \mathcal{I}_\rho([s, t])$ such that $f\kappa|_{[s, t]} = f^{\text{rev}}$. Then*

$$\dim \mathcal{F}_{st}^\kappa(x) = \sum_{s \leq u \leq v \leq t} \left\lfloor \frac{\rho_{uv}}{2} \right\rfloor + 1.$$

Proof. We prove the statement by induction on ρ_{st} .

If $s = t$, the statement is trivial.

Let $t \in P$ and assume that the statement holds for every $u \leq v$ such that $v < t$. Without loss of generality, we may also assume $P = [s, t]$. Given any $u < t$, we have

$$\sum_{u \leq v < t} f_{uv}(x)\kappa_{vt}(x) = f_{ut}^{\text{rev}}(x) - f_{ut}(x). \quad (4.1)$$

By Lemma 4.5, the left-hand side is antisymmetric. This implies that the coefficients of x^i in f_{ut} for $0 \leq i \leq \lfloor \frac{\rho_{ut}}{2} \rfloor$ determine uniquely a polynomial f_{ut} satisfying equation 4.1. Hence, we have $\lfloor \frac{\rho_{ut}}{2} \rfloor + 1$ degrees of freedom in choosing f_{ut} . We conclude the proof by using the induction hypothesis. $\square$

We may now finally define the left and right KLS functions associated to a kernel.

Theorem 4.8. *Let $\kappa \in \mathcal{I}_\rho(P)$ be a (P, ρ) -kernel. There exists a unique element $f \in \mathcal{I}_\rho(P)$ satisfying the properties*

1. $f_{ss}(x) = 1$ for all $s \in P$;
2. $\deg f_{st}(x) < \frac{1}{2}\rho_{st}$ for all $s < t$;
3. $f^{\text{rev}} = \kappa f$.

Similarly, there exists a unique element $g \in \mathcal{I}_\rho(P)$ satisfying the properties

1. $g_{ss}(x) = 1$ for all $s \in P$;
2. $\deg g_{st}(x) < \frac{1}{2}\rho_{st}$ for all $s < t$;
3. $g^{\text{rev}} = g\kappa$.

Proof. We show only the first part of the statement, as the second part follows analogously. For every $s < t$ consider the polynomial $f_{st}(x) = a_0 + a_1x + \dots + a_mx^m$ where $m = \rho_{st}$. As we have seen in the proof of Proposition 4.7, the polynomial $f_{st}(x)$ is uniquely determined by the choice of $a_m, a_{m-1}, \dots, a_{\lceil m/2 \rceil}$, hence we must have $a_m = a_{m-1} = \dots = a_{\lceil m/2 \rceil} = 0$ to ensure $\deg f_{st}(x) \leq \lfloor \frac{1}{2}(m-1) \rfloor < \frac{1}{2}\rho_{st}$, so f exists and it is unique. $\square$

We call f [resp. g] the *right* [resp. *left*] *Kazhdan–Lusztig–Stanley (KLS) function* associated to κ .

If P is bounded, we call $f_P(x) := f_{\hat{0}\hat{1}}(x)$ [resp. $g_P(x) := g_{\hat{0}\hat{1}}(x)$] the *right* [resp. *left*] *Kazhdan–Lusztig–Stanley polynomial* of P .

Left and right KLS functions associated to a specific kernel have often very different behaviours, but their relation may be expressed in terms of duality.

Example 4.9. As observed at the start of the section, the characteristic function χ of a weakly ranked poset P is always a (P, ρ) -kernel. By definition, its left KLS function is ζ . Its right KLS function, instead, is a more subtle object which is much more difficult to characterize in the general case.

Remark 4.10. Let (P, ρ) be a weakly ranked poset. The dual poset P^* is also weakly graded by ρ^* where $\rho_{ts}^* = \rho_{st}$ for every $s \leq t$ in P . Given $\kappa \in \mathcal{I}_\rho(P)$ a (P, ρ) -kernel, define the (P^*, ρ^*) -kernel $\kappa^* \in \mathcal{I}_{\rho^*}(P^*)$ given by $\kappa_{ts}^* = \kappa_{st}$ for every $s \leq t$ in P .

If g^* is the left KLS function associated to κ^* and f is the right KLS function associated to κ , then

$$f = g^*.$$

Thus, one can go between the left and right KLS function of a poset by taking the dual.

One of the motivating examples for the introduction of KLS functions is the study of Coxeter groups, as we observe below. We show a pair of examples of kernels with their respective KLS functions.

Example 4.11. Let W be a Coxeter group and let S be its set of simple reflections. We endow W with the partial order given by the (strong) Bruhat order. Thus, W is known to be a locally finite graded poset whose rank function ρ is the length function. The R -polynomial of an interval in W may be defined from the function $R \in \mathcal{I}(W)$, recursively described as

$$R_{uv}(x) = \begin{cases} 1 & \text{if } u = v \\ R_{us,vs}(x) & \text{if } \rho(us) < \rho(u) \text{ and } \rho(vs) < \rho(v) \text{ for some } s \in S \\ (x-1)R_{u,vs}(x) + xR_{us,vs}(x) & \text{if } \rho(us) > \rho(u) \text{ and } \rho(vs) < \rho(v) \text{ for some } s \in S, \end{cases}$$

for all $u \leq v$ in W . It is known from [35, Lemma 2.1 (i, ii)] that R is a W -kernel. Also, [35, (2.2.a)] implies that the Kazhdan–Lusztig polynomial $P_{u,v}$ is exactly f_{uv} , where f is the right KLS function with respect to R .

Independently from the kernel of choice, KLS functions have some common properties. The following lemma explicit their constant term.

Lemma 4.12. *Let κ be a (P, ρ) -kernel and let f, g be respectively their associated right and left KLS function. We have the following equalities of coefficients*

$$[x^0] g_{st}(x) = [x^{\rho_{st}}] g_{st}^{\text{rev}}(x) = [x^{\rho_{st}}] \kappa_{st}(x),$$

$$[x^0] f_{st}(x) = [x^{\rho_{st}}] f_{st}^{\text{rev}}(x) = [x^{\rho_{st}}] \kappa_{st}(x),$$

for any $s \leq t$. In particular, the constant term of f and g is non-zero if and only if κ is non-degenerate.

Proof. The third property of Theorem 4.8 of the left KLS function is equivalent the following equation

$$g_{st}^{\text{rev}}(x) - g_{st}(x) = \kappa_{st}(x) + \sum_{s < w < t} g_{sw}(x) \kappa_{wt}(x)$$

for any $s < t$.

The claim follows from observing that every term in the sum on the right-hand side has degree smaller than ρ_{st} . If $s = t$, the statement is trivial. The proof for f is identical. $\square$

§4.1.1 The Z -Function

Let us define one of the possible KLS-invariants of a weakly graded poset, the Z -function.

Definition 4.13. Let $\kappa \in \mathcal{I}_\rho(P)$ be a (P, ρ) -kernel and let f, g be respectively their associated right and left KLS function. The Z -function associated to κ is defined as the element $Z \in \mathcal{I}_\rho(P)$ given by

$$Z := g\kappa f.$$

We notice that the Z -function can be expressed only in terms of the KLS functions in the following way.

Proposition 4.14. Let $\kappa \in \mathcal{I}_\rho(P)$ be a (P, ρ) -kernel and let f, g be respectively their associated right and left KLS function. Then the Z -function associated to κ is

$$Z = g^{\text{rev}} f = g f^{\text{rev}}.$$

It follows that the Z -function is symmetric, which means that it satisfies the identity $Z^{\text{rev}} = (g^{\text{rev}} f)^{\text{rev}} = g f^{\text{rev}} = Z$. We may also prove a converse of the previous Proposition.

Proposition 4.15. Suppose that $f, g \in \mathcal{I}_\rho(P)$ satisfy the following properties:

1. $f_{ss}(x) = g_{ss}(x) = 1$ for all $s \in P$;
2. $\deg f_{st}(x), \deg g_{st}(x) < \frac{1}{2}\rho_{st}$ for all $s < t$.

Then f and g are the right and left KLS functions of the same (P, ρ) -kernel κ if and only if $g^{\text{rev}} f = g f^{\text{rev}}$.

Proof. Let $\kappa_f := f^{\text{rev}} f^{-1}$ and $\kappa_g := g^{-1} g^{\text{rev}}$. By Theorem 4.8 we know that f is the right KLS function of κ_f and g is the left KLS function of κ_g . We notice that $g^{\text{rev}} f = g \kappa_g f$ and $g f^{\text{rev}} = g \kappa_f f$, so we have the desired statement. $\square$

Remark 4.16. By Remark 4.10, we have that the Z -function associated to the kernel κ^* of the dual poset (P^*, ρ^*) is Z^* such that

$$Z_{ts}^*(x) = Z_{st}(x),$$

for all $s \leq t$ in P .

Remark 4.17. If κ is non-degenerate, then $\deg Z_{st}(x) = \rho_{st}$ for all $s \leq t$ in P , in fact we have

$$Z_{st}(x) = \kappa_{st}(x) + \sum_{\substack{s \leq u \leq v \leq t \\ (u,v) \neq (s,t)}} g_{su}(x) \kappa_{uv}(x) f_{vt}(x)$$

where every term on the right-hand side except $\kappa_{st}(x)$ has degree strictly smaller than ρ_{st} .

Example 4.18. Nicholas Proudfoot, Yuan Xu and Ben Young have introduced the notion of the Z -polynomial of a matroid as follows [42]. For any matroid M , they define the Z -polynomial as

$$Z_M(x) := \sum_{F \text{ flat of } M} x^{\text{rk } F} P_{M^F}(x),$$

where M^F is the contraction of M at F and $P_N(t)$ is the right KLS polynomial associated to the characteristic function χ in the lattice of flats $\mathcal{L}(N)$ of the matroid N . Since the left KLS function of χ is ζ , then $Z_M(x)$ is the Z -polynomial associated to the $\mathcal{L}(M)$ -kernel χ .

§4.1.2 The Chow Function

In 2004, Eva Maria Feichtner and Sergey Yuzvinsky [26] constructed a family of graded algebras associated to a finite atomic lattice. These objects have the property of being isomorphic to the Chow Ring of an associated smooth noncomplete toric variety. Furthermore, it is always possible to associate to a realizable loopless matroid one of these algebras which is isomorphic to the cohomology ring of the associated De Concini-Procesi wonderful compactification of the hyperplane arrangement complement [21]. Later, Karim Adiprasito, June Huh and Eric Katz used the construction made by Feichtner and Yuzvinsky to define the Chow ring of any loopless matroid [1].

Inspired by this line of research, Luis Ferroni, Jacob P. Matherne and Lorenzo Vecchi have thus introduced in [27] the *Chow function* associated to a (P, ρ) -kernel. When P is the lattice of flats of a loopless matroid and the kernel is the characteristic kernel, the resulting Chow polynomial coincides with the Hilbert–Poincaré series of the matroid’s Chow ring. This framework extends the polynomial invariant beyond the matroid setting and raises the question of whether analogous graded algebras can be associated to more general posets. In this subsection, we introduce the Chow function associated to a general kernel and study its properties.

Let us prove some preliminary results before providing the definition of the Chow function.

Lemma 4.19. *Let κ be a (P, ρ) -kernel. Then, for every $s < t$ in P , the polynomial $\kappa_{st}(x)$ is divisible by $x - 1$.*

Proof. We prove it by induction on the cardinality of $[s, t]$. If $s \lessdot t$, the identity $\kappa \kappa^{\text{rev}} = \delta$ implies

$$\delta_{st}(x) = 0 = \kappa_{ss}(x) \kappa_{st}^{\text{rev}}(x) + \kappa_{st}(x) \kappa_{tt}^{\text{rev}}(x) = \kappa_{st}^{\text{rev}}(x) + \kappa_{st}(x).$$

By evaluating it at $x = 1$, we get $\kappa_{st}(1) = -\kappa_{st}(1)$ which also returns that 1 is a root of $\kappa_{st}(x)$.

If $\#[s, t] > 2$, we get

$$\kappa_{st}(x) + \kappa_{st}^{\text{rev}}(x) = - \sum_{s < w < t} \kappa_{sw}(x) \kappa_{wt}^{\text{rev}}(x).$$

By induction, if we evaluate it at $x = 1$ we get

$$\kappa_{st}(1) + \kappa_{st}^{\text{rev}}(1) = 0$$

which concludes the proof. $\square$

Definition 4.20. Let κ be a (P, ρ) -kernel. The *reduced (P, ρ) -kernel* $\bar{\kappa} \in \mathcal{I}_\rho(P)$ is the function given by

$$\bar{\kappa}_{st}(x) = \begin{cases} (x - 1)^{-1} \kappa_{st}(x) & s < t \\ -1 & s = t. \end{cases}$$

Definition 4.21. Let κ be a (P, ρ) -kernel. The *Chow function* associated to κ is the element $H \in \mathcal{I}_\rho(P)$ defined by

$$H = -(\bar{\kappa})^{-1}.$$

We can equivalently define the Chow function recursively. For any $s \in P$ we have $H_{ss}(x) = 1$; if $s < t$ in P , then

$$H_{st}(x) = \sum_{s < w \leq t} \bar{\kappa}_{sw}(x) H_{wt}(x) \quad \text{or, dually,} \quad (4.2)$$

$$H_{st}(x) = \sum_{s \leq w < t} H_{sw}(x) \bar{\kappa}_{wt}(x). \quad (4.3)$$

In the following example, we recall how this definition is connected to the Chow ring of a loopless matroid.

Example 4.22. The *Chow ring* of a loopless matroid M on a ground set E is the graded algebra

$$\underline{\text{CH}}(M) := \frac{\mathbb{Q}[x_F : F \in \mathcal{L}(M) \setminus \{\emptyset, E\}]}{I_M + J_M},$$

where $\deg x_F = 1$, $\mathcal{L}(M)$ is the lattice of flats of M and

$$I_M = \langle x_F x_G : F, G \in \mathcal{L}(M) \setminus \{\emptyset, E\} \text{ are incomparable} \rangle,$$

$$J_M = \left\langle \sum_{\substack{F \in \mathcal{L}(M) \setminus \{\emptyset, E\} \\ i \in F}} x_F - \sum_{\substack{F \in \mathcal{L}(M) \setminus \{\emptyset, E\} \\ j \in F}} x_F : i, j \in E \right\rangle.$$

This is the Chow ring that arises from $\mathcal{L}(M)$ using the construction of Feichtner-Yuzvinsky [26] and taking the maximal building set $\mathcal{L}(M) \setminus \{\emptyset, E\}$.

Consider the uniform matroid $M = U_{2,3}$ on $E = [3]$, whose independent sets are the subsets of cardinality at most two. Its lattice of flats is

$$\mathcal{L}(M) = \{\emptyset, \{1\}, \{2\}, \{3\}, E\}.$$

Writing $x_i := x_{\{i\}}$, the three non-empty proper flats are pairwise incomparable, so

$$\underline{\text{CH}}(M) = \frac{\mathbb{Q}[x_1, x_2, x_3]}{\langle x_1 x_2, x_1 x_3, x_2 x_3, x_1 - x_2, x_1 - x_3 \rangle},$$

which is isomorphic to $\mathbb{Q}[z]/\langle z^2 \rangle$, hence its Hilbert–Poincaré series is

$$\text{Hilb}(\underline{\text{CH}}(M), x) := \sum_{k \geq 0} \dim_{\mathbb{Q}}(\underline{\text{CH}}^k(M)) x^k = 1 + x.$$

We can also compute the characteristic Chow polynomial of $\mathcal{L}(M)$ directly and notice that

$$H_{\mathcal{L}(M)}(x) = 1 + x = \text{Hilb}(\underline{\text{CH}}(M), x).$$

It may be proved [27, Theorem 4.1] that the characteristic Chow polynomial of the lattice of flats of a loopless matroid M always equals the Hilbert–Poincaré series of the Chow ring of M .

In the following proposition, we prove some of the properties of the Chow function.

Proposition 4.23. Let κ be a (P, ρ) -kernel, and let $H \in \mathcal{I}_\rho(P)$ be the corresponding Chow function. The following properties hold.

(i) For every $s < t$, we have that

$$[x^{\rho_{st}-1}] H_{st}(x) = [x^{\rho_{st}}] \kappa_{st}(x).$$

In particular, if κ is non-degenerate, then $\deg H_{st}(x) = \rho_{st} - 1$ for every $s < t$.

(ii) The Chow function is symmetric, i.e.

$$H_{st}(x) = x^{\rho_{st}-1} H_{st}(x^{-1}) \quad \text{for every } s < t.$$

Proof. By equation 4.2, we write

$$H_{st}(x) = \bar{\kappa}_{st}(x) + \sum_{s < w < t} \bar{\kappa}_{sw}(x) H_{wt}(x). \quad (4.4)$$

We prove both statements by induction on the cardinality of $[s, t]$. If $s \lessdot t$, the identity above simplifies to

$$H_{st}(x) = \bar{\kappa}_{st}(x)$$

which implies $[x^{\rho_{st}-1}] H_{st}(x) = [x^{\rho_{st}-1}] \bar{\kappa}_{st}(x) = [x^{\rho_{st}}] \kappa_{st}(x)$. Another identity that has a simpler version in this case is $(\kappa \kappa^{\text{rev}})_{st} = (\delta)_{st}$, which is equivalent to

$$\kappa_{st}^{\text{rev}}(x) + \kappa_{st}(x) = 0.$$

This can be rewritten as

$$0 = x^{\rho_{st}} \kappa_{st}(x^{-1}) + \kappa_{st}(x) = x^{\rho_{st}}(x^{-1} - 1) \bar{\kappa}_{st}(x^{-1}) + (x - 1) \bar{\kappa}_{st}(x).$$

Since $H_{st}(x) = \bar{\kappa}_{st}(x)$, we get

$$0 = x^{\rho_{st}-1} (1 - x) H_{st}(x^{-1}) + (x - 1) H_{st}(x)$$

that implies $x^{\rho_{st}-1} H_{st}(x^{-1}) = H_{st}(x)$.

Instead, if $\# [s, t] > 2$, we get that the right-hand side of equation 4.4 is the sum of $\bar{\kappa}_{st}(x)$ with terms of degree at most $\rho_{sw} - 1 + \rho_{wt} - 1 = \rho_{st} - 2$, therefore

$$[x^{\rho_{st}-1}] H_{st}(x) = [x^{\rho_{st}-1}] \bar{\kappa}_{st}(x) = [x^{\rho_{st}}] \kappa_{st}(x).$$

Furthermore, by multiplying equation 4.4 by $(x - 1)$ and subtracting $\kappa_{st}(x)$ from both sides we get

$$\begin{aligned} (x - 1) H_{st}(x) - \kappa_{st}(x) &= \sum_{s < w < t} \kappa_{sw}(x) H_{wt}(x) \\ &= \sum_{s < w < t} \kappa_{sw}(x) x^{\rho_{wt}-1} H_{wt}(x^{-1}). \end{aligned} \quad (4.5)$$

Since κ is a (P, ρ) -kernel, we have $\kappa\kappa^{\text{rev}} = \delta$, from which we obtain

$$\begin{aligned}
 (x-1)H_{st}(x) - \kappa_{st}(x) &= \sum_{s < w < t} \left(- \sum_{s \leq u < w} \kappa_{su}(x) x^{\rho_{uw}} \kappa_{uw}(x^{-1}) \right) x^{\rho_{wt}-1} H_{wt}(x^{-1}) \\
 &= - \sum_{s \leq u < t} \kappa_{su}(x) x^{\rho_{ut}-1} \left(\sum_{u < w < t} \kappa_{uw}(x^{-1}) H_{wt}(x^{-1}) \right) \\
 &= - \sum_{s \leq u < t} \kappa_{su}(x) x^{\rho_{ut}-1} ((x^{-1} - 1) H_{ut}(x^{-1}) - \kappa_{ut}(x^{-1})) \\
 &= \frac{1}{x} \sum_{s \leq u < t} \kappa_{su}(x) x^{\rho_{ut}} \kappa_{ut}(x^{-1}) \\
 &\quad - \frac{1-x}{x} \sum_{s \leq u < t} \kappa_{su}(x) x^{\rho_{ut}-1} H_{ut}(x^{-1}) \\
 &= -\frac{1}{x} \kappa_{st}(x) + \frac{x-1}{x} \sum_{s < u < t} \kappa_{su}(x) x^{\rho_{ut}-1} H_{ut}(x^{-1}).
 \end{aligned}$$

Hence, by moving terms in the identity we get

$$xH_{st}(x) = \kappa_{st}(x) + \sum_{s \leq u < t} \kappa_{su}(x) x^{\rho_{ut}-1} H_{ut}(x^{-1}).$$

By induction, we know that $x^{\rho_{ut}-1} H_{ut}(x^{-1}) = H_{ut}(x)$ if $s < u < t$, so we can rewrite

$$xH_{st}(x) = \kappa_{st}(x) + x^{\rho_{st}-1} H_{st}(x^{-1}) + \sum_{s < u < t} \kappa_{su}(x) H_{ut}(x). \quad (4.6)$$

We also have that by adding $H_{st}(x)$ to both sides of equation 4.5

$$xH_{st}(x) = \sum_{s \leq w \leq t} \kappa_{sw}(x) H_{wt}(x),$$

and by plugging this identity into equation 4.6, we prove

$$H_{st}(x) = x^{\rho_{st}-1} H_{st}(x^{-1}). \quad \square$$

The Chow function associated to a kernel may also be defined by the following universal properties.

Proposition 4.24. *Let κ be a (P, ρ) -kernel. The Chow function H is the unique element in $\mathcal{I}_\rho(P)$ such that*

1. $H_{ss}(x) = 1$ for all $s \in P$;
2. for every $s < t$, the polynomial $H_{st}(x)$ is symmetric with centre of symmetry $\frac{1}{2}(\rho_{st} - 1)$;
3. $\kappa H = H^{\text{rev}}$ or $H\kappa = H^{\text{rev}}$.

Proof. We have already shown that the Chow functions satisfy the first two properties, we now prove that they also satisfy the third one. Let $s < t$, if we multiply by $x-1$ the formula that defines the Chow function in equation 4.2 we get

$$(x-1)H_{st}(x) = \sum_{s < w \leq t} \kappa_{sw}(x) H_{wt}(x).$$

If we add $H_{st}(x)$ to both sides of the identity we obtain

$$xH_{st}(x) = \sum_{s \leq w \leq t} \kappa_{sw}(x)H_{wt}(x) = (\kappa H)_{st}(x).$$

We now notice by Proposition 4.23 that

$$(\kappa H)_{st}(x) = xH_{st}(x) = x^{\rho_{st}}H_{st}(x^{-1}) = H_{st}^{\text{rev}}(x).$$

Similarly, by starting from the recursive definition of Chow function given by equation 4.3 we would get $H\kappa = H^{\text{rev}}$.

We now prove the uniqueness. Any element $\widetilde{H}$ satisfying the properties of the statement is such that, for any $s < t$ in P

$$x\widetilde{H}_{st}(x) = (\kappa\widetilde{H})_{st}(x),$$

since we can repeat the reasoning above in reverse, using only properties (1), (2) and (3). If we subtract $\widetilde{H}_{st}(x)$ from both sides and then divide by $x - 1$, we get

$$\widetilde{H}_{st}(x) = \sum_{s < w \leq t} \bar{\kappa}_{sw}(x)\widetilde{H}_{wt}(x)$$

which is equivalent to $\bar{\kappa}\widetilde{H} = -\delta$. We conclude by the uniqueness of the inverse in $\mathcal{I}(P)$. $\square$

We may also prove a converse of the previous Proposition.

Proposition 4.25. *Let H be an element of $\mathcal{I}_\rho(P)$ such that $H_{ss}(x) = 1$ for all $s \in P$, and such that for every $s < t$ the polynomial $H_{st}(x)$ is symmetric with centre of symmetry $\frac{1}{2}(\rho_{st} - 1)$. There exists a unique (P, ρ) -kernel κ such that H is the associated Chow function.*

Proof. By Proposition 4.24, we have that any (P, ρ) -kernel κ that satisfies $\kappa H = H^{\text{rev}}$ must have H as a Chow function. Since H is invertible, κ is uniquely determined as

$$\kappa = H^{\text{rev}}H^{-1},$$

which is always a (P, ρ) -kernel by Theorem 4.4. $\square$

§4.1.3 The Augmented Chow Function

In 2020, a paper from Braden, Huh, Matherne, Proudfoot and Wang [15] introduced the *augmented Chow ring* of a loopless matroid. Motivated by this construction and its role in the singular Hodge theory of matroids, the authors of the paper [27] have introduced the augmented Chow functions associated to a kernel. Just as the KLS functions, their construction distinguishes between left and right augmented Chow functions. In particular, they prove that the left augmented Chow function of (P, χ) , where P is the lattice of flats of a loopless matroid M , is the Hilbert series of the augmented Chow ring of M .

Let us introduce the definitions of the augmented Chow functions.

Definition 4.26. Let κ be a (P, ρ) -kernel. We call

$$F = Hf^{\text{rev}}, \quad G = g^{\text{rev}}H$$

respectively the *right augmented Chow function* and the *left augmented Chow function* associated to κ .

In the following example, we explain the connection between the left augmented Chow functions and the augmented Chow ring of a loopless matroid.

Example 4.27. The *augmented Chow ring* of a loopless matroid M on a ground set E is the graded algebra

$$\mathrm{CH}(M) := \frac{\mathbb{Q}[x_F, y_i : F \in \mathcal{L}(M) \setminus \{E\}, i \in E]}{I_M + J_M},$$

where $\deg x_F = \deg y_i = 1$ and

$$\begin{aligned} I_M &= \langle x_F x_G : F, G \in \mathcal{L}(M) \setminus \{E\} \text{ are incomparable} \rangle \\ &\quad + \langle y_i x_F : F \in \mathcal{L}(M) \setminus \{E\}, i \in E \setminus F \rangle, \\ J_M &= \left\langle y_i - \sum_{\substack{F \in \mathcal{L}(M) \setminus \{E\} \\ i \notin F}} x_F : i \in E \right\rangle. \end{aligned}$$

Consider again the uniform matroid $M = U_{2,3}$ on $E = [3]$. Writing $x_0 := x_\emptyset$ and $x_i := x_{\{i\}}$ for $i \in [3]$, the linear relations in $\mathrm{CH}(M)$ give us the identity

$$y_i = x_0 + \sum_{j \neq i} x_j.$$

Set $z := x_0 + x_1 + x_2 + x_3$ so that $y_i = z - x_i$. For every distinct pair $i, j \in [3]$ we have $x_i x_j = 0$ and $y_i x_j = 0$, thus $z x_j = 0$. At the same time, we have $y_i x_0 = 0$ for every $i \in [3]$ and this implies $z^2 + x_i^2 = 0$. Hence

$$\mathrm{CH}(M) \cong \frac{\mathbb{Q}[z, x_1, x_2, x_3]}{\langle z x_i, x_i x_j, x_i^2 + z^2 : i, j \in [3], i \neq j \rangle},$$

and in particular

$$\mathrm{Hilb}(\mathrm{CH}(M), x) := \sum_{k \geq 0} \dim_{\mathbb{Q}} \mathrm{CH}^k(M) x^k = 1 + 4x + x^2.$$

We can also compute the left augmented characteristic Chow polynomial of $\mathcal{L}(M)$ directly and notice that

$$\begin{aligned} G_{\mathcal{L}(M)}(x) &= \sum_{F \in \mathcal{L}(M)} x^{\mathrm{rk}(F)} H_{FE}(x) \\ &= (1 + x) + 3x + x^2 \\ &= 1 + 4x + x^2 \\ &= \mathrm{Hilb}(\mathrm{CH}(M), x). \end{aligned}$$

It may be proved [27, Theorem 4.11] that the left augmented characteristic Chow polynomial of the lattice of flats of a loopless matroid M always equals the Hilbert–Poincaré series of the augmented Chow ring of M .

Despite being more subtle invariants than the Chow functions, the augmented Chow functions also satisfy some interesting properties.

Proposition 4.28. *Let κ be a (P, ρ) -kernel and let F, G be the augmented Chow functions. The following properties hold true.*

1. For every $s \leq t$ in P ,

$$[x^{\rho_{st}}] F_{st}(x) = [x^{\rho_{st}}] G_{st}(x) = [x^{\rho_{st}}] \kappa_{st}(x).$$

In particular, if κ is non-degenerate then $\deg F_{st}(x) = \deg G_{st}(x) = \rho_{st}$ for all $s \leq t$.

2. The augmented Chow functions are symmetric, i.e. they satisfy

$$F = F^{\text{rev}}, \quad G = G^{\text{rev}}.$$

Proof. We prove the statement only for the left augmented Chow function G , as the proof for F is similar. Let $s \leq t$ and rewrite $G_{st}(x)$ as

$$G_{st}(x) = g_{st}^{\text{rev}}(x) + \sum_{s \leq w < t} g_{sw}^{\text{rev}} H_{wt}(x)$$

and notice that it is the sum of $g_{st}^{\text{rev}}(x)$ with terms of degree at most $\rho_{st} - 1$, hence by Lemma 4.12

$$[x^{\rho_{st}}] G_{st}(x) = [x^{\rho_{st}}] g_{st}^{\text{rev}}(x) = [x^{\rho_{st}}] \kappa_{st}(x).$$

Furthermore, by Proposition 4.24 and Proposition 4.8, we conclude

$$G = g^{\text{rev}} H = g \kappa H = g H^{\text{rev}} = (g^{\text{rev}} H)^{\text{rev}} = G^{\text{rev}}. \quad \square$$

§4.2 Eulerian KLS Theory

The KLS theory of posets can be specialized to study Eulerian posets. In fact, there is a specific element of the incidence algebra that is a kernel if and only if the associated poset is Eulerian.

Proposition 4.29. *Let P be a finite graded bounded poset. Then P is Eulerian if and only if the function $\varepsilon \in \mathcal{I}_\rho(P)$ given by*

$$\varepsilon_{st}(x) := (x - 1)^{\rho(t) - \rho(s)}$$

for every $s \leq t$ in P is a (P, ρ) -kernel.

Proof. Let $s \leq t$ in P , then

$$\begin{aligned} (\varepsilon^{\text{rev}} \varepsilon)_{st}(x) &= \sum_{s \leq u \leq t} x^{\rho_{su}} (x^{-1} - 1)^{\rho_{su}} (x - 1)^{\rho_{ut}} \\ &= (x - 1)^{\rho_{st}} \sum_{s \leq u \leq t} (-1)^{\rho_{su}} \end{aligned}$$

hence ε is a (P, ρ) -kernel if and only if

$$\sum_{s \leq u \leq t} (-1)^{\rho_{su}} = \delta_{st}(x)$$

for all $s \leq t$ in P , which by Möbius inversion is equivalent to

$$\mu_{st}(x) = (-1)^{\rho_{st}}$$

for all $s \leq t$ in P . □

Whenever P is Eulerian, we will refer to ε as the *Eulerian P -kernel*. We will also refer to their relative KLS invariants as the *Eulerian Z -function*, *Eulerian Chow function* and *Eulerian augmented Chow functions* of P . We can provide a combinatorial and geometric interpretation to each of these invariants. Before diving into that, let us analyse the KLS functions in the Eulerian case.

Theorem 4.30. *Let P be an Eulerian poset. Then*

1. *the left KLS polynomial $g_{\hat{0}\hat{1}}$ associated to the Eulerian P -kernel is the toric g -polynomial of P ;*
2. *the right KLS polynomial $f_{\hat{0}\hat{1}}$ associated to the Eulerian P -kernel is the toric g -polynomial of the dual P^* .*

Proof. Let $s \leq t$ in P , then $[s, t]$ is an Eulerian poset. We prove that the toric g -polynomial of P satisfies the universal properties of Theorem 4.8. Indeed, the g -polynomial of $[s, s]$ is always 1, and the degree of the g -polynomial of $[s, t]$ is at most $\lfloor (\rho_{st} - 1)/2 \rfloor < \rho_{st}/2$. Furthermore

$$\begin{aligned} \sum_{s \leq u \leq t} g([s, u], x)(x-1)^{\rho_{ut}} &= (x-1)h([s, t], x) + g([s, t], x) \\ &= g^{\text{rev}}([s, t], x), \end{aligned}$$

where the second equality comes from the palindromic property of the toric h -vector. The second statement follows from Remark 4.10. $\square$

Remark 4.31. Let P be an Eulerian poset. Using the previous theorem, we may give a non-recursive definition of the toric g - and h - polynomial of P . In particular, the toric g -polynomial of P could be defined as the left KLS function $g_P(x)$ associated to the Eulerian P -kernel, while the toric h -polynomial of P could be defined as

$$h_P(x) := \frac{g_P^{\text{rev}}(x) - g_P(x)}{x-1}.$$

§4.2.1 Eulerian Z -Function and the Interval Poset

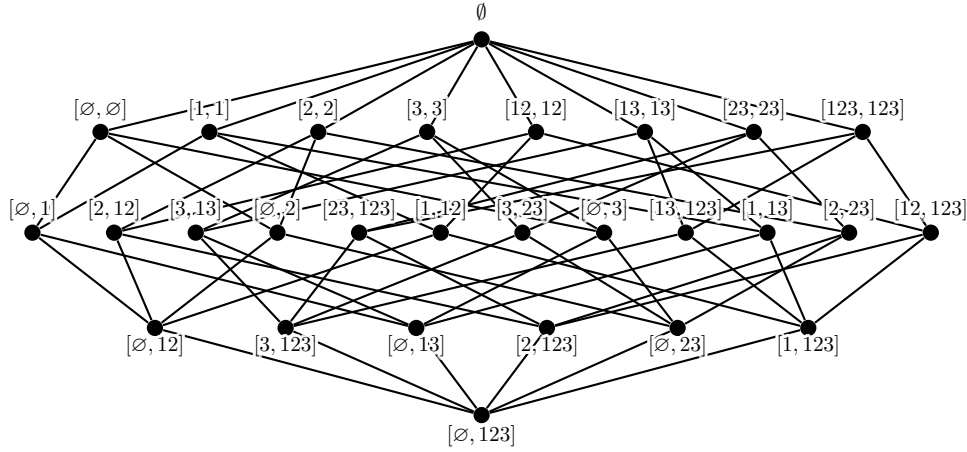
In this subsection, we give a combinatorial and geometric interpretation to the Eulerian Z -polynomial. We construct the *interval poset* and then prove that the interval poset of an Eulerian poset is still Eulerian. Consequently, we show that the Eulerian Z -polynomial of a poset equals the toric h -vector of its interval poset. We conclude by briefly discussing the relation between the interval poset and the notion of the geometric antiprism of a poset.

Let us give the definition of interval poset.

Definition 4.32. Let P be a poset and let $\text{Int}(P)$ be the set of its closed intervals. We *interval poset* of P is $I(P) := \text{Int}(P) \cup \{\emptyset\}$, ordered by reverse inclusion.

Many authors define the interval poset of P as $I(P)^*$, or equivalently, they define it by ordering $\text{Int}(P) \cup \{\emptyset\}$ with respect to inclusion.

Example 4.33. Let B_n be the boolean poset on $[n]$, for some $n \in \mathbb{N}$. The following picture shows the Hasse diagram of $I(B_3)$.


 Figure 4.1: The interval poset $I(B_3)$.

The interval poset $I(B_n)^*$ is isomorphic to $\mathcal{F}(\mathcal{Q}_n)$, where $\mathcal{Q}_n$ denotes the n -cube $[-1; 1]^n$.

Indeed, we notice that $\mathcal{F}(\mathcal{Q}_n) \setminus \{\emptyset\}$ is in bijection with the words $s_1 s_2 \dots s_n$ of length n in the alphabet $\{-1, 1, *\}$. Namely, the face corresponding to the string $s = s_1 s_2 \dots s_n$ is $F_s = \{(x_1, \dots, x_n) \in \mathcal{Q}_n \mid \forall i \in [n] : \text{if } s_i \neq * \text{ then } x_i = s_i\}$.

For each string $s := s_1 s_2 \dots s_n$, set $a(s) := \{i \in [n] \mid s_i = 1\} \in B_n$ and $b(s) := \{i \in [n] \mid s_i \neq -1\}$. Notice that $a(s) \subseteq b(s)$. The map $\Phi : \{\hat{0}\} \sqcup \{-1, 1, *\}^n \rightarrow I(B_n)^*$ given by

$$\begin{aligned} \Phi : \{\hat{0}\} \sqcup \{-1, 1, *\}^n &\rightarrow I(B_n)^* \\ \hat{0} &\mapsto \emptyset \\ s \in \{-1, 1, *\}^n &\mapsto [a(s), b(s)] \end{aligned}$$

is an isomorphism of posets, if $\{-1, 1, *\}^n$ has the poset structure induced by the bijection with $\mathcal{F}(\mathcal{Q}_n) \setminus \{\emptyset\}$ and $\hat{0}$ is an adjoined minimum element.

The interval poset preserves many properties of the starting poset. We prove in the following that it preserves graded and Eulerian posets.

Lemma 4.34. *Let P be a graded bounded poset with a rank function ρ . Then $I(P)$ is graded and its rank function is given by*

$$\hat{\rho}_{[u,v]}[s,t] = \rho_{us} + \rho_{tv} \quad \text{and} \quad \hat{\rho}_{[u,v]}\emptyset = 1 + \rho_{uv},$$

for any $[s,t], [u,v] \in \text{Int}(P)$ such that $[s,t] \subseteq [u,v]$.

Proof. Let $\mathcal{C}$ be a maximal chain in $I(P)$, and let us call its elements as follows

$$\mathcal{C} := [P = [s_1, t_1] \supsetneq [s_2, t_2] \supsetneq \dots \supsetneq [s_n, t_n] \supsetneq [s_{n+1}, t_{n+1}] \supsetneq \emptyset],$$

Since $\mathcal{C}$ is saturated, we have

$$s_i = s_{i+1}, t_{i+1} < t_i \quad \text{or} \quad s_i < s_{i+1}, t_i = t_{i+1}$$

for every $i = 1, \dots, n$. Set $q_{n+1} := s_{n+1} = t_{n+1}$, and for each $i = 2, \dots, n+2$ let $q_{i-1} := t_{i-1}$ if we are in the first case, and let $q_{i-1} := s_{i-1}$ if we are in the second case. Since $\mathcal{C}$ is saturated, $\{q_i\}_{i \in [n+1]}$, up to reordering the indices, form a maximal chain

$$\hat{0} = q_1 < q_2 < \dots < q_n < q_{n+1} = \hat{1}.$$

Since P is graded, we have that n is the rank of P and also that

$$\rho(I(P)) = \rho(P) + 1.$$

The same argument proves that $\widehat{\rho}_{[u,v]}\emptyset = 1 + \rho_{uv}$ for any $[u, v] \in \text{Int}(P)$, as the interval from $[u, v]$ to $\emptyset$ in $I(P)$ is isomorphic to $I([u, v])$.

Let $[s, t], [u, v] \in \text{Int}(P)$ be any two intervals such that $[s, t] \subseteq [u, v]$. Since the interval between these two elements in $I(P)$ is isomorphic to $[u, s] \times [t, v]^*$, it is a graded poset of rank $\rho_{us} + \rho_{tv}$, hence

$$\widehat{\rho}_{[u,v][s,t]} = \rho_{us} + \rho_{tv}. \quad \square$$

Proposition 4.35. *If P is Eulerian, then $I(P)$ is also Eulerian.*

Proof. Let $[s, t], [u, v] \in \text{Int}(P)$ be any two intervals such that $[s, t] \subseteq [u, v]$. Since the interval between these two elements in $I(P)$ is isomorphic to $[u, s] \times [t, v]^*$, we have

$$\mu_{I(P)}([u, v], [s, t]) = \mu_P(u, s)\mu_P(t, v)$$

which is exactly $(-1)^{\widehat{\rho}([s,t],[u,v])}$ by Lemma 4.34.

Let $[u, v] \in \text{Int}(P)$ be any closed interval in P . Then

$$\begin{aligned} \mu_{I(P)}([u, v], \emptyset) &= - \sum_{\substack{x, y \in P \text{ such that} \\ u \leq x \leq y \leq v}} \mu_{I(P)}([u, v], [x, y]) \\ &= - \sum_{\substack{x, y \in P \text{ such that} \\ u \leq x \leq y \leq v}} \mu_P(u, x)\mu_P(y, v) \\ &= - \sum_{u \leq y \leq v} \mu_P(y, v) \sum_{u \leq x \leq y} \mu_P(u, x) \\ &= - \sum_{u \leq y \leq v} \mu_P(y, v)\delta(u, y) \\ &= -\mu_P(u, v), \end{aligned}$$

which is exactly $(-1)^{\widehat{\rho}([u,v],\emptyset)}$ by Lemma 4.34. $\square$

In a recent paper by Luis Ferroni and Roberto Riccardi [28], the Z -polynomial of a poset has been characterized in terms of the interval poset. We prove this characterization in the following theorems.

Given any Eulerian poset P , let us denote by $\widehat{\varepsilon}$ the Eulerian kernel of $I(P)$, and by $\widehat{g}$ and $\widehat{h}$ the toric g - and h -polynomials of $I(P)$.

Theorem 4.36. *Let P be an Eulerian poset. If $[s, t] \subseteq [u, v]$ in P , then*

$$\widehat{g}_{[u,v][s,t]}(x) = g_{us}(x)f_{tv}(x).$$

Proof. Consider the function $z \in \mathcal{I}_\rho(I(P))$ given by

$$z_{[u,v][s,t]}(x) = g_{us}(x)f_{tv}(x), \quad z_{[u,v]}\emptyset(x) = \widehat{g}_{[u,v]}\emptyset(x)$$

for every $[s, t], [u, v] \in \text{Int}(P)$ such that $[s, t] \subseteq [u, v]$. Set $J := I(P) \setminus \{\emptyset\}$. It is sufficient to prove that $z|_J$ is the left KLS function associated to $\widehat{\varepsilon}|_J$. Indeed, the restriction of a KLS function is the KLS function of the restricted kernel: by construction, it implies that if $z|_J$ is the left KLS function associated to $\widehat{\varepsilon}|_J$, then z is the left KLS function associated to $\widehat{\varepsilon}$.

We proceed by showing that $z|_J$ satisfies the universal properties of the left KLS function. It is trivial to check $z_{ww}(x) = 1$ for every $w \in I(P)$. Furthermore, for every $[s, t], [u, v] \in \text{Int}(P)$ such that $[s, t] \subsetneq [u, v]$ we have that

$$\deg z_{[u,v][s,t]}(x) = \deg g_{us}(x) + \deg f_{tv}(x) < \frac{1}{2} \widehat{\rho}_{[u,v][s,t]}.$$

To conclude the proof we need to prove that $z|_J^{\text{rev}} = (z\widehat{\varepsilon})|_J$. For every $[s, t], [u, v] \in \text{Int}(P)$ such that $[s, t] \subseteq [u, v]$, we have

$$\begin{aligned} z_{[u,v][s,t]}^{\text{rev}}(x) &= x^{\widehat{\rho}_{[u,v][s,t]}} z_{[u,v][s,t]}(x^{-1}) \\ &= x^{\rho_{us}} g_{us}(x^{-1}) x^{\rho_{tv}} f_{tv}(x^{-1}) \\ &= g_{us}^{\text{rev}}(x) f_{tv}^{\text{rev}}(x) \\ &= (g\varepsilon)_{us}(x) (\varepsilon f)_{tv}(x), \end{aligned}$$

and if we expand the products, we get

$$\begin{aligned} z_{[u,v][s,t]}^{\text{rev}}(x) &= \left(\sum_{u \leq w \leq s} g_{uw}(x) (x-1)^{\rho_{ws}} \right) \left(\sum_{t \leq y \leq v} (x-1)^{\rho_{ty}} f_{yv}(x) \right) \\ &= \sum_{[s,t] \subseteq [w,y] \subseteq [u,v]} g_{uw}(x) f_{yv}(x) (x-1)^{\rho_{ws} + \rho_{ty}} \\ &= \sum_{[s,t] \subseteq [w,y] \subseteq [u,v]} g_{uw}(x) f_{yv}(x) (x-1)^{\widehat{\rho}_{[w,y][s,t]}} \\ &= (z\widehat{\varepsilon})_{[u,v][s,t]}(x). \end{aligned} \quad \square$$

Having a formula for the left KLS function of $I(P)$, we can prove the following.

Theorem 4.37. *Let P be an Eulerian poset and let Z be its Eulerian Z -function. Then, for each pair of elements $u \leq v$ in P , we have*

$$Z_{uv}(x) = \widehat{h}_{[u,v]\emptyset}(x).$$

Proof. For $u \leq v$ in P , we calculate

$$\begin{aligned} Z_{uv}(x) &= (g^{\text{rev}} f)_{uv} \\ &= \sum_{u \leq t \leq v} g_{ut}^{\text{rev}}(x) f_{tv}(x) \end{aligned}$$

and since $g^{\text{rev}} = g\varepsilon$

$$\begin{aligned} Z_{uv}(x) &= \sum_{u \leq t \leq v} \left(\sum_{u \leq s \leq t} g_{us}(x) (x-1)^{\rho_{st}} \right) f_{tv}(x) \\ &= \sum_{[s,t] \subseteq [u,v]} g_{us}(x) f_{tv}(x) (x-1)^{\rho_{st}} \\ &= \sum_{[s,t] \subseteq [u,v]} \widehat{g}_{[u,v][s,t]}(x) (x-1)^{\widehat{\rho}_{[s,t]\emptyset} - 1} \\ &= (x-1)^{-1} \sum_{[s,t] \subseteq [u,v]} \widehat{g}_{[u,v][s,t]}(x) (x-1)^{\widehat{\rho}_{[s,t]\emptyset}} \\ &= (x-1)^{-1} (\widehat{g\widehat{\varepsilon}} - \widehat{g})_{[u,v]\emptyset}(x) \\ &= (x-1)^{-1} (\widehat{g^{\text{rev}}} - \widehat{g})_{[u,v]\emptyset}(x). \end{aligned}$$

Since $\hat{g}$ is the toric g -vector of $I(P)$, we have

$$Z_{uv}(x) = \hat{h}_{[u,v]}(x). \quad \square$$

Luis Ferroni and Roberto Riccardi also note the following fact.

Given a d -dimensional polytope $\mathcal{P} \subseteq \mathbb{R}^d$, we say that it is *admissible* if the origin lies in the relative interior and, for every face F of $\mathcal{P}$, the orthogonal projection of the origin onto the affine hull of F lies in the relative interior of F . We also define the polytope

$$\mathcal{Q} = \text{conv}(\mathcal{P} \times \{0\}, \mathcal{P}^* \times \{1\}).$$

A theorem of Mark N. Broadie [19, Theorem 2.1] states that the face lattice of $\mathcal{Q}$ is isomorphic to $I(\mathcal{P})$ if and only if $\mathcal{P}$ is admissible. In this case, we will say that $\mathcal{Q}$ is the *geometric antiprism* of $\mathcal{P}$. Therefore, we have the following theorem.

Theorem 4.38 ([28, Theorem 1.3]). *Let $\mathcal{P}$ be an admissible polytope. The Z -polynomial of $\mathcal{P}$ is the toric h -polynomial of the face lattice of the geometric antiprism of $\mathcal{P}$.*

Example 4.39. Let $\Delta_{n-1} \subseteq \mathbb{R}^n$ be an $(n-1)$ -dimensional simplex. Up to a translation we may assume that it is admissible, hence the face lattice of $\text{conv}(\Delta_{n-1} \times \{0\}, \Delta_{n-1}^* \times \{1\})$ is isomorphic to $I(\mathcal{F}(\Delta_{n-1})) = I(B_n)$, where B_n denotes the boolean poset on $[n]$. By Example 4.33, we know that this poset is isomorphic to $\mathcal{F}(\mathcal{Q}_n^*)$, where $\mathcal{Q}_n$ is the n -cube. The dual of the n -cube is also known as the n -dimensional *cross-polytope*. Since the n -cube is simplicial, the n -dimensional cross-polytope is simplicial, therefore its toric h -vector coincides with the usual h -vector. By Example 1.39 we also know that the f -polynomial of the n -cube is

$$t^n + \frac{(2t+1)^n - 1}{t}.$$

Hence, the f -polynomial of the n -dimensional cross-polytope is $(2+t)^n$, and as a consequence, its h -polynomial is

$$(1+x)^n.$$

Therefore, the Z -polynomial of $\mathcal{F}(\Delta_{n-1}) = B_n$ is

$$Z_{B_n}(x) = h_{I(B_n)}(x) = (1+x)^n.$$

§4.2.2 Eulerian Chow Function and the Order Complex

In this subsection, we give a combinatorial and geometric interpretation to the Eulerian Chow polynomial. In particular, we show that it is equal to the h -vector of the order complex of the proper part of the poset. Furthermore, we see its relation with the values of the function ζ .

Theorem 4.40. *Let P be an Eulerian poset. The Eulerian Chow polynomial of P is*

$$H_{\hat{0}\hat{1}}(x) = h(\Delta(P), x).$$

where $\Delta(P)$ is the order complex of P .

Proof. We prove it by induction on the rank of P . If $\text{rank } P = 0$, then it is trivially true. Let P be of rank $d+1 > 0$. Notice that

$$\zeta^n(\hat{0}, \hat{1}) = \sum_{i \geq 0} \sum_{\substack{S \subseteq [d] \\ \#S=i}} \binom{n}{i+1} \alpha_P(S)$$

since the number of multichains of length n whose elements in $P \setminus \{\hat{0}, \hat{1}\}$ are $t_1 < \dots < t_i$ is $\binom{i+2}{(n-1)-i} = \binom{n}{i+1}$. Furthermore,

$$\begin{aligned} \sum_{\substack{S \subseteq [d] \\ \#S=i}} \alpha_P(S) &= \sum_{\substack{S \subseteq [d] \\ \#S=i}} \sum_{T \subseteq S} \beta_P(T) \\ &= \sum_{j \geq 0} \sum_{\substack{T \subseteq [d] \\ \#T=j}} \binom{d-j}{d-i} \beta_P(T) \end{aligned}$$

hence for $n \geq 1$,

$$\begin{aligned} \zeta^n(\hat{0}, \hat{1}) &= \sum_{i,j \geq 0} \sum_{\substack{T \subseteq [d] \\ \#T=j}} \binom{n}{i+1} \binom{d-j}{d-i} \beta_P(T) \\ &= \sum_{j \geq 0} \sum_{\substack{T \subseteq [d] \\ \#T=j}} \binom{d+n-j}{d+1} \beta_P(T) \\ &= [x^n] \frac{\sum_{j=0}^d \sum_{\#T=j} \beta_P(T) x^{j+1}}{(1-x)^{d+2}}. \end{aligned}$$

Recall that, by Remark 1.50, the h -polynomial of $\Delta(P)$ is

$$h(\Delta(P)) = \sum_{i \geq 0} \left(\sum_{\substack{S \subseteq [d] \\ \#S=i}} \beta_P(S) \right) x^i,$$

hence

$$\begin{aligned} \zeta^n(\hat{0}, \hat{1}) &= [x^n] \frac{x h(\Delta(P), x)}{(1-x)^{d+2}} \\ &= [x^{n-1}] \frac{h(\Delta(P), x)}{(1-x)^{d+2}}. \end{aligned}$$

Thus we have the formula

$$h(\Delta(P), x) = (1-x)^{\text{rank} P+1} \sum_{n \geq 0} \zeta^{n+1}(\hat{0}, \hat{1}) x^n. \quad (4.7)$$

We may rewrite the Chow function using equation 4.2 as

$$H_{\hat{0}\hat{1}}(x) = \sum_{\hat{0} < t \leq \hat{1}} (x-1)^{\rho_{\hat{0}t}-1} H_{t\hat{1}}(x)$$

which by the induction hypothesis is

$$\begin{aligned} H_{\hat{0}\hat{1}}(x) &= \sum_{\hat{0} < t \leq \hat{1}} (x-1)^{\rho_{\hat{0}t}-1} (1-x)^{\rho_{\hat{0}\hat{1}}-\rho_{\hat{0}t}+1} \sum_{n \geq 0} \zeta^{n+1}(t, \hat{1}) x^n \\ &= (1-x)^{\text{rank} P+1} \sum_{\hat{0} < t \leq \hat{1}} (-1)^{\rho_{\hat{0}t}-1} \frac{1}{1-x} \sum_{n \geq 0} \zeta^{n+1}(t, \hat{1}) x^n \\ &= (1-x)^{\text{rank} P+1} \sum_{n, \ell \geq 0} \sum_{\hat{0} < t \leq \hat{1}} (-1)^{\rho_{\hat{0}t}-1} \zeta^{n+1}(t, \hat{1}) x^{n+\ell} \\ &= (1-x)^{\text{rank} P+1} \sum_{m \geq 0} \left(\sum_{n=0}^m \sum_{\hat{0} < t \leq \hat{1}} (-1)^{\rho_{\hat{0}t}-1} \zeta^{n+1}(t, \hat{1}) \right) x^m. \end{aligned}$$

By equation 4.7, we observe that if we have

$$\sum_{n=0}^m \sum_{\hat{0} < t \leq \hat{1}} (-1)^{\rho_{\hat{0}t}-1} \zeta^{n+1}(t, \hat{1}) = \zeta^{m+1}(\hat{0}, \hat{1}),$$

then we have proved the theorem. The identity above is a consequence of the Eulerianity of P , in fact

$$\begin{aligned} \sum_{n=0}^m \sum_{\hat{0} < t \leq \hat{1}} (-1)^{\rho_{\hat{0}t}-1} \zeta^{n+1}(t, \hat{1}) &= - \sum_{n=1}^{m+1} \sum_{\hat{0} < t \leq \hat{1}} \mu(\hat{0}, t) \zeta^n(t, \hat{1}) \\ &= - \sum_{n=1}^{m+1} ((\mu \zeta^n) - \zeta^n)(\hat{0}, \hat{1}) \\ &= \sum_{n=1}^{m+1} (\zeta^n(\hat{0}, \hat{1}) - \zeta^{n-1}(\hat{0}, \hat{1})) \\ &= \zeta^{m+1}(\hat{0}, \hat{1}). \end{aligned} \quad \square$$

Equation 4.7 implies also the following corollary.

Corollary 4.41. *Let P be an Eulerian poset. The Eulerian Chow polynomial of P satisfies the identity*

$$\frac{H_P(x)}{(1-x)^{\rho(P)+1}} = \sum_{n=0}^{\infty} \zeta_P^{n+1}(\hat{0}, \hat{1}) x^n.$$

Let us see as an example the case of Boolean posets.

Example 4.42. Let B_n be the Boolean poset on $[n]$. The Chow polynomial of B_n is the h -polynomial of $\Delta(B_n)$, which is also the boundary complex of the dual of the permutahedron, as it follows from its face lattice characterization. For example, for $n = 5$ we have

$$H_{B_5}(x) = 1 + 26x + 66x^2 + 26x^3 + x^4.$$

In general, by [47, Corollary 3.13.2], we have that

$$H_{B_n}(x) = A_n(x),$$

given that A_n is the n -th *Eulerian polynomial* defined as $A_n(x) := \sum_{\sigma \in S_n} x^{\text{des}(\sigma)}$, where $\text{des}(\sigma) := \#\{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$ for every $\sigma \in S_n$.

§4.2.3 Eulerian Augmented Chow Function and the Stellahedral Transform

In this subsection, we provide an interpretation to the Augmented Chow Function in the Eulerian case. The following discussion is the result of joint work with Luis Ferroni and Tommaso Faustini [25].

We define the *stellahedral transform* of a poset and then demonstrate that it can be realized as a Bier poset and that it preserves Eulerianity. We subsequently prove that the toric h -vector of the stellahedral transform of an Eulerian poset is the Eulerian right augmented Chow polynomial of the poset. Finally, we show that it preserves the property of being the face lattice of a polytope by defining the *dual stellahedral transform* of a polytope.

Definition 4.43. Let P be a poset with minimum element $\hat{0}$. We define the *stellahedral transform* of P as the poset

$$\text{St}(P) = \{(c, y) : c = \{\hat{0} = c_0 < c_1 < \dots < c_k\} \text{ chain in } P, y \in P, c_k \leq y\} \sqcup \{\hat{1}_{\text{St}}\}$$

where the partial order is given by imposing $\hat{1}_{\text{St}}$ as the maximum element and taking

$$(c, y) \leq (c', y') \iff c \subseteq c' \text{ and } y \geq y' \text{ in } P.$$

This poset transformation preserves several features of the poset one feeds into it. The following lemma shows that if a poset P is bounded and graded with rank n , its stellahedral transform is bounded and graded with rank $n + 1$.

Lemma 4.44. *Let P be a graded bounded poset with $\hat{0}$ and $\hat{1}$ respectively the minimum element and the maximum element. Then, the poset $\text{St}(P)$ is:*

- *bounded with minimum element $(\{\hat{0}\}, \hat{1})$ and maximum element $\hat{1}_{\text{St}}$;*
- *graded by the rank function*

$$\rho_{(c,y),(c',y')} = \rho_{y'y} + |c'| - |c|,$$

$$\rho_{(c,y),\hat{1}_{\text{St}(P)}} = \rho_{\hat{0}y} - |c| + 2$$

for every $(c, y) \leq (c', y') \in \text{St}(P)$.

Proof. By construction, $\text{St}(P)$ is bounded by $(\{\hat{0}\}, \hat{1})$ and $\hat{1}_{\text{St}(P)}$. To show that $\text{St}(P)$ is graded by the rank function described in the statement, consider a covering relation $(c, y) \lessdot (c', y')$ for some $(c, y), (c', y') \in \text{St}(P)$. We have that either y covers y' in P and $c = c'$, or $y = y'$ and c' covers c in the poset of chains, which means $c \subseteq c'$ and $|c'| = |c| + 1$. In both cases, $\rho_{y'y} + |c'| - |c| = 1$. For a top cover, $|c| = \rho_{\hat{0}y} + 1$. This proves the stated rank formulas. $\square$

The stellahedral transform of a poset might exhibit a subtle behavior even when the starting poset is quite simple: the condition $c_k \leq y$ interlaces the internal structure of P with that of its chains, making $\text{St}(P)$ significantly different from P , its order complex $\Delta(P)$, and their direct product.

Example 4.45. Let A_n be the antichain of n elements. Call P the poset obtained by adjoining a minimum element $\hat{0}$ to A_n . Then $\text{St}(P)$ has the Hasse diagram shown in Figure 4.2.

Example 4.46. Let $P = C_n = \{0, 1, \dots, n\}$ be the chain of rank n . Then $\text{St}(P) \setminus \{\hat{1}_{\text{St}}\}$ is isomorphic to the following subposet of $B_n \times C_n$

$$\{(S, x) \in B_n \times C_n \mid S \neq \emptyset \Rightarrow x \leq n - \max(S)\}$$

where $B_n = \mathcal{P}([n])$ is the boolean lattice of rank n .

The construction of the stellahedral transform of a poset can be reinterpreted using the concept of a *Bier poset*, originally defined in [14].

Definition 4.47. Let P be a bounded poset of finite length and let $I \subseteq P$ be a proper ideal. Then the poset $\text{Bier}(P, I)$ is defined as

$$\text{Bier}(P, I) = \{[x, y] \subseteq P \mid x \in I, y \notin I\} \sqcup \{\hat{1}_{\text{Bier}}\}$$

by imposing $\hat{1}_{\text{Bier}}$ as the maximum and ordering the other elements by reverse inclusion.

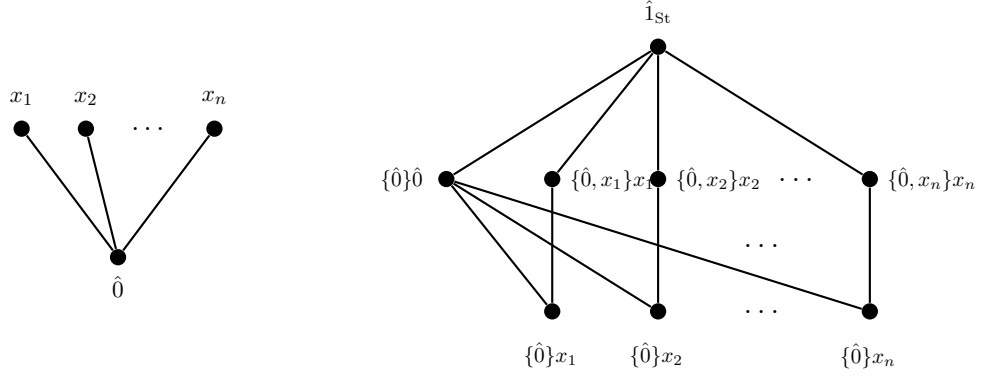


Figure 4.2: On the left, the Hasse diagram of $P = A_n \sqcup \{\hat{0}\}$; on the right, the Hasse diagram of $\text{St}(P)$.

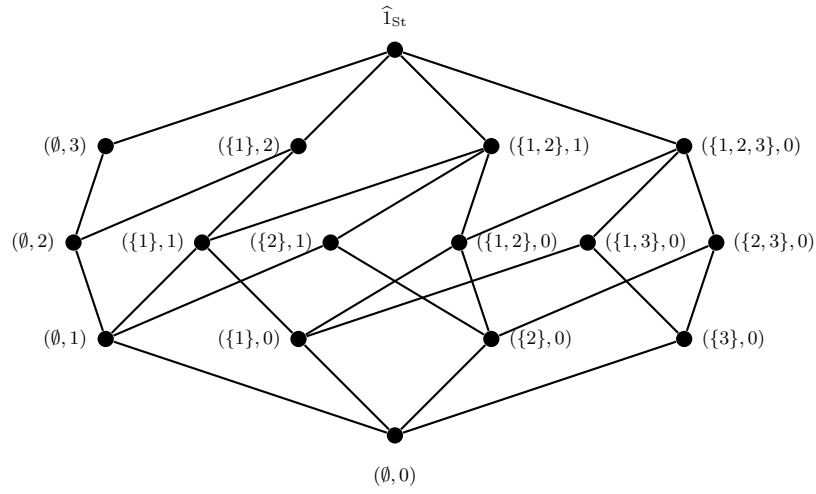


Figure 4.3: Hasse diagram of $\text{St}(C_3)$.

Given a bounded poset P , let $I_P := \{\{\hat{0} = y_0 < y_1 < \cdots < y_r\} \text{ chain in } P\}$ be the set of all chains of P anchored at the bottom element $\hat{0}$. This is naturally a poset using the partial order given by set inclusion. We now define an intermediate construction that will turn out to be useful in our discussion.

Definition 4.48. Let P be a bounded poset. The *chain lift* of P is the poset on the set $P^\uparrow := I_P \sqcup P$, with the order given by

$$p \leq_\uparrow q \iff \begin{cases} p \subseteq q & p, q \in I_P, \\ p \leq q & p, q \in P, \\ \max p \leq q & p \in I_P, q \in P. \end{cases}$$

Notice that if P is a graded poset, then the chain lift $P^\uparrow$ is graded by

$$\rho_{p,q}^\uparrow = \begin{cases} |q| - |p| & \text{if } p, q \in I_P, \\ \rho_{p,q} & \text{if } p, q \in P, \\ \rho_{\hat{0},q} - |p| + 2 & \text{if } p \in I_P \text{ and } q \in P. \end{cases}$$

From this setup, we obtain the following identity

$$\text{St}(P) = \text{Bier}(P^\uparrow, I_P). \quad (4.8)$$

We now prove some properties of $P^\uparrow$. Then, using known results on Bier posets, it will be possible to transfer them to the stellahedral transform $\text{St}(P)$.

Lemma 4.49. *If P is an Eulerian poset, then $P^\uparrow$ is also Eulerian.*

Proof. Let $J = [p, q] \subseteq P^\uparrow$ be an interval. We prove that $[p, q]$ has the same number of elements of even and odd rank when $p < q$. We divide the proof into four cases.

1. If $p, q \in P \subseteq P^\uparrow$, then $J \subseteq P$ is Eulerian because P is Eulerian.
2. If $p, q \in I_P$, then J is isomorphic to a boolean lattice, which is Eulerian.
3. If $p \in I_P$, $q \in P$ and $\max p < q$, consider the map

$$\begin{aligned} f : J \cap I_P &\rightarrow J \cap I_P \\ S &\mapsto S \cup \{q\} \quad \text{if } q \notin S \\ S &\mapsto S \setminus \{q\} \quad \text{if } q \in S. \end{aligned}$$

Notice that f is a rank parity reversing involution, hence $J \cap I_P$ has the same number of elements of even and odd rank. Since $J \cap P \cong [\max p, q]$, J has also the same number of elements of even and odd rank.

4. If $p \in I_P$, $q \in P$ and $\max p = q$, then

$$J = \{c \in I_{[\hat{0}, q]} \mid p \subseteq c\} \sqcup \{q\}.$$

Let C_q be the poset of nonempty chains of $(\hat{0}, q)$. If $|p| > 2$, J is isomorphic to the interval $[p \setminus \{\hat{0}, q\}, \hat{1}]_{C_q \sqcup \{\hat{0}, \hat{1}\}}$. Similarly, if $|p| = 2$, the poset J is isomorphic to $C_q \sqcup \{\hat{0}, \hat{1}\}$. In both cases, we may conclude by Exercise 3.141(c) of [47] that J is Eulerian. Otherwise, if $|p| = 1$ (equiv. $\hat{0} = q$), then J is the unique Eulerian poset of rank 1. $\square$

From now on, we assume that P is an Eulerian poset. All the KLS functions, Chow functions and augmented Chow functions that we will consider are those associated to the Eulerian kernel.

The following key result characterizes particular values of the left KLS function of $\text{St}(P)$ via the right KLS functions of P . For the remainder of this subsection, let $\tilde{\rho}$ denote the rank function of $\text{St}(P)$, and let $\tilde{\varepsilon}$ be the Eulerian kernel in its incidence algebra. Correspondingly, we will use $\tilde{g}$ and $\tilde{h}$ to represent the toric g - and h -functions of $\text{St}(P)$.

Proposition 4.50. *For any $(c, y) \leq (c', y') < \hat{1}_{\text{St}(P)}$ in $\text{St}(P)$, we have the following formula:*

$$\tilde{g}_{(c, y), (c', y')}(x) = f_{y'y}(x),$$

where f is the right KLS function of P .

Proof. We start by defining the function:

$$z_{(c, y), (c', y')}(x) = \begin{cases} f_{y'y}(x) & \text{if } (c', y') \neq \hat{1}_{\text{St}(P)} \\ \tilde{g}_{(c, y), \hat{1}_{\text{St}(P)}}(x) & \text{otherwise} \end{cases}$$

KLS functions are determined interval by interval. By Theorem 4.8, in order to show the validity of our formula, it suffices to show that z satisfies the following three properties on intervals ending below the top:

$$(i') \quad z_{(c,y),(c,y)} = 1,$$

$$(ii') \quad \deg z_{(c,y),(c',y')}(x) < \frac{1}{2} \tilde{\rho}_{(c,y),(c',y')} \text{ for any } (c,y) < (c',y'), \text{ and}$$

$$(iii') \quad z^{\text{rev}} = z \cdot \tilde{\varepsilon} \text{ whenever } (c',y') < \hat{1}_{\text{St}(P)}.$$

Note that (i') is immediate from the fact that f is the KLS function of P , hence $z_{(c,y),(c,y)}(x) = f_{yy}(x) = 1$.

Similarly, (ii') follows again from the fact that f is the KLS function of P . Suppose $(c,y) < (c',y')$. We have $\deg(z) = \deg(f_{y'y}(x))$. If $y' < y$ we have

$$\deg(f_{y'y}(x)) < \frac{1}{2} \rho_{y'y} \leq \frac{1}{2} \tilde{\rho}_{(c,y),(c',y')}.$$

On the other hand if $y' = y$ we must have $c \subsetneq c'$, so $|c'| - |c| \geq 1$ and we have

$$\deg(f_{yy}(x)) = 0 < \frac{|c'| - |c|}{2} = \frac{1}{2} \tilde{\rho}_{(c,y),(c',y')}.$$

Now we prove (iii'):

$$\begin{aligned} (z \cdot \tilde{\varepsilon})_{(c,y),(c',y')} &= \sum_{(d,w) \in [(c,y),(c',y')]} z_{(c,y),(d,w)}(x) \cdot \tilde{\varepsilon}_{(d,w),(c',y')}(x) \\ &= \sum_{(d,w) \in [(c,y),(c',y')]} f_{wy}(x) (x-1)^{\rho_{y'w} + |c'| - |d|} \\ &= \sum_{w \in [y',y]} \sum_{d \in [c,c']} f_{wy}(x) (x-1)^{\rho_{y'w} + |c'| - |d|} \\ &= \left(\sum_{w \in [y',y]} f_{wy}(x) (x-1)^{\rho_{y'w}} \right) \left(\sum_{d \in [c,c']} (x-1)^{|c'| - |d|} \right). \end{aligned}$$

The first factor is the convolution $(\varepsilon \cdot f)_{y'y}$, which equals $f_{y'y}^{\text{rev}}(x)$ since f is the right KLS function of P . The second factor is the convolution of ζ and the Eulerian kernel in the boolean poset $[c, c']$, which equals ζ^{rev} in the boolean poset, evaluating to $x^{|c'| - |c|}$. Thus,

$$(\varepsilon \cdot f)_{y'y} x^{|c'| - |c|} = x^{\rho_{y'y} + |c'| - |c|} f_{y'y}(x^{-1}) = z_{(c,y),(c',y')}^{\text{rev}}. \quad \square$$

Now, having the above formula for the left KLS function of $\text{St}(P)$, we can prove the following theorem.

Theorem 4.51. *Let P be an Eulerian poset. The right augmented Chow polynomial of P equals the toric h -polynomial of $\text{St}(P)$:*

$$F_P(x) = h_{\text{St}(P)}(x).$$

Proof. Let n be the rank of P . By the definition of the toric h -polynomial, we have

$$h_{\text{St}(P)}(x) = \frac{\tilde{g}_{(\{\hat{0}\}, \hat{1}), \hat{1}_{\text{St}(P)}}^{\text{rev}} - \tilde{g}_{(\{\hat{0}\}, \hat{1}), \hat{1}_{\text{St}(P)}}}{x-1}.$$

Since $\tilde{g}^{\text{rev}} = \tilde{g} \cdot \tilde{\varepsilon}$, we can write

$$h_{\text{St}(P)}(x) = \frac{1}{x-1} \sum_{(\{\hat{0}\}, \hat{1}) \leq (d,y) < \hat{1}_{\text{St}(P)}} \tilde{g}_{(\{\hat{0}\}, \hat{1}), (d,y)}(x) (x-1)^{\tilde{\rho}_{(d,y), \hat{1}_{\text{St}(P)}}}.$$

From Lemma 4.44, we know that the co-rank of an arbitrary element (d, y) is

$$\tilde{\rho}_{(d,y), \hat{1}_{\text{St}(P)}} = \rho_{\hat{0},y} - |d| + 2,$$

hence using Proposition 4.50 we have that the toric h -polynomial of $\text{St}(P)$ is

$$\sum_{(\{\hat{0}\}, \hat{1}) \leq (d,y) < \hat{1}_{\text{St}(P)}} f_{y\hat{1}}(x)(x-1)^{\rho_{\hat{0},y}-|d|+1}.$$

We can split the sum by isolating the case $y = \hat{0}$. If $y = \hat{0}$, the only valid chain is $d = \{\hat{0}\}$, so $|d| = 1$ and $\rho_{\hat{0}\hat{0}} = 0$. The corresponding term evaluates to $f_{\hat{0}\hat{1}}(x)$. For $y > \hat{0}$, we split the sum over chains d bounded by y and containing $\hat{0}$ into two sets: those that do not contain y and those that contain y . Let $c = d \setminus \{\hat{0}, y\}$, so c is a face in the order complex $\Delta(\hat{0}, y)$. If $y \notin d$, then $|d| = |c| + 1$; if $y \in d$, then $|d| = |c| + 2$. Thus, we have

$$\begin{aligned} h_{\text{St}(P)}(x) &= f_{\hat{0}\hat{1}}(x) + \sum_{y > \hat{0}} f_{y\hat{1}}(x) \sum_{c \in \Delta(\hat{0}, y)} \left[(x-1)^{\rho_{\hat{0},y}-|c|} + (x-1)^{\rho_{\hat{0},y}-|c|-1} \right] \\ &= f_{\hat{0}\hat{1}}(x) + \sum_{y > \hat{0}} f_{y\hat{1}}(x) \sum_{c \in \Delta(\hat{0}, y)} x(x-1)^{\rho_{\hat{0},y}-|c|-1}. \end{aligned}$$

Using the definition of the h -polynomial of simplicial complexes (where $\dim \Delta(\hat{0}, y) = \rho_{\hat{0},y} - 2$) we can write:

$$h_{\text{St}(P)}(x) = f_{\hat{0}\hat{1}}(x) + \sum_{y > \hat{0}} f_{y\hat{1}}(x) x h_{\Delta(\hat{0}, y)}(x) = f_{\hat{0}\hat{1}}(x) + \sum_{y > \hat{0}} f_{y\hat{1}}(x) x H_{\hat{0}, y}(x).$$

Now we use that $xH_{s,t}(x) = H_{s,t}^{\text{rev}}(x)$ for every $s < t$, as proved during the proof of Proposition 3.6 of [27]

$$h_{\text{St}(P)}(x) = f_{\hat{0}\hat{1}}(x) + \sum_{y > \hat{0}} f_{y\hat{1}}(x) H_{\hat{0}, y}^{\text{rev}}(x) = \sum_{y \in P} f_{y\hat{1}} H_{\hat{0}, y}^{\text{rev}},$$

and then we conclude using [27, Prop. 3.6 and Prop. 3.15]

$$h_{\text{St}(P)}(x) = (H^{\text{rev}} \cdot f)_P = (H \cdot \varepsilon \cdot f)_P = (H \cdot f^{\text{rev}})_P = F_P. \quad \square$$

The stellahedral transform also preserves the property of being the face lattice of a polytope. The following discussion aims to construct the operation on polytopes that corresponds to the stellahedral transform on their respective face posets.

Given a polytope $\mathcal{P}$ at a non-empty proper face F is the polytope defined by taking the convex hull of $\mathcal{P}$ and v , where $v \in \mathbb{R}^n$ is any point that is beyond F . Although the resulting polytope depends on the choice of v , its face poset does not. Combinatorially, it is obtained by removing the proper faces containing F and replacing them with the closure of the set

$$\{v \star G \mid \exists H \in \mathcal{F}(\mathcal{P}) \setminus \{F\} : F, G \leq H, F \not\leq G\}.$$

Definition 4.52. Let $\mathcal{P}$ be a polytope. Up to combinatorial equivalence, we call the *dual stellahedral transform* of $\mathcal{P}$ the polytope $\text{DStell}(\mathcal{P})$ that is constructed by first taking the pyramid over $\mathcal{P}$ and then applying a stellar subdivision to every proper face strictly containing the apex of the pyramid, in order of decreasing dimension.

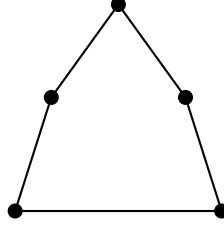


Figure 4.4: Dual stellahedral transform of the 1-simplex.

The following theorem connects the previous construction to the stellahedral transform of posets.

Theorem 4.53. *Let $\mathcal{P}$ be a polytope and S be the poset of pairs $(\mathcal{C}, F)$ where*

- (i) $F \in \mathcal{F}(\mathcal{P})$ is a face of $\mathcal{P}$,
- (ii) $\mathcal{C} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m\}$ is a (possibly empty) strict chain in $\mathcal{F}(\mathcal{P}) \setminus \{\mathcal{P}\}$,
- (iii) If $\mathcal{C} \neq \emptyset$, then $F \subseteq F_1$,

such that

$$(\mathcal{C}, F) \leq (\mathcal{C}', F') \iff F \subseteq F' \text{ and } \mathcal{C} \subseteq \mathcal{C}'.$$

If we add a maximum element $\hat{1}_S$ to S , the face poset of $\text{DStell}(\mathcal{P})$ is isomorphic to $S \sqcup \{\hat{1}_S\}$. In particular, there is an isomorphism given by the function

$$\begin{aligned} \Phi : S \sqcup \{\hat{1}_S\} &\rightarrow \mathcal{F}(\text{DStell}(\mathcal{P})) \\ (\mathcal{C}, F) &\mapsto \text{conv}(F, v_{F_1}, \dots, v_{F_m}) \\ \hat{1}_S &\mapsto \hat{1}_{\text{DStell}(\mathcal{P})} \end{aligned}$$

where $\mathcal{C} = \{F_1 \subsetneq \cdots \subsetneq F_m\}$ and F is seen as embedded in the base of $\text{DStell}(\mathcal{P})$. Here, $v_{\hat{0}}$ denotes the apex of the pyramid over $\mathcal{P}$, while v_G denotes the vertex introduced when performing the stellar subdivision of the face $v_{\hat{0}} \star G \leq \text{Pyr}(\mathcal{P})$ for every nonempty proper face G of $\mathcal{P}$.

The map Φ induces a labeling of the faces of the dual stellahedral transform. In particular, the vertices of $\text{DStell}(\mathcal{P})$ and the rank-1 elements of S are such that

- the vertices v of $\mathcal{P} \subseteq \text{DStell}(\mathcal{P})$ are in bijection with the pairs $(\emptyset, v)$ for all vertices $v \in \mathcal{P}$;
- the apex $v_{\hat{0}}$ of the pyramid over $\mathcal{P}$ is the image of the element $(\{\hat{0}\}, \hat{0})$;
- the vertices v_F are in bijection with the pairs $(\{F\}, \hat{0})$ for all faces $F \in \mathcal{F}(\mathcal{P}) \setminus \{\mathcal{P}\}$.

If $\mathcal{P}$ is a square, we may see in Figure 4.5 its vertex labels.

The dual stellahedral transform of a polytope always contains a face that is affinely equivalent to the polytope itself, since we do not subdivide any face of the base of the pyramid. In Figure 4.6, we may see some labels of faces of $\text{DStell}(\mathcal{P})$. Notice that the faces that do not intersect the base are all simplices.

Proof of Theorem 4.53. In order to prove that Φ is an isomorphism, it is sufficient to show that it is well-defined, bijective, and order-reflecting, since it is order-preserving by construction.

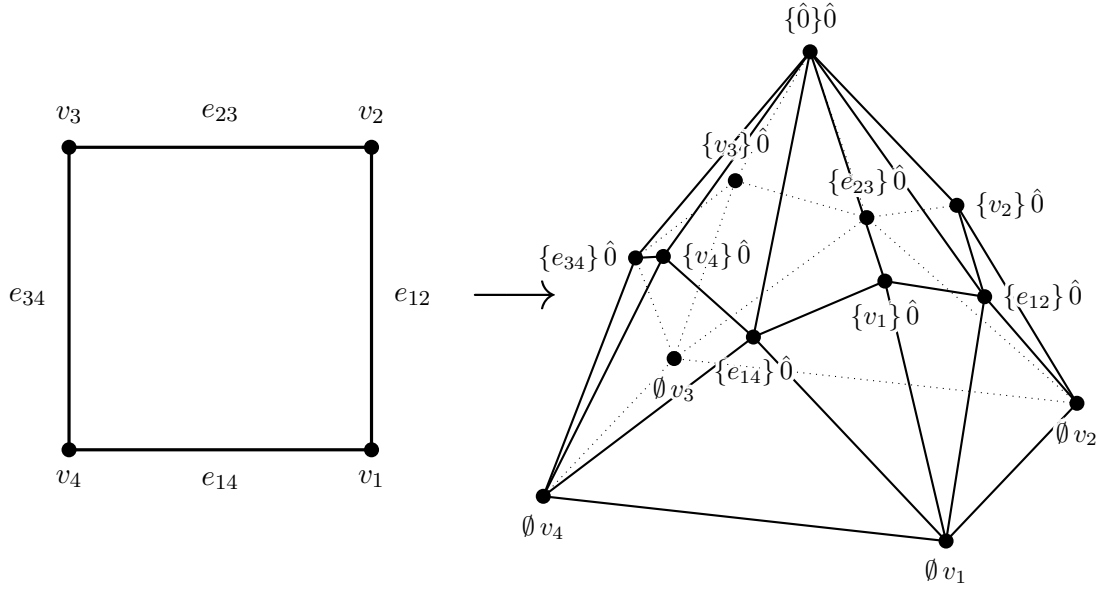


Figure 4.5: Vertices of the dual stellated transform of a square.

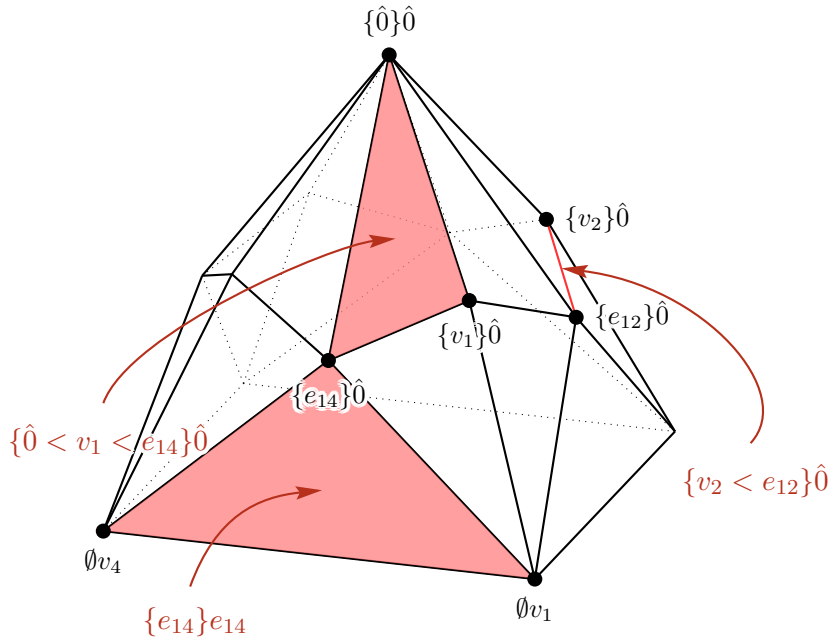


Figure 4.6: Faces of the dual stellated transform of a square.

Let Q be any face of $\text{DStell}(\mathcal{P})$ that is disjoint from the base $\mathcal{P}$. If one performs stellar subdivisions on all nonempty proper faces of the pyramid $\text{Pyr}(\mathcal{P})$ in decreasing order of dimension, one would obtain a polytope whose boundary complex is barycentric subdivision of the boundary complex of $\text{Pyr}(\mathcal{P})$. In the construction of $\text{DStell}(\mathcal{P})$ we only skip subdivisions of faces contained in the base, hence the poset structure of the faces disjoint from $\mathcal{P}$ does not change. Consequently, every face of $\text{DStell}(\mathcal{P})$ disjoint from the base $\mathcal{P}$ is a simplex of the form

$$v_{F_1} \star \cdots \star v_{F_m}$$

where $F_1 \subsetneq \cdots \subsetneq F_m \neq \hat{1}$ is a chain of faces of $\mathcal{P}$. It follows that Φ is well-defined on the pairs $(\mathcal{C}, F) \in S$ such that $F = \hat{0}$, which are in bijection with the faces that are disjoint from the base since Φ restricts to the function

$$(\{F_1 \subsetneq \cdots \subsetneq F_m\}, \hat{0}) \mapsto v_{F_1} \star \cdots \star v_{F_m}.$$

Now, consider any proper face Q of $\text{DStell}(\mathcal{P})$ that does intersect the base $\mathcal{P}$. Every proper face containing both the apex and a proper face of $\mathcal{P}$ has been subdivided during the construction, so Q cannot contain the apex $v_{\hat{0}}$. Therefore, Q is the convex hull of a face $F \in \mathcal{F}(\mathcal{P})$ and a (possibly empty) set of vertices $v_{G_1}, \dots, v_{G_m}$, for some non-empty proper faces $G_1, \dots, G_m \in \mathcal{F}(\mathcal{P})$. Notice that all the faces of $\text{Pyr}(\mathcal{P})$ not contained in the base have been removed and replaced with faces of the form $v_G \star H$ where G is a proper face of $\mathcal{P}$, and H is a proper face of a face H' containing $v_{\hat{0}} \star G$ but that does not itself contain $v_{\hat{0}} \star G$. We observe that H' must satisfy $H' \cap \mathcal{P} = (v_{\hat{0}} \star G) \cap \mathcal{P} = G$. Otherwise, during the subdivision step of $v_{\hat{0}} \star G$, the intermediate polytope of the construction would contain as a subset the face $v_{\hat{0}} \star G'$ for some $G \subsetneq G'$, which is impossible since the order of the subdivisions is by decreasing dimension. For this reason, we must have $F \subseteq G_i$ for every $i \in [m]$. Assume without loss of generality that $v_{G_1}, \dots, v_{G_m}$ are ordered such that $i < j$ if and only if v_{G_i} has been introduced later than v_{G_j} in a subdivision step. We prove by induction on m that $G_1 \subsetneq \cdots \subsetneq G_m$. If $m = 1$, the statement is trivially true. If $m > 1$, we notice that the face Q must be $v_{G_1} \star H$ where H is a face of $\text{DStell}(\mathcal{P})$ and in particular it is $H = v_{G_2} \star \cdots \star v_{G_m} \star F$ such that $G_2 \subsetneq \cdots \subsetneq G_m$ by the induction hypothesis. This implies that there exists an edge of $\text{DStell}(\mathcal{P})$ between v_{G_1} and v_{G_2} , from which it follows that $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m$ by the previous discussion on faces that do not intersect the base. This implies that $Q = \Phi(\{G_1 \subsetneq \cdots \subsetneq G_m\}, F)$. Furthermore, we have that for any chain of faces $\emptyset \neq F \subseteq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m \subsetneq \mathcal{P}$ in $\mathcal{F}(\mathcal{P})$ with $m \geq 1$, the set $Q = \text{conv}(F, v_{G_1}, \dots, v_{G_m})$ is a face of $\text{DStell}(\mathcal{P})$. Indeed, let

$$Q_j := (v_{\hat{0}} \star G_j) \star v_{G_{j+1}} \star \cdots \star v_{G_m}$$

for any $j = 1, \dots, m$. For $j = m$, the set $Q_m = v_{\hat{0}} \star G_m$ is a face of $\text{Pyr}(\mathcal{P})$, before having applied any stellar subdivision. For $j < m$, we see by induction that Q_j is a face of the intermediate polytope obtained during the construction of $\text{DStell}(\mathcal{P})$, just after applying a stellar subdivision to the face $v_{\hat{0}} \star G_{j+1}$. The set Q is obtained as a face of $\text{DStell}(\mathcal{P})$ from $Q_1 = (v_{\hat{0}} \star G_1) \star v_{G_2} \star \cdots \star v_{G_m}$ after a stellar subdivision of the face $v_{\hat{0}} \star G_1$, and remains unchanged during the rest of the construction of $\text{DStell}(\mathcal{P})$. Hence, we have a bijection between the faces that intersect the base and the pairs $(\mathcal{C}, F)$ of S such that $F \neq \emptyset$. Ultimately, we have that Φ is a bijection such that

$$\begin{aligned} \Phi : S \sqcup \{\hat{1}_S\} &\rightarrow \mathcal{F}(\text{DStell}(\mathcal{P})) \\ \{(\mathcal{C}, F) \in S \mid F = \emptyset\} &\leftrightarrow \{F \in \mathcal{F}(\text{DStell}(\mathcal{P})) \mid F \cap \mathcal{P} = \emptyset\} \\ \{(\mathcal{C}, F) \in S \mid F \neq \emptyset\} &\leftrightarrow \{F \in \mathcal{F}(\text{DStell}(\mathcal{P})) \mid F \cap \mathcal{P} \neq \emptyset\} \setminus \{\text{DStell}(\mathcal{P})\} \\ \hat{1}_S &\leftrightarrow \hat{1}_{\text{DStell}(\mathcal{P})}, \end{aligned}$$

Face inclusion is order-reflecting by vertex labels, which concludes the proof. $\square$

The previous theorem proves that the stellahedral transform preserves the property of a poset of being the face poset of a polytope. In particular, we have the following corollary.

Corollary 4.54. *Let $\mathcal{P}$ be a polytope. Then $\mathcal{F}(\text{DStell}(\mathcal{P})) \cong \text{St}(\mathcal{F}(\mathcal{P}^*))$.*

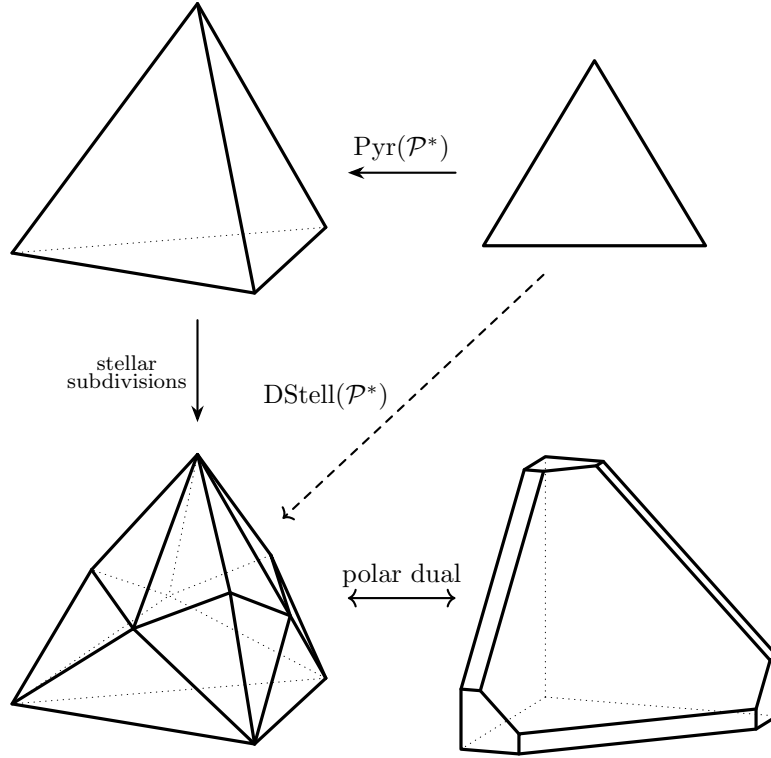


Figure 4.7: Construction, up to polar duality, of the 3-dimensional stellahedron from a 2-simplex.

The reason for choosing the name *stellahedral transform* becomes transparent from the previous corollary taking the case of a simplex. Up to polar duality, the procedure described in Corollary 4.54 sends simplices of dimension r to stellahedra of dimension $r + 1$. Figure 4.7 depicts the construction when $r = 2$; here we use that the polar dual of a simplex is again a simplex. In the following example, we look closer at the case of simplexes.

Example 4.55. Let B_n the Boolean poset on $[n]$. The Eulerian left and right KLS functions of B_n are equal to ζ , by Theorem 4.30 (which also implies that the Eulerian kernel is equal to the characteristic kernel). Since the Eulerian Chow function of B_n is the Eulerian polynomial $A_n(x)$ by Example 4.42, we have

$$\begin{aligned} G_{B_n}(x) &= \sum_{u \in B_n} g_{0u}^{\text{rev}}(x) H_{u\hat{1}}(x) \\ &= \sum_{i=0}^r \binom{n}{i} A_{n-i}(x) x^i. \end{aligned}$$

By Theorem 4.51 and Corollary 4.54 we have that $G_{B_n}(x)$ is the toric h -polynomial of the dual of the stellahedron. This polynomial is also known as the n -th *binomial Eulerian polynomial* [44].

By [27, Theorem 1.3], for the characteristic kernel, Chow and left augmented Chow functions are known to be nonnegative (and, in fact, also unimodal). It is thus natural to wonder if the same property holds for Chow and augmented Chow polynomials in the Eulerian case. For Eulerian Chow polynomials, this was already raised by Ferroni, Matherne, and Vecchi in [27, Question 5.5]. The answer to both questions is that they

may have negative coefficients without further conditions on the poset.

Theorem 4.56. *There exist Eulerian posets whose Chow polynomial and augmented Chow polynomials have negative coefficients.*

The specific question for Chow polynomials was also raised in MathOverflow over two years ago¹ receiving no answers.

Example 4.57. We make use of the constructions and the notation of Bayer and Heteyi in [8]. Let P be the horizontal double of $P(n, \mathcal{I}, N)$ for $n = 8$, $\mathcal{I} = \{[1, 2], [3, 6], [5, 8]\}$ and $N = 19$. Since $\mathcal{I}$ is an even interval system on $[1, n]$, P is an Eulerian poset. One may calculate that its Eulerian Chow polynomial is

$$1 + 1664x + 128620x^2 + 810560x^3 - 125786x^4 + 810560x^5 + 128620x^6 + 1664x^7 + x^8$$

and its right augmented Chow polynomial is

$$1 + 1702x + 136410x^2 + 1182162x^3 - 1108x^4 - 1108x^5 + 1182162x^6 + 136410x^7 + 1702x^8 + x^9.$$

Both of the above polynomials have negative coefficients.

§4.2.4 Relation to the cd -Index

In this subsection, we will calculate the Eulerian KLS invariants that we have presented in the previous discussion, by employing similar techniques to those used by Bayer and Ehrenborg for the toric h -polynomial.

In order to do that, we recall from Section 3.1.2 that we have defined a linear function $g : \mathbb{Q}\langle c, d \rangle \rightarrow \mathbb{Q}[x]$ such that, for any Eulerian poset P , the polynomial $g(\Phi_P(c, d))$ equals the toric g -polynomial of P . We constructed some polynomials to determine the value of this function.

Set $p(n, k) := \binom{n}{k} - \binom{n}{k-1}$. Given a positive integer n ,

$$\begin{aligned} Q_n(x) &:= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k p(n-1, k) x^k, \\ R_n(x) &:= x^{n-1} Q_n(x^{-1}). \end{aligned}$$

For odd n , we also define

$$T_n(x) := (-1)^{(n-1)/2} p(n-1, (n-1)/2) x^{(n-1)/2}$$

while for even n we let $T_n(x) := 0$.

For any n positive integer, and for any cd -monomials $\mathbf{u}$ and $\mathbf{v}$, this map satisfies the following identities

$$g(c^n) = Q_{n+1}(x), \quad g(d^n) = x^n, \quad g(c^n d) = x T_{n+1}(x), \quad g(\mathbf{u} d \mathbf{v}) = g(\mathbf{u} d) g(\mathbf{v}).$$

These results will be used throughout the entire discussion.

¹See <https://mathoverflow.net/questions/475156/eulerian-posets-and-order-complexes>.

Eulerian $\mathcal{Z}$ -Polynomial

For any ab -monomial $\mathbf{u} \in \mathbb{Q}\langle a, b \rangle$, let $\mathcal{Z}, \bar{g} : \mathbb{Q}\langle a, b \rangle \rightarrow \mathbb{Q}[x]$ be the linear maps defined by

$$\begin{aligned}\bar{g}(\mathbf{u}, x) &:= x^{\deg(\mathbf{u})+1}g(\mathbf{v}, x^{-1}), \\ \mathcal{Z}(\mathbf{u}, x) &:= \bar{g}(\mathbf{u}, x) + g(\mathbf{u}^*, x) + \sum_{\mathbf{v}} \bar{g}(\mathbf{v}_{(1)}, x)g(\mathbf{v}_{(2)}^*, x).\end{aligned}$$

In the next proposition we find a recursive identity for $\mathcal{Z}$ for every cd -monomial which is not a power of c .

Proposition 4.58. *Let $\mathbf{u}$ and $\mathbf{v}$ be any cd -monomials. The map $\mathcal{Z}$ satisfies the identity*

$$\mathcal{Z}(\mathbf{u}d\mathbf{v}) = x^{-1}\bar{g}(\mathbf{u}d)\mathcal{Z}(\mathbf{v}) + g(\mathbf{v}^*d)\mathcal{Z}(\mathbf{u}).$$

Proof. Since $g(\mathbf{u}d\mathbf{v}) = g(\mathbf{u}d)g(\mathbf{v})$, by expanding the definition of $\mathcal{Z}$ we get

$$\begin{aligned}\mathcal{Z}(\mathbf{u}d\mathbf{v}) &= \bar{g}(\mathbf{u}d\mathbf{v}) + g(\mathbf{v}^*d\mathbf{u}^*) + \sum_{\mathbf{u}} \bar{g}(\mathbf{u}_{(1)})g(\mathbf{v}^*d\mathbf{u}_{(2)}^*) + \bar{g}(\mathbf{u}c)g(\mathbf{v}^*) + \bar{g}(\mathbf{u})g(\mathbf{v}^*c) \\ &\quad + \sum_{\mathbf{v}} \bar{g}(\mathbf{u}d\mathbf{v}_{(1)})g(\mathbf{v}_{(2)}^*) \\ &= x^{-1}\bar{g}(\mathbf{u}d)\bar{g}(\mathbf{v}) + g(\mathbf{v}^*d)g(\mathbf{u}^*) + \sum_{\mathbf{u}} \bar{g}(\mathbf{u}_{(1)})g(\mathbf{v}^*d)g(\mathbf{u}_{(2)}^*) + \bar{g}(\mathbf{u}c)g(\mathbf{v}^*) \\ &\quad + \bar{g}(\mathbf{u})g(\mathbf{v}^*c) + x^{-1}\sum_{\mathbf{v}} \bar{g}(\mathbf{u}d)\bar{g}(\mathbf{v}_{(1)})g(\mathbf{v}_{(2)}^*).\end{aligned}$$

By collecting the terms which are multiples of $x^{-1}\bar{g}(\mathbf{u}d)$ and of $g(\mathbf{v}^*d)$, we may also rewrite it as

$$\begin{aligned}\mathcal{Z}(\mathbf{u}d\mathbf{v}) &= x^{-1}\bar{g}(\mathbf{u}d)(\mathcal{Z}(\mathbf{v}) - g(\mathbf{v}^*)) + g(\mathbf{v}^*d)(\mathcal{Z}(\mathbf{u}) - \bar{g}(\mathbf{u})) + \bar{g}(\mathbf{u}c)g(\mathbf{v}^*) + g(\mathbf{v}^*c)\bar{g}(\mathbf{u}) \\ &= x^{-1}\bar{g}(\mathbf{u}d)\mathcal{Z}(\mathbf{v}) + g(\mathbf{v}^*d)\mathcal{Z}(\mathbf{u}) \\ &\quad + (\bar{g}(\mathbf{u}c) - x^{-1}\bar{g}(\mathbf{u}d))g(\mathbf{v}^*) + (g(\mathbf{v}^*c) - g(\mathbf{v}^*d))\bar{g}(\mathbf{u})\end{aligned}$$

Since $g(\mathbf{v}d) = (x-1)g(\mathbf{v}) + g(\mathbf{v}c)$, we also have $\bar{g}(\mathbf{v}d) = x(1-x)\bar{g}(\mathbf{v}) + x\bar{g}(\mathbf{v}c)$, which implies that

$$\begin{aligned}\mathcal{Z}(\mathbf{u}d\mathbf{v}) &= x^{-1}\bar{g}(\mathbf{u}d)\mathcal{Z}(\mathbf{v}) + g(\mathbf{v}^*d)\mathcal{Z}(\mathbf{u}) - (1-x)\bar{g}(\mathbf{u})g(\mathbf{v}^*) + (1-x)\bar{g}(\mathbf{u})g(\mathbf{v}^*) \\ &= x^{-1}\bar{g}(\mathbf{u}d)\mathcal{Z}(\mathbf{v}) + g(\mathbf{v}^*d)\mathcal{Z}(\mathbf{u}).\end{aligned}$$

□

As a consequence of this proposition, for any cd -word $\mathbf{w} = c^{k_1}dc^{k_2}d \dots c^{k_{r-1}}dc^{k_r}$, the polynomial $\mathcal{Z}(\mathbf{w}, x)$ satisfies

$$\begin{aligned}\mathcal{Z}(c^{k_1}dc^{k_2}d \dots c^{k_{r-1}}dc^{k_r}) &= x^{-1}\bar{g}(c^{k_1}d)\mathcal{Z}(c^{k_2}d \dots c^{k_{r-1}}dc^{k_r}) + g(c^{k_r}d \dots c^{k_3}dc^{k_2}d)\mathcal{Z}(c^{k_1}) \\ &= x^{-1}(x^{k_1+3} \cdot x^{-1}T_{k_1+1}(x^{-1}))\mathcal{Z}(c^{k_2}d \dots c^{k_{r-1}}dc^{k_r}) \\ &\quad + x^{r-1} \prod_{j=2}^r T_{k_j+1}(x)\mathcal{Z}(c^{k_1}) \\ &= xT_{k_1+1}(x)\mathcal{Z}(c^{k_2}d \dots c^{k_{r-1}}dc^{k_r}) + x^{r-1} \prod_{j=2}^r T_{k_j+1}(x)\mathcal{Z}(c^{k_1}).\end{aligned}$$

From this recursion, we derive the following theorem.

Theorem 4.59. *Let P be an Eulerian poset with cd-index $\Phi_P(c, d) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} \mathbf{w}$, then the Eulerian Z -polynomial is given by $Z_P(x) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} \mathcal{Z}(\mathbf{w}, x)$, where for each cd-word $\mathbf{w} = c^{k_1} d c^{k_2} d \dots c^{k_r} d c^{k_{r+1}}$, the polynomial $\mathcal{Z}(\mathbf{w}, x)$ equals*

$$x^r \sum_{i=1}^{r+1} \mathcal{Z}(c^{k_i}) \prod_{\substack{j=1, \dots, r+1 \\ j \neq i}} T_{k_j+1}(x).$$

The only unknown terms in the formula are the values of the family $\{\mathcal{Z}(c^n)\}_n$, the Z -polynomials of the butterfly posets (see the definition in Example 2.29). By expanding the definition of $\mathcal{Z}(c^n)$, we get the expression

$$\begin{aligned} \mathcal{Z}(c^n) &= g(c^n) + \bar{g}(c^n) + 2 \sum_{i=0}^{n-1} \bar{g}(c^i) g(c^{n-1-i}) \\ &= Q_{n+1}(x) + x R_{n+1}(x) + 2x \sum_{i=0}^{n-1} R_{i+1}(x) Q_{n-i}(x). \end{aligned} \quad (4.9)$$

This formula may be made more explicit, as shown in the following proposition.

Proposition 4.60. *The value of the Z -polynomial on the butterfly poset of rank $n+1$ is*

$$\mathcal{Z}(c^n) = \begin{cases} (1+x)^2(1-x)^{n-1} & \text{if } n \text{ is odd} \\ (1+x) \left((1-x)^n + 2x \sum_{i=1}^{n/2} T_{2i-1}(x) (1-x)^{n-2i} \right) & \text{if } n \text{ is even} \end{cases}.$$

Proof. Let $Z(z)$, $T(z)$, $Q(z)$ and $R(z)$ be the ordinary generating functions of $\{\mathcal{Z}(c^n)\}_{n \in \mathbb{N}}$, $\{T_n(x)\}_{n \geq 1}$, $\{Q_n(x)\}_{n \geq 1}$ and $\{R_n(x)\}_{n \geq 1}$ respectively. By equation 4.9, $Z(z)$ equals

$$Z(z) = \frac{Q(z) + xR(z) + 2xQ(z)R(z)}{z}.$$

Recall from Remark 3.14 that

$$Q(z) = \frac{z + xz^2 C(-xz^2)}{1 - z + xz}, \quad R(z) = \frac{z + z^2 C(-xz^2)}{1 + z - xz},$$

where $C(z)$ is the ordinary generating function of Catalan numbers. We claim that

$$Q(z)R(z) = \frac{Q(z) - R(z)}{1 - x}.$$

A simple calculation shows that the ordinary generating function of the Catalan numbers satisfies the identity $C(z) = 1 + zC(z)^2$. Then we may write the product $Q(z)R(z)$ as

$$\begin{aligned} & z^2 \frac{(1 + xzC(-xz^2))(1 + zC(-xz^2))}{1 - (1-x)^2 z^2} \\ &= z^2 \frac{1 + (z + xz)C(-xz^2) + xz^2 C(-xz^2)^2}{1 - (1-x)^2 z^2} \\ &= z^2 \frac{2 + (xz + z - 1)C(-xz^2)}{1 - (1-x)^2 z^2} \end{aligned}$$

By multiplying both the numerator and the denominator by $1 - x$, we obtain

$$z^2 \frac{2 - 2x + (1-x)(xz + z - 1)C(-xz^2)}{(1-x)(1 - (1-x)^2 z^2)}$$

which is equal to

$$\begin{aligned}
 & z \frac{2z - 2xz + (1-x)z(xz + z - 1)C(-xz^2)}{(1-x)(1-(1-x)^2z^2)} \\
 &= \frac{1}{(1-x)} \left(\frac{(1+z(1-x))z(1+xzC(-xz^2))}{1-(1-x)^2z^2} - \frac{(1-z(1-x))z(1+zC(-xz^2))}{1-(1-x)^2z^2} \right) \\
 &= \frac{Q(z) - R(z)}{1-x}.
 \end{aligned}$$

Thus, we may rewrite $Z(z)$ as follows

$$\begin{aligned}
 Z(z) &= \frac{Q(z) + xR(z) + \frac{2x}{1-x}(Q(z) - R(z))}{z} \\
 &= \frac{(1+x)Q(z) - x(1+x)R(z)}{z(1-x)} \\
 &= \frac{1+x}{1-x} \cdot \frac{Q(z) - xR(z)}{z} \\
 &= \frac{1+x}{1-x} \cdot \frac{(1+(1-x)z)(1+xzC(-xz^2)) - x(1-(1-x)z)(1+zC(-xz^2))}{1-(1-x)^2z^2}
 \end{aligned}$$

By using again the identity $C(z) = 1 + zC(z)^2$, we get

$$\begin{aligned}
 Z(z) &= \frac{1+x}{1-x} \cdot \frac{(-x^2z + z - x + 1) + 2x(1-x)z^2C(-xz^2)}{1-(1-x)^2z^2} \\
 &= \frac{1+x}{1-x} \cdot \frac{(1-x)(xz + z + 1) + 2x(1-x)z^2C(-xz^2)}{1-(1-x)^2z^2} \\
 &= (1+x) \cdot \frac{(xz + z + 1) + 2xz^2C(-xz^2)}{1-(1-x)^2z^2}
 \end{aligned}$$

Since $T(z) = zC(-xz^2)$ by Remark 3.12, we conclude

$$Z(z) = \frac{(1+x)^2z}{1-(1-x)^2z^2} + \frac{(1+x)(2xzT(z) + 1)}{1-(1-x)^2z^2}.$$

In the Taylor expansion of $Z(z)$, we have that $\frac{(1+x)^2z}{1-(1-x)^2z^2}$ has all the terms that have odd degree in z , while $\frac{(1+x)(2xzT(z)+1)}{1-(1-x)^2z^2}$ has only even degree terms in z . This implies that for every $k \geq 0$ we have

$$\mathcal{Z}(c^{2k+1}) = [z^{2k+1}] \frac{(1+x)^2z}{1-(1-x)^2z^2} = [z^{2k+1}] \left((1+x)^2z \sum_{h \geq 0} (1-x)^{2h} k z^{2h} \right) = (1+x)^2(1-x)^{2k},$$

and

$$\begin{aligned}
 \mathcal{Z}(c^{2k}) &= [z^{2k}] \frac{(1+x)(2xzT(z) + 1)}{1-(1-x)^2z^2} \\
 &= (1+x)[z^{2k}] \left(\left(1 + 2xz \sum_{h \geq 0} T_{2h+1}(x) z^{2h+1} \right) \sum_{\ell \geq 0} (1-x)^{2\ell} k z^{2\ell} \right) \\
 &= (1+x) \left((1-x)^{2k} + 2x \sum_{i=1}^k T_{2i-1}(x) (1-x)^{2k-2i} \right). \quad \square
 \end{aligned}$$

In the following table, we show the values of $\mathcal{Z}(c^n)$ for $n = 0, 1, \dots, 10$.

n	$\mathcal{Z}(c^n)$
0	$1 + x$
1	$1 + 2x + x^2$
2	$1 + x + x^2 + x^3$
3	$1 - 2x^2 + x^4$
4	$1 - x - 2x^2 - 2x^3 - x^4 + x^5$
5	$1 - 2x - x^2 + 4x^3 - x^4 - 2x^5 + x^6$
6	$1 - 3x + x^2 + 5x^3 + 5x^4 + x^5 - 3x^6 + x^7$
7	$1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8$
8	$1 - 5x + 8x^2 - 14x^4 - 14x^5 + 8x^7 - 5x^8 + x^9$
9	$1 - 6x + 13x^2 - 8x^3 - 14x^4 + 28x^5 - 14x^6 - 8x^7 + 13x^8 - 6x^9 + x^{10}$
10	$1 - 7x + 19x^2 - 21x^3 - 6x^4 + 42x^5 + 42x^6 - 6x^7 - 21x^8 + 19x^9 - 7x^{10} + x^{11}$

Table 4.1: The first values of $\mathcal{Z}(c^n)$.

Corollary 4.61. *Let P be an Eulerian poset with cd-index $\Phi_P(c, d) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} \mathbf{w}$ and even rank $n + 1$. For each cd-word $\mathbf{w} = c^{k_1} d c^{k_2} d \dots c^{k_r} d c^{k_{r+1}}$, then polynomial $Z(\mathbf{w}, x)$ equals 0 if and only if there are two odd exponents k_i and k_j for $i \neq j$.*

Let $\mathbf{w} = c^{k_1} d c^{k_2} d \dots c^{k_r} d c^{k_{r+1}}$ be a cd-word for which the only odd exponent is k_s , for some $s \in [r + 1]$. Then the polynomial $Z(\mathbf{w}, x)$ equals

$$\left((-1)^{(n-k_s)/2} \prod_{\substack{j=1, \dots, r+1 \\ j \neq s}} -C_{k_j/2} \right) x^{(n-k_s)/2} (1+x)^2 (1-x)^{k_s-1},$$

where C_k denotes the k -th Catalan number.

Eulerian Chow Polynomial

In Section 3.1.2, we have defined the map κ given by

$$\begin{aligned} \kappa : \mathbb{Q}\langle a, b \rangle &\rightarrow \mathbb{Q}[x] \\ a &\mapsto x - 1, \\ b &\mapsto 0. \end{aligned}$$

For any ab -monomial $\mathbf{v} \in \mathbb{Q}\langle a, b \rangle$, let $H : \mathbb{Q}\langle a, b \rangle \rightarrow \mathbb{Q}[x]$ be the algebra morphism defined by

$$H(\mathbf{v}) = \kappa(\mathbf{v}) + \sum_{\mathbf{v}} \kappa(\mathbf{v}_{(1)}) H(\mathbf{v}_{(2)}).$$

We show that we can use this function to calculate the Eulerian Chow polynomial of P .

Proposition 4.62. *Let P be an Eulerian poset, $\Psi(P)$ its ab -index and H the Eulerian Chow function of P . Then*

$$H_P(x) = H(\Psi(P)).$$

Proof. We prove it by induction on the rank of P .

If $\text{rank}(P) = 1$, then $H_P(x) = 1 = H(1)$.

Let $\text{rank}(P) = n + 1$ for $n \geq 0$. By the induction hypothesis and equation 4.2, we have

$$\begin{aligned} H_P(x) &= \sum_{\hat{0} < s \leq \hat{1}} \bar{\varepsilon}_{\hat{0}s}(x) H_{s\hat{1}}(x) \\ &= \bar{\varepsilon}_P(x) + \sum_P \bar{\varepsilon}_{P_{(1)}}(x) H_{P_{(2)}}(x) \\ &= \kappa(\Psi(P)) + \sum_P \kappa(\Psi(P_{(1)})) H(\Psi(P_{(2)})). \end{aligned}$$

Since the ab -index is a Newtonian coalgebra morphism, we verify the first identity

$$H_P(x) = \kappa(\Psi(P)) + \sum_P \kappa((\Psi(P))_{(1)}) H((\Psi(P))_{(2)}) = H(\Psi(P)). \quad \square$$

Theorem 4.63. *The map H can be equivalently defined as*

$$\begin{aligned} H : \mathbb{Q}\langle a, b \rangle &\rightarrow \mathbb{Q}[x] \\ a &\mapsto x, \\ b &\mapsto 1. \end{aligned}$$

Hence, the Chow polynomial of an Eulerian poset P is a specialization of its ab -index $\Psi_P(a, b)$, in particular

$$H_P(x) = \Psi_P(x, 1).$$

Proof. Let $\mathbf{v} \in \mathbb{Q}\langle a, b \rangle$ be an ab -monomial. By expanding the definition of κ , we observe that

$$\begin{aligned} H(a\mathbf{v}) &= \kappa(a\mathbf{v}) + \kappa(1)H(\mathbf{v}) + \sum_{\mathbf{v}} \kappa(a\mathbf{v}_{(1)})H(\mathbf{v}_{(2)}) \\ &= (x-1)\kappa(\mathbf{v}) + H(\mathbf{v}) + (x-1) \sum_{\mathbf{v}} \kappa(\mathbf{v}_{(1)})H(\mathbf{v}_{(2)}) \\ &= (x-1)H(\mathbf{v}) + H(\mathbf{v}) \\ &= xH(\mathbf{v}), \end{aligned}$$

and also

$$\begin{aligned} H(b\mathbf{v}) &= \kappa(b\mathbf{v}) + \kappa(1)H(\mathbf{v}) + \sum_{\mathbf{v}} \kappa(b\mathbf{v}_{(1)})H(\mathbf{v}_{(2)}) \\ &= H(\mathbf{v}). \end{aligned}$$

This pair of identities proves the statement. $\square$

Theorem 4.64. *The map H on the cd -monomials is*

$$\begin{aligned} H : \mathbb{Q}\langle c, d \rangle &\rightarrow \mathbb{Q}[x] \\ c &\mapsto 1 + x, \\ d &\mapsto 2x. \end{aligned}$$

Hence, the Chow polynomial of an Eulerian poset P is a specialization of its cd -index $\Phi_P(c, d)$, in particular

$$H_P(x) = \Phi_P(1 + x, 2x).$$

Eulerian Augmented Chow Polynomial

Similarly to the previous KLS invariants, we define a linear map $L : \mathbb{Q}\langle a, b \rangle \rightarrow \mathbb{Q}[x]$ such that for any ab -monomial $\mathbf{v} \in \mathbb{Q}\langle a, b \rangle$ it is defined as

$$L(\mathbf{v}) := H(\mathbf{v}) + \sum_{\mathbf{v}} g(\mathbf{v}_{(1)})H(\mathbf{v}_{(2)}).$$

Proposition 4.65. *Given P be an Eulerian poset, let $\Psi(P)$ be its ab -index. Let G be the left augmented Eulerian Chow function of P and g be the Eulerian left KLS function of P . Then*

$$G_P(x) = g_P(x) + xL(\Psi(P)).$$

Proof. Recall that $G^{\text{rev}} = G$ and $H^{\text{rev}} = xH + (1-x)\delta$, hence by the definition of the augmented Chow function we can write

$$\begin{aligned} (G - g)_P(x) &= (G^{\text{rev}} - g)_P(x) \\ &= (g \cdot H^{\text{rev}} - g)_P(x) \\ &= \left(g \cdot (xH + (1-x)\delta) - g \right)_P(x) \\ &= (xg \cdot H - xg)_P(x) \\ &= x \left(\sum_P g_{P_{(1)}}(x)H_{P_{(2)}}(x) + H_P(x) \right) \\ &= x \left(\sum_P g(\Psi(P_{(1)}))H(\Psi(P_{(2)})) + H(\Psi(P)) \right). \end{aligned}$$

Analogously to the proof of Proposition 4.62, we have

$$(G - g)_P(x) = x \left(\sum_P g(\Psi(P_{(1)}))H(\Psi(P_{(2)})) + H(\Psi(P)) \right) = xL(\Psi(P)). \quad \square$$

In order to find an explicit description of L (and of G), we prove the following lemma.

Lemma 4.66. *Let n be a natural number and let $\mathbf{u}, \mathbf{v}$ be any cd -monomials. The map L satisfies the identities*

$$\begin{aligned} L(c^n) &= (1+x)^n + 2 \sum_{i=0}^{n-1} (1+x)^i Q_{n-i}(x), \\ L(c^n d) &= 2x \left((1+x)^n + 2 \sum_{i=0}^{n-1} (1+x)^i Q_{n-i}(x) \right) + Q_{n+2}(x) + (1+x)Q_{n+1}(x), \\ L(\mathbf{u}d\mathbf{v}) &= (L - g)(\mathbf{u}d)H(\mathbf{v}) + g(\mathbf{u}d)L(\mathbf{v}). \end{aligned}$$

Proof. We start with the first statement. Since $g(c^k) = Q_{k+1}(x)$ for every natural number k , we have

$$\begin{aligned} L(c^n) &= H(c^n) + 2 \sum_{i=0}^{n-1} g(c^{n-1-i})H(c^i) \\ &= (1+x)^n + 2 \sum_{i=0}^{n-1} (1+x)^i Q_{n-i}(x). \end{aligned}$$

Similarly, for the second statement

$$\begin{aligned} L(c^n d) &= H(c^n d) + 2 \sum_{i=0}^{n-1} g(c^{n-1-i}) H(c^i d) + g(c^{n+1}) + (1+x)g(c^n) \\ &= 2x \left((1+x)^n + 2 \sum_{i=0}^{n-1} (1+x)^i Q_{n-i}(x) \right) + Q_{n+2}(x) + (1+x)Q_{n+1}(x). \end{aligned}$$

To prove the third statement, recall that $g(\mathbf{w}_1 d \mathbf{w}_2) = g(\mathbf{w}_1 d)g(\mathbf{w}_2)$ for any pair of cd -monomials $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{Q}\langle c, d \rangle$. Hence

$$\begin{aligned} L(\mathbf{u} d \mathbf{v}) &= H(\mathbf{u} d \mathbf{v}) + \sum_{\mathbf{u}} g(\mathbf{u}_{(1)}) H(\mathbf{u}_{(2)} d \mathbf{v}) + g(\mathbf{u} c) H(\mathbf{v}) + g(\mathbf{u}) H(c \mathbf{v}) \\ &\quad + \sum_{\mathbf{v}} g(\mathbf{u} d \mathbf{v}_{(1)}) H(\mathbf{v}_{(2)}) \\ &= \left((1+x)g(\mathbf{u}) + g(\mathbf{u} c) + 2x \sum_{\mathbf{u}} g(\mathbf{u}_{(1)}) H(\mathbf{u}_{(2)}) + 2x H(\mathbf{u}) \right) H(\mathbf{v}) \\ &\quad + g(\mathbf{u} d) \sum_{\mathbf{v}} g(\mathbf{v}_{(1)}) H(\mathbf{v}_{(2)}) \\ &= g(\mathbf{u} d) (L - H)(\mathbf{v}) + \left((1+x)g(\mathbf{u}) + g(\mathbf{u} c) + 2x L(\mathbf{u}) \right) H(\mathbf{v}) \\ &= g(\mathbf{u} d) (L - H)(\mathbf{v}) + L(\mathbf{u} d) H(\mathbf{v}) \\ &= (L - g)(\mathbf{u} d) H(\mathbf{v}) + g(\mathbf{u} d) L(\mathbf{v}). \end{aligned} \quad \square$$

Using the previous proposition, we can calculate the Eulerian right augmented Chow polynomial of the butterfly poset (see the definition in Example 2.29). Given a natural number k , let

$$\begin{aligned} G_k(x) &:= G(c^k, x) \\ &= x(1+x)^k + 2x \sum_{i=0}^{k-1} (1+x)^i Q_{k-i}(x) + Q_{k+1}(x). \end{aligned}$$

Notice that $G_k(x)$ satisfies the recursion

$$G_k(x) = (1+x)G_{k-1}(x) + xT_k(x), \quad (4.10)$$

for all $k \geq 1$. In fact, we may calculate

$$\begin{aligned} G_k(x) &= xL(c^k) + g(c^k) \\ &= x \left((1+x)H(c^{k-1}) + 2 \sum_{i=0}^{k-1} (1+x)^i Q_{k-i}(x) \right) + Q_{k+1}(x) \\ &= (1+x)x \left(H(c^{k-1}) + 2 \sum_{i=0}^{k-2} (1+x)^i Q_{k-1-i}(x) \right) + 2xQ_k(x) + Q_{k+1}(x) \\ &= (1+x)(G_{k-1}(x) - Q_k(x)) + 2xQ_k(x) + Q_{k+1}(x) \\ &= (1+x)G_{k-1}(x) + (x-1)Q_k(x) + Q_{k+1}(x). \end{aligned}$$

Since we have $Q_{k+1}(x) = (1-x)Q_k(x) + xT_k(x)$ by Proposition 3.13, this implies the desired identity.

Recall $G_0(x) = 1 + x$. By iterating the recursion in equation 4.10, we can derive the formula

$$G_k(x) = (1 + x)^{k+1} + x \sum_{i=0}^{k-1} (1 + x)^i T_{k-i}(x).$$

In the following table, we show the values of $G_k(x)$ for $n = 0, 1, \dots, 8$.

k	$G_k(x)$
0	$1 + x$
1	$1 + 3x + x^2$
2	$1 + 4x + 4x^2 + x^3$
3	$1 + 5x + 7x^2 + 5x^3 + x^4$
4	$1 + 6x + 12x^2 + 12x^3 + 6x^4 + x^5$
5	$1 + 7x + 18x^2 + 26x^3 + 18x^4 + 7x^5 + x^6$
6	$1 + 8x + 25x^2 + 44x^3 + 44x^4 + 25x^5 + 8x^6 + x^7$
7	$1 + 9x + 33x^2 + 69x^3 + 83x^4 + 69x^5 + 33x^6 + 9x^7 + x^8$
8	$1 + 10x + 42x^2 + 102x^3 + 152x^4 + 152x^5 + 102x^6 + 42x^7 + 10x^8 + x^9$

Table 4.2: The first values of $G_k(x)$.

Let n be a natural number, note that the polynomial $L(c^n d, x) - g(c^n d, x)$ equals

$$\begin{aligned} & 2x \left((1 + x)^n + 2 \sum_{i=0}^{n-1} (1 + x)^i Q_{n-i}(x) \right) + Q_{n+2}(x) + (1 + x)Q_{n+1}(x) - xT_{n+1}(x) \\ &= 2 \left(x(1 + x)^n + 2x \sum_{i=0}^{n-1} (1 + x)^i Q_{n-i}(x) \right) + \left((1 - x)Q_{n+1}(x) + xT_{n+1}(x) \right) \\ & \quad + (1 + x)Q_{n+1}(x) - xT_{n+1}(x). \end{aligned}$$

By applying again Proposition 3.13, we calculate

$$\begin{aligned} L(c^n d, x) - g(c^n d, x) &= 2 \left(x(1 + x)^n + 2x \sum_{i=0}^{n-1} (1 + x)^i Q_{n-i}(x) \right) \\ & \quad + Q_{n+2}(x) + (1 + x)Q_{n+1}(x) - xT_{n+1}(x) \\ &= 2 \left(x(1 + x)^n + 2x \sum_{i=0}^{n-1} (1 + x)^i Q_{n-i}(x) \right) + 2Q_{n+1}(x) \\ &= 2G_n(x). \end{aligned}$$

Hence, using Lemma 4.66, we have the following corollary.

Corollary 4.67. *Let k_1 be a natural number and let $\mathbf{v}$ be any cd-monomials. The map G satisfies the recursion*

$$G(c^{k_1} d \mathbf{v}, x) = 2xG_{k_1}(x)H(\mathbf{v}) + xT_{k_1+1}(x)G(\mathbf{v}, x).$$

As a consequence, we get the following theorem.

Theorem 4.68. *Let P be an Eulerian poset with cd-index $\Phi_P(c, d) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} \mathbf{w}$. Then, the left augmented Eulerian Chow polynomial is given by $G_P(x) = \sum_{\mathbf{w}} \xi_{\mathbf{w}} G(\mathbf{w}, x)$, where for each cd-word $\mathbf{w} = c^{k_1} d c^{k_2} d \dots c^{k_r} d c^{k_{r+1}}$, the polynomial $G(\mathbf{w}, x)$ equals*

$$x^r \left(\sum_{i=0}^r 2^{r-i} (1+x)^{\sum_{h \geq 2} k_{i+h}} G_{k_{i+1}}(x) \prod_{j=1}^i T_{k_j+1}(x) \right).$$

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