Anisotropic min-max via phase transitions

Antonio De Rosa and Alessandro Pigati

Abstract

We develop a PDE-based approach to the min-max construction of nontrivial integer rectifiable varifolds that are stationary with respect to anisotropic surface energies on closed Riemannian manifolds, in codimension one. Specifically, we study the anisotropic analogue of the Allen–Cahn energy and establish a Modica-type gradient bound for its critical points. Using this in conjunction with certain estimates for stable solutions, we then prove that the energy densities of stable or bounded-Morse-index critical points of its rescalings concentrate along an integer rectifiable varifold that is stationary for the underlying anisotropic integrand. As a consequence, we construct a (possibly singular) anisotropic min-max hypersurface via Allen–Cahn, obtaining an analogue of the result of Hutchinson–Tonegawa in the anisotropic setting.

1   Introduction

The Allen–Cahn equation has become a central tool in the study of minimal hypersurfaces and related variational problems in geometric analysis. Originally introduced by Allen and Cahn [6] as a diffuse-interface model for phase transitions, this reaction-diffusion equation reads

$$\partial_t u - \Delta u + \frac{W'(u)}{\varepsilon^2} = 0.$$

Here W denotes a double-well potential with global minima at ± 1, the prototypical example being $W(u) = \frac{1}{4}(1 - u^2)^2$. Modica and Mortola [45] observed that the associated energy functional

$$\int \left[\varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right], \tag{1.1}$$

Γ-converges, as the interfacial width parameter $\varepsilon \to 0$, to the area functional of sets of finite perimeter. This observation revealed a deep connection between the Allen–Cahn variational framework and the theory of minimal hypersurfaces in geometric measure theory, as originally suggested by De Giorgi.

In particular, critical points of (1.1), that is, functions u satisfying

$$-\varepsilon \Delta u + \frac{W'(u)}{\varepsilon} = 0,$$

have been shown to concentrate on minimal hypersurfaces as $\varepsilon \to 0$. This phenomenon dates back to the foundational work of Modica [44] and Sternberg [57], and has since developed into a well-established field. We refer the reader to the surveys [46, 52, 59] for an overview.

Building on this idea and on the works of Ilmanen [34], Hutchinson and Tonegawa [33], Tonegawa [58], Tonegawa and Wickramasekera [60], and Wickramasekera [61], Guaraco [32] established an existence theorem for closed minimal hypersurfaces in arbitrary closed Riemannian manifolds by adapting the mountain-pass construction to the Allen–Cahn functional (1.1). His work provided a PDE-based alternative to the Almgren–Pitts min-max theory from [7, 51]. The Almgren–Pitts theory has had a profound impact on geometric analysis and was later refined by Marques and Neves [40, 41] in their proof of the Willmore conjecture, leading also to the recent resolution of Yau's conjecture by Song [54]. On the other hand, the PDE-based min-max theory has since been extended to a variety of settings and proved capable of achieving more refined results which predated analogues in the Almgren–Pitts framework, such as the multiplicity-one result of [14]. Collectively, these developments have established the Allen–Cahn

framework as a flexible and powerful analytic alternative to the Almgren–Pitts one, allowing geometric-measure-theoretic ideas to be realized within a purely elliptic PDE setting, with promising analogues in higher codimension: see [38, 35, 1, 39, 11, 56, 13] for the Ginzburg–Landau model with no magnetic field and [50, 48] for the abelian Higgs model (as well as [55, 49] for slightly different settings).

Parallel to these isotropic developments, a significant body of work has emerged on minimal surfaces with respect to anisotropic surface energies, which naturally arise in crystalline surface tension models, capillarity problems, and in the modeling of interfaces between distinct materials. We refer to the survey [24] for an overview of the theory of anisotropic minimal surfaces. The anisotropic area functional generalizes the classical area by introducing a convex, one-homogeneous integrand depending on both position and normal direction. While existence and regularity of minimizers for anisotropic energies are by now well understood [8, 9, 18, 21, 23], a satisfactory existence and regularity theory for stationary solutions remains incomplete [5, 25, 26], due largely to the absence of a monotonicity formula for density ratios [3]. Allard [4] conjectured the existence of closed anisotropic minimal hypersurfaces in closed Riemannian manifolds, but an anisotropic counterpart of the Almgren–Pitts min-max theory had remained elusive until recently. In [19, 22] De Philippis, the first-named author, and Li have now resolved this problem by developing an anisotropic version of the Almgren–Pitts theory, proving the existence of closed anisotropic minimal hypersurfaces with essentially optimal regularity in any closed Riemannian manifold.

The goal of the present paper is to provide a PDE-based alternative to the latter approach to the anisotropic min-max construction. More precisely, we develop an anisotropic version of the isotropic Allen–Cahn min-max construction by Guaraco [32]. Let (M^n, g) be a closed Riemannian manifold with $n \geq 2$. We consider a smooth even function

$$F : UT(M) \to (0, \infty), \quad F(x, v) = F(x, -v),$$

defined on the unit tangent bundle, and extend F 1-homogeneously to the tangent bundle:

$$F : TM \to [0, \infty), \quad F(x, \lambda v) = \lambda F(x, v) \text{ for all } \lambda \geq 0, \ (x, v) \in TM.$$

We assume that F is uniformly convex in directions orthogonal to the radial one, namely

$$D^2 F_x(v)[w, w] \geq \lambda |w|^2 \quad \text{for } v \in UT_x(M) \text{ and } w \in T_x M, \ w \perp v,$$

for all $x \in M$, where F_x denotes the restriction of F to $T_x M$. The anisotropic surface area of an embedded hypersurface $\Sigma^{n-1} \subset M^n$ is defined by

$$\mathcal{F}(\Sigma) := \int_\Sigma F(x, \nu_x) \, d\mathcal{H}_g^{n-1}(x),$$

and the anisotropic Allen–Cahn energy of a function $u : M \to \mathbb{R}$ by

$$E_\varepsilon(u) := \int_M \left[\varepsilon \frac{F(x, \nabla u(x))^2}{2} + \frac{W(u(x))}{\varepsilon} \right] d\operatorname{vol}_g(x).$$

All the relevant assumptions on the double-well potential W will be recalled in (2.5)–(2.6) in Section 3. The energy functional E_ε has been recently investigated by Cicalese, Nagase, and Pisante [16], and the associated parabolic equation has been the subject of an extensive literature [27, 28, 31, 37, 42]. Bouchitté [12] proved in a more general framework that E_ε Γ-converges to $c_W \cdot \mathcal{F}$, where $c_W := \int_{-1}^1 \sqrt{2W}$. We provide a shorter proof of this Γ-convergence in Theorem 4.1, tailored for E_ε. We also remark that in recent years there has been a growing literature on the Γ-convergence to anisotropic perimeters of heterogeneous variants of the Allen–Cahn energy functional in a periodic medium [15, 17, 29].

On the contrary, nothing was known in the literature for the convergence of stable (or bounded-Morse-index) critical points for E_ε to anisotropic minimal hypersurfaces, as $\varepsilon \to 0$; actually, as explained in the body of the paper, in order to have a C^2 functional we need to perturb F slightly in the variational construction, although ultimately we still obtain critical points for the original E_ε. Establishing this convergence is a crucial step to extend the isotropic min-max construction of Guaraco [32] to the anisotropic Allen–Cahn framework. The main obstruction is the lack of a suitable monotonicity formula in the anisotropic setting [3], which makes it a challenging task to prove that the limiting varifold has

no diffuse part, and in particular that it is rectifiable. For the isotropic Allen–Cahn functional, the monotonicity formula is deduced by a celebrated gradient bound proved by Modica [43]. Our first main contribution is the following anisotropic version of Modica's bound (see Theorem 5.1 for the details).

Theorem 1.1. *Let $u : M \to [-1, 1]$ be a critical point of E_ε. Then*

$$\frac{\varepsilon F^2(\nabla u)}{2} \leq \frac{W(u)}{\varepsilon} + C(M^n, g, F). \tag{1.2}$$

Compared to the isotropic counterpart, its proof is much more delicate due to the appearance of additional terms when the Euler–Lagrange equation is differentiated, which need to be controlled by exploiting convexity and homogeneity of F at various places. An additional challenge is the lack of effective bounds on the Hessian $D^2 u$ away from $\{\nabla u = 0\}$, as globally u is guaranteed to belong only to $C^{1,\alpha}$ for some $\alpha \in (0, 1)$; terms involving the Hessian naturally appear in the setting of inherently non-autonomous integrands F on closed manifolds.

Theorem 1.1 is somewhat surprising: in the isotropic Allen–Cahn case, the validity of (1.2) implies the monotonicity formula [34, 33], which is known to fail in the anisotropic setting [3]. Hence, while (1.2) remains valid, it no longer entails monotonicity. Nonetheless, Theorem 1.1 serves as a key ingredient in proving our main result.

Theorem 1.2. *There exists a nontrivial F-stationary integral $(n-1)$-varifold V in M, whose weight $c_W \|V\|$ arises as the limit, as $\varepsilon \to 0$, of the energy densities*

$$e_\varepsilon(u_\varepsilon) := \frac{\varepsilon}{2} F(\nabla u_\varepsilon)^2 + \frac{1}{\varepsilon} W(u_\varepsilon)$$

of suitable critical points u_ε of E_ε, constructed via a min-max procedure.

Remark 1.3. While we exhibit such V from the simplest mountain-pass construction, the analysis contained in the present paper applies to any min-max scheme. We conjecture that, in fact, the energy density of *any* sequence (u_ε) of critical points with bounded energy concentrates along an integral F-stationary varifold as $\varepsilon \to 0$, up to a subsequence, regardless of stability conditions.

While we do not attempt to develop any regularity theory beyond integrality of V, we hope to do so in a future work, exploiting once again the stability of u_ε. It would also be interesting to investigate analogues in codimension two or higher, e.g. by devising appropriate anisotropic versions of [50].

Below, we briefly discuss the proof strategy of Theorem 1.2, which is obtained combining Proposition 3.5 with Theorem 7.2 and Theorem 7.10.

First of all, we construct a family of min-max solutions (u_ε) satisfying uniform (in ε) lower and upper bounds of $E_\varepsilon(u_\varepsilon)$ and with Morse index ≤ 1 (up to a regularizing approximation of F); see Proposition 3.5. Hence, the energy densities of u_ε will subconverge in the sense of measures to a finite Radon measure μ on M.

To encode the geometry of the level sets of u_ε, we introduce the $(n-1)$-varifolds $\tilde{V}_\varepsilon$ in (6.6), which heuristically are weighted averages of the level sets of u_ε. The weight measure $\|\tilde{V}_\varepsilon\|(M)$ can be bounded above (up to a constant) with $E(u_\varepsilon)$, and hence we can extract a subsequential limit $\tilde{V}_0$ (as $\varepsilon \to 0$) with $\|\tilde{V}_0\| \leq C\mu$. Moreover, using the Morse index bound of (the approximations of) u_ε, we are also able to deduce a local bound on the isotropic first variation of $\tilde{V}_0$ away from a finite set $\mathcal{S}$: see Corollary 6.6. To this aim, we derive a diffuse version of the stability inequality for anisotropic minimal hypersurfaces, stated in Theorem 6.4. This result parallels those obtained in the isotropic case by Padilla and Tonegawa [47] and Tonegawa [58], although in our setting the PDE degenerates at points where $\nabla u_\varepsilon = 0$, as F^2 is not C^2 at the zero section (unless F is a quadratic norm). To handle this, we perform our estimates first for suitable smooth approximations of F^2 (the same ones used to regularize E_ε) and then pass to the limit.

By means of the Modica-type estimate (1.2) in Theorem 1.1 and another application of stability, we can also control the weight $\|\tilde{V}_0\|$ from below with μ on $M \setminus \mathcal{S}$ and, more importantly, we can prove that $\tilde{V}_0$ and μ are rectifiable: see Theorem 6.7. A crucial ingredient in the proof of Theorem 6.7 is to show that the upper density $\Theta^{n-1,*}(\|\tilde{V}_0\|, x) > 0$ for $\|\tilde{V}_0\|$-a.e. x. This, combined with the local bound on the isotropic first variation of $\tilde{V}_0$, provides the rectifiability of $\tilde{V}_0$ and hence of μ on $M \setminus \mathcal{S}$, by Allard's rectifiability theorem [2, Section 5].

3

Our next step involves extracting an F-stationary varifold V and, as a byproduct of the analysis, showing that μ is a rectifiable measure on all M, as stated in Theorem 7.2. This is achieved by analyzing the stress-energy tensor T_ε of u_ε defined in (7.2), and proving its convergence as $\varepsilon \to 0$ to a tensor-valued measure T_0 whose divergence is controlled. A careful analysis of the directions of invariance of the tangent measures of T_0 provides the desired rectifiability of μ, with an argument in the spirit of [20]. Also, writing $d\mu = \theta\, d(\mathcal{H}^{n-1}\llcorner\Sigma)$, where $\Sigma \subset M$ is a rectifiable Borel set with σ-finite $\mathcal{H}^{n-1}$ measure and $\theta : \Sigma \to (0, \infty)$, we show that the $(n-1)$-dimensional varifold

$$dV(x,\nu) := \frac{\theta(x)}{F(x,\nu_x)}\delta_{\nu_x}(\nu) \otimes d(\mathcal{H}^{n-1}\llcorner\Sigma)(x)$$

is rectifiable and F-stationary, where $\nu_x \perp T_x\Sigma$ is the unit normal.

Our candidate varifold claimed in Theorem 1.2 is $c_W^{-1}V$: we are just left to prove that $\frac{\theta(x)}{c_W F(x,\nu_x)}$ is an integer for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$. This is done by a one-dimensional slicing argument in the normal directions to Σ. Fix a generic point $x_0 \in \Sigma$ and assume $\nu_x = e_n$ in a chart. On small cylinders of the form $B_\varepsilon^{n-1}(x_0) \times (-r,r)$, one rescales the functions u_ε to obtain limit profiles defined on Euclidean cylinders with almost flat metric and almost autonomous integrand F_{x_0}. Using the convergence of stress-energy tensors, we show in (7.12) the smallness of the tangential gradient $\partial_{x_i}u_\varepsilon$ for $i < n$ in an L^2 sense, that is, the fact that u_ε is almost one-dimensional. Such smallness, combined with the Modica-type inequality (1.2), is then proved to imply that $E_\varepsilon(u_\varepsilon)$ is close to ($F(x_0, e_n)$ times) a multiple of the energy $c_W = \int_{-1}^{1}\sqrt{2W}$ of the heteroclinic solution, unless u_ε is essentially constant: see Lemma 7.8. From this, we deduce that along almost every normal line the energy of u_ε on that line converges to a multiple of $c_W F(x_0, e_n)$. This in turn implies that $\frac{\theta(x_0)}{c_W F(x,\nu_{x_0})}$ is an integer, as asserted in Theorem 7.10.

2 Preliminaries

2.1 Terminology

Throughout the paper, we fix a smooth closed (i.e., compact and without boundary) n-dimensional Riemannian manifold (M^n, g) with $n \geq 2$. We note that the fixed background Riemannian metric plays essentially no role in what follows, since it can be absorbed into the integrand. It is however useful in order to fix a background volume form and to identify n-dimensional planes with their normals.

We assume the reader to be familiar with the standard notions in geometric measure theory [53], and we adopt the following notation and conventions.

- $UT(M)$: the unit tangent bundle of M, namely,

$$UT(M) := \{(x, v) \in TM : \|v\|_{g_x} = 1\}.$$

- $G_{n-1}(M)$: the unoriented hyperplane bundle of M, namely,

$$G_{n-1}(M) := \{(x, T) : x \in M,\ T \text{ is an } (n-1)\text{-dimensional linear subspace of } T_x M\}.$$

  By means of the background metric, we can identify $G_{n-1}(M)$ with $UT(M)/\sim$, where we have set the equivalence relation $(x, v) \sim (x, -v)$.

- $\partial^* E$: the reduced boundary of a Caccioppoli set E.

- $\mathcal{V}(M) = \mathcal{V}_{n-1}(M^n)$: the space of $(n-1)$-varifolds on M, namely, the space of nonnegative Radon measure on $G_{n-1}(M)$.

- $\llbracket K \rrbracket$: the integral varifold of multiplicity one associated with an $(n-1)$-rectifiable set K with $\mathcal{H}^{n-1}(K) < \infty$.

- $\|V\|$: the weight of $V \in \mathcal{V}(M)$, i.e., the Radon measure on M associated with V.

2.2 Anisotropic energies

In the sequel, an *anisotropic integrand* will be a smooth function $F : G_{n-1}(M) \to (0, \infty)$. Note that, according to the identification above, we can consider it as a function $F : UT(M) \to (0, \infty)$, which is an even function in the second variable. When no confusion arises, we will often switch between these viewpoints without further comment. Also, for the fixed background metric g, we will just use the symbol $|v|$ for $|v|_{g_x} = \sqrt{g_x(v,v)}$ and $\mathcal{H}^{n-1}$ for $\mathcal{H}^{n-1}_g$. It will also be useful to extend F 1-homogeneously in the second variable as

$$F(x, v) = |v| F\left(x, \frac{v}{|v|}\right), \quad F(x, 0) = 0.$$

We will always assume this extension has been made whenever we consider derivatives of F.

We assume that F is uniformly convex in directions orthogonal to the radial direction, namely

$$D^2 F_x(v)[w, w] \geq \lambda |w|^2 \quad \text{for } v \in UT_x(M) \text{ and } w \in T_x M, \ w \perp v$$

at all points $x \in M$, where F_x denotes the restriction of F to $T_x M$. Note that, by 1-homogeneity of F, the uniform convexity assumption gives

$$D^2 F_x(v)[w, w] \geq \lambda \frac{|w|^2}{|v|} \quad \text{for } v \in T_x M \setminus \{0\} \text{ and } w \in T_x M, \ w \perp v \tag{2.1}$$

and the 2-homogeneous function F^2 is automatically uniformly convex, namely

$$D^2(F_x^2) \geq 2\lambda g_x \quad \text{on } T_x M \setminus \{0\} \tag{2.2}$$

in the sense of quadratic forms at each point $x \in M$, up to possibly decreasing $\lambda > 0$. Possibly further decreasing $\lambda \in (0, 1)$, we can also assume that

$$\lambda |v| \leq F_x(v) \leq \frac{|v|}{\lambda} \quad \text{for all } v \in T_x M. \tag{2.3}$$

For a finite perimeter set E, we denote the *anisotropic perimeter* of E by

$$\mathcal{F}(E) := \int_{\partial^* E} F(x, \nu_x) \, d\mathcal{H}^{n-1}(x),$$

where here ν_x denotes the exterior measure-theoretic normal of E. Actually, the assumption that F is even implies that the sign of the unit normal ν_x is irrelevant.

For a varifold $V \in \mathcal{V}(M)$, we define its *F-anisotropic energy*, and respectively its localized energy to a Borel subset $U \subset M$, as

$$\mathcal{F}(V) := \int_{G_{n-1}(M)} F(x, T) \, dV(x, T), \quad \text{and respectively} \quad \mathcal{F}(V; U) := \int_{G_{n-1}(U)} F(x, T) \, dV(x, T),$$

where $G_{n-1}(U)$ is the restriction of the Grassmannian bundle to U. Note that, by the identification of the $(n-1)$-dimensional Grassmanian with the unit sphere, we can equivalently think an $(n-1)$-dimensional varifold as a measure on $UT(M)$ invariant under reflection $(x, v) \mapsto (x, -v)$. In that case we will write

$$\mathcal{F}(V) = \int_{UT(M)} F(x, \nu) \, dV(x, \nu).$$

The *first variation* of the F-anisotropic energy is defined as

$$\delta_F V(X) := \frac{d}{dt}\Big|_{t=0} \mathcal{F}\left((\varphi_t)_\# V\right)$$

where $V \in \mathcal{V}(M)$, $X \in C^1(M, TM)$, and φ_t is the flow of X (i.e., $\frac{d\varphi_t}{dt} = X(\varphi_t)$ and $\varphi_0 = \mathrm{id}_M$). Referring to [20] for the general expression, here we record that when $M = \mathbb{R}^{n+1}$ and $F(x, \nu)$ is autonomous, i.e., independent of the spatial variable x (so that we can view it as an even function $\mathbb{R}^{n+1} \to \mathbb{R}$), we have the following formula:

$$
\begin{aligned}
\delta_F V(X) &= \int_{UT(M)} [F(\nu) \operatorname{div} X - \langle DF(\nu), DX^\top \nu \rangle] \, dV(x, \nu) \\
&= \int_{UT(M)} \langle F(\nu)I - \nu \otimes DF(\nu), DX \rangle \, dV(x, \nu),
\end{aligned}
\tag{2.4}
$$

where $DX^\top$ is the transpose of DX. For the general formula of $\delta_F V(X)$, one needs to add a term which depends on $D_x F$ (see [20]). Back to the case of closed M, a varifold $V \in \mathcal{V}(M)$ is said to have *bounded F-anisotropic first variation* if there exists $C > 0$ such that for all $X \in C^1(M, TM)$

$$
|\delta_F V(X)| \leq C \|X\|_{L^\infty}.
$$

Equivalently, an $(n-1)$-varifold $V \in \mathcal{V}(M)$ has bounded F-anisotropic first variation if $\delta_F V$ is a $(TM$-valued) Radon measure. We will say that V is *F-stationary* if $\delta_F V \equiv 0$. In the isotropic setting $F_x(v) = |v|$, we will simply write δV to denote the first variation of a varifold V.

2.3 Anisotropic Allen–Cahn

We now generalize the Allen–Cahn energy to the anisotropic setting by letting

$$
E_\varepsilon(u) := \int_M \left[\varepsilon \frac{F(x, \nabla u(x))^2}{2} + \frac{W(u(x))}{\varepsilon} \right] d\operatorname{vol}_g(x),
$$

where W is a fixed double-well potential vanishing at ± 1; we assume that $W : \mathbb{R} \to [0, \infty)$ is smooth with

$$
W > 0 \text{ on } \mathbb{R} \setminus \{\pm 1\}, \quad W(\pm 1) = 0, \quad W''(\pm 1) > 0, \quad (\sqrt{W})'' \leq -c < 0 \text{ on } (-1, 1), \tag{2.5}
$$

as well as

$$
-Cs \leq W'(s) < 0 \text{ on } (-\infty, -1), \quad 0 < W'(s) \leq Cs \text{ on } (1, \infty), \quad cs^2 \leq W(s) \leq Cs^2 \text{ on } \mathbb{R} \setminus [-2, 2], \tag{2.6}
$$

for two constants $c, C > 0$. Two standard choices are $W(s) = \frac{(1 - s^2)^2}{4}$ or $W(s) = 1 + \cos(\pi s)$, suitably modified outside of the interval $[-1, 1]$. The requirements (2.6) on $\mathbb{R} \setminus [-1, 1]$ are actually irrelevant in the construction of critical points u, since we can always modify W on this set in order to satisfy them, and the latter will imply that $|u| \leq 1$.

We will denote by

$$
e_\varepsilon(u(x)) := \varepsilon \frac{F(x, \nabla u(x))^2}{2} + \frac{W(u(x))}{\varepsilon} \tag{2.7}
$$

the energy density of $u : M \to \mathbb{R}$ with respect to E_ε.

We observe that, although F^2 is smooth away from the zero section of TM, it is in general only C^1 on TM. Indeed, if F_x^2 is C^2 at 0, then by Taylor expansion there exists a symmetric matrix Q such that

$$
F_x^2(v) = \langle Qv, v \rangle + o(|v|^2).
$$

From the 2-homogeneity of F_x^2 we obtain

$$
F_x^2(v) = \lim_{t \to 0} \frac{F_x^2(tv)}{t^2} = \langle Qv, v \rangle.
$$

We deduce that F_x^2 is C^2 if and only if F_x is the Euclidean norm up to linear changes of coordinates. Having a C^1 functional will be enough for first-order considerations and in particular to check the Palais–Smale condition. However, to give precise meaning to stability (or to Morse index bounds) of critical

points, it will be useful to consider smooth approximations F_δ converging smoothly to F away from the zero section of TM, as $\delta \to 0$; see also Remark 3.2 below.

To this aim, for each $\delta \in (0,1)$, we define for every $x \in M$

$$G_\delta(x,v) := (F_x^2 * \eta_\delta)(v) - (F_x^2 * \eta_\delta)(0) \quad \text{and} \quad F_\delta(x,v) := \sqrt{G_\delta(x,v)},$$

where η_δ is the standard radial mollifier supported in the ball $B_\delta(0) \subset T_x M \cong \mathbb{R}^n$ (using the metric g for the latter identification). Note that $G_\delta(x,v) \geq 0$ as it is convex and even in v, which forces $v = 0$ to be the minimum point of $G_\delta(x,\cdot)$. Hence, F_δ^2 is smooth, $F_\delta^2 \to F^2$ in $C^1_{loc}(TM)$ as $\delta \to 0$, and for all $\delta \in (0,1)$

$$F_\delta \text{ satisfies } (2.2)–(2.3), \text{ with a smaller } \lambda' > 0 \text{ uniform in } \delta \text{ in place of } \lambda. \tag{2.8}$$

In fact, the validity of (2.3) for F_δ, namely $(\lambda'|v|)^2 \leq G_\delta(x,v) \leq (|v|/\lambda')^2$, is immediate to check for fixed δ (if $|v| \geq 2$ it follows from (2.3) for F, while near the origin it follows from (2.2) and smoothness of $G_\delta(x,\cdot)$), and its uniformity follows from the fact that $G_\delta(x,v) = \delta^2 G_1(x,v/\delta)$. As an immediate consequence of (2.2), the differential of $(F_\delta)_x^2$ is monotone, i.e.,

$$\langle D(F_\delta)_x^2(v) - D(F_\delta)_x^2(w), v - w \rangle \geq 2\lambda'|v - w|^2. \tag{2.9}$$

We define the anisotropic Allen-Cahn energy associated with F_δ as follows:

$$E_{\varepsilon,\delta}(u) := \int_M \left[\varepsilon \frac{F_\delta(x, \nabla u(x))^2}{2} + \frac{W(u(x))}{\varepsilon} \right] d\operatorname{vol}_g .$$

As in (2.7), we denote by

$$e_{\varepsilon,\delta}(u) := \varepsilon \frac{F_\delta(x, \nabla u(x))^2}{2} + \frac{W(u(x))}{\varepsilon}$$

the energy density of $u : M \to \mathbb{R}$ with respect to $E_{\varepsilon,\delta}$.

3 Existence of nontrivial solutions via min-max

The main purpose of this section is to check the Palais–Smale condition for the functional $E_{\varepsilon,\delta}$ and to deduce the existence of nontrivial critical points for E_ε.

Proposition 3.1. *The functional $E_{\varepsilon,\delta}$ is finite and of class C^2 on $H^1(M)$, with*

$$DE_{\varepsilon,\delta}(u)[v] = \int_M \left[\frac{\varepsilon}{2} D(F_\delta)_x^2(\nabla u)[\nabla v] + \frac{1}{\varepsilon} W'(u)v \right],$$

$$D^2 E_{\varepsilon,\delta}(u)[v,w] = \int_M \left[\frac{\varepsilon}{2} D^2(F_\delta)_x^2(\nabla u)[\nabla v, \nabla w] + \frac{1}{\varepsilon} W''(u)vw \right].$$

Moreover, it satisfies the Palais–Smale condition: if $u_k \in H^1(M)$ is a sequence such that $E_{\varepsilon,\delta}(u_k)$ is bounded uniformly in k and $DE_{\varepsilon,\delta}(u_k) \to 0$ strongly in $H^1(M)$ (note that $DE_{\varepsilon,\delta}(u_k) \in H^1(M)^* \cong H^1(M)$), then (u_k) admits a strongly converging subsequence.

Proof. The finiteness of $E_{\varepsilon,\delta}$ follows from the quadratic growth of W, while the C^2 regularity and the formulas for its derivatives are standard calculations. Since by (2.6) and (2.8)

$$(F_\delta)_x^2(\nabla u) \geq \lambda'|\nabla u|^2, \quad W(u) \geq c|u|^2 - C,$$

given a sequence (u_k) as in the statement, we immediately deduce that it is bounded in $H^1(M)$; by the Rellich–Kondrachov compactness theorem, there exists $u \in H^1(M)$ such that u_k converges to u, weakly in $H^1(M)$ and strongly in $L^2(M)$, along a subsequence. We observe that

$$\lim_{k \to \infty} DE_{\varepsilon,\delta}(u)[u_k - u] = \lim_{k \to \infty} \int_M \left[\frac{\varepsilon}{2} D(F_\delta)_x^2(\nabla u)[\nabla u_k - \nabla u] + \frac{1}{\varepsilon} W'(u)(u_k - u) \right] = 0, \tag{3.2}$$

where for the first term on the right-hand side we use that $D(F_\delta)^2_x(\nabla u) \in L^2$ (as $|D(F_\delta)^2_x(v)| \le C|v|$) and $\nabla u_k - \nabla u \rightharpoonup 0$ in L^2, while for the second term we use that $|W'(u)| \le C|u| + C \in L^2$ by (2.6) and $u_k \to u$ strongly in L^2.

Since $DE_{\varepsilon,\delta}(u_k) \to 0$ strongly in H^1 and u_k is bounded in H^1 uniformly in k, we deduce that

$$\lim_{k \to \infty} DE_{\varepsilon,\delta}(u_k)[u_k - u] = 0$$

and hence

$$DE_{\varepsilon,\delta}(u_k)[u_k - u] - DE_{\varepsilon,\delta}(u)[u_k - u] \to 0. \tag{3.3}$$

Arguing again as for (3.2), we have $W'(u_k)(u_k - u) \to 0$ in L^1, hence we deduce that

$$\int_M [D(F_\delta)^2_x(\nabla u_k)[\nabla u_k - \nabla u] - D(F_\delta)^2_x(\nabla u)[\nabla u_k - \nabla u]] \to 0.$$

Using (2.9), this implies that

$$\int_M |\nabla u_k - \nabla u|^2 \to 0,$$

and hence $u_k \to u$ strongly in H^1, as desired. $\qquad\square$

Remark 3.2. The original functional E_ε is of class C^1 and the previous proof still applies to show that E_ε satisfies the Palais–Smale condition. On the other hand, E_ε cannot be used to give an immediate meaning to stability or to a bound on the Morse index of critical points.

Proposition 3.3. *Any critical point u of $E_{\varepsilon,\delta}$ is smooth and satisfies $|u| \le 1$. Also, there exists $\alpha \in (0,1)$ depending only on F such that, for every family of critical points $(u_{\varepsilon,\delta})_{\delta \in (0,1)}$ for the perturbed functionals $E_{\varepsilon,\delta}$ with $\sup_\delta E_{\varepsilon,\delta}(u_{\varepsilon,\delta}) < \infty$, along a subsequence $u_{\varepsilon,\delta}$ converges in $C^{1,\alpha}(M)$ as $\delta \to 0$ to a critical point*

$$u_\varepsilon = \lim_{\delta \to 0} u_{\varepsilon,\delta} \in C^{1,\alpha}(M)$$

for E_ε, with $E_\varepsilon(u_\varepsilon) = \lim_{\delta \to 0} E_{\varepsilon,\delta}(u_{\varepsilon,\delta})$. Moreover, u_ε is smooth on the open set $\{|\nabla u_\varepsilon| \ne 0\}$.

Proof. For each $u = u_{\varepsilon,\delta}$ the Euler–Lagrange equation for the functional $E_{\varepsilon,\delta}$ reads

$$\operatorname{div}(a(\nabla u)) = \varepsilon^{-2} W'(u), \tag{3.4}$$

where in local coordinates $a(\nabla u) = g^{ij}\partial_i((F_\delta)^2_x)(\nabla u)\partial_j$. Note that the vector field a is Lipschitz and, by (2.9), a is also monotone, in the sense that

$$\langle a(v) - a(w), v - w \rangle \ge \lambda'|v - w|^2, \quad \text{for all } v, w \in T_x M.$$

The bound $u \le 1$ readily follows by testing the integral form of (3.4) with $\varphi := (u-1)^+$ and using the strict inequality $0 < W'$ on $(1, \infty)$ in (2.6). The bound $u \ge -1$ is obtained analogously. The fact that u is smooth follows by expanding (3.4) into a linear second-order elliptic PDE, using the fact that F_δ^2 is smooth.

We can apply the results of [36, Chapter 4] to the PDE (3.4) in divergence form to get the uniform bound

$$\|u_{\varepsilon,\delta}\|_{C^{1,\alpha}} \le C,$$

where C may depend on ε and the total energy $E_{\varepsilon,\delta}(u_{\varepsilon,\delta})$, but not on δ, as $\sup_\delta E_{\varepsilon,\delta}(u_{\varepsilon,\delta}) < \infty$. By Arzelà–Ascoli, we deduce the existence of a subsequential limit u_ε in $C^{1,\alpha}$, up to slightly decreasing α. The limit u_ε is thus a critical point of E_ε. The smoothness of u_ε on $\{\nabla u_\varepsilon \ne 0\}$ follows by standard Schauder theory, using that F is smooth away from the zero section of TM. $\qquad\square$

Remark 3.4. In fact, any critical point u_ε of E_ε is $C^{1,\alpha}$, as one can see as follows: for a fixed small ball B, we consider a minimizer $u_{\varepsilon,\delta}$ of $E_{\varepsilon,\delta}$ on B, with trace equal to $u_\varepsilon|_{\partial B}$, and deduce that $\|u_{\varepsilon,\delta}\|_{C^{1,\alpha}(B')} \le C$ on a smaller concentric ball B'. As $\delta \to 0$, any subsequential limit in $H^1(B)$ must coincide with u_ε, as shown by the uniqueness result in [36, Section 4], showing that u_ε is $C^{1,\alpha}$ on B', and hence on M.

We are now ready to construct nontrivial critical points for E_ε, satisfying uniform upper and lower bounds on the energy, together with a stability property. By (2.8) and (2.3), we deduce that the isotropic Allen–Cahn energy

$$\tilde{E}_\varepsilon(u) := \int_M \left[\varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right]$$

satisfies

$$(\lambda')^2 \tilde{E}_\varepsilon(u) \le E_{\varepsilon,\delta}(u) \le (\lambda')^{-2} \tilde{E}_\varepsilon(u), \quad \text{for all } u \in H^1(M). \tag{3.5}$$

Letting

$$\Gamma := \{ \gamma : [-1,1] \to H^1(M) : \gamma \text{ continuous}, \ \gamma(-1) \equiv -1, \ \gamma(1) \equiv 1 \}$$

and defining the mountain-pass values

$$c_{\varepsilon,\delta} := \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} E_{\varepsilon,\delta}(\gamma(t)), \quad \tilde{c}_{\varepsilon,\delta} := \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} \tilde{E}_\varepsilon(\gamma(t))$$

as in [32], we deduce from [32, Proposition 5.2] that

$$0 < \liminf_{\varepsilon \to 0} \tilde{c}_\varepsilon \le \limsup_{\varepsilon \to 0} \tilde{c}_\varepsilon < \infty.$$

Combining this with (3.5), we conclude that there exist $0 < \beta \le \beta'$ and a small $\varepsilon_0 > 0$ such that

$$\beta \le c_{\varepsilon,\delta} \le \beta' \quad \text{for all } \delta \in (0,1), \ \varepsilon \in (0,\varepsilon_0).$$

We can now deduce the following.

Proposition 3.5. *For $\varepsilon \in (0,\varepsilon_0)$ there exists a critical point u_ε for E_ε, with*

$$E_\varepsilon(u_\varepsilon) \in [\beta, \beta'].$$

Moreover, u_ε is a limit in $C^{1,\alpha}(M)$ of critical points $u_{\varepsilon,\delta}$ for $E_{\varepsilon,\delta}$ with Morse index ≤ 1, along a sequence $\delta \to 0$ depending on ε.

Proof. The existence of critical points $u_{\varepsilon,\delta}$ with energy in $[\beta, \beta']$ and Morse index ≤ 1 follows from classical min-max theory on Banach spaces (see, e.g., [30, Chapter 10]). The convergence along a subsequence to a critical point for E_ε now follows from Proposition 3.3. $\square$

The main contribution of the present work is to show that, in such a situation (and even for more general min-max problems), the energy density of u_ε concentrates along an integral F-stationary varifold.

4 Γ-convergence of E_ε to $\mathcal{F}$

As a preliminary result, in fact not needed in the next sections, we prove in this section that E_ε Γ-converges to $c_W \cdot \mathcal{F}$, where $c_W := \int_{-1}^1 \sqrt{2W}$ is a positive constant depending only on W. This was first proved by Bouchitté [12] in a more general framework. We provide a shorter proof in Theorem 4.1, tailored to E_ε. In the following we denote by 1_S the indicator function of a set S, i.e., $1_S = 1$ on S and $1_S = 0$ on $M \setminus S$.

Theorem 4.1. *Given a sequence $\varepsilon_k \to 0$ and maps $u_k \in H^1(M)$ with $\liminf_{k\to\infty} E_{\varepsilon_k}(u_k) < \infty$, there exists a finite perimeter set $S \subseteq M$ such that*

$$u_k \to 1_S - 1_{M \setminus S}$$

pointwise a.e. and in $L^2(M)$ up to a subsequence, as well as

$$c_W \cdot \mathcal{F}(S) \le \liminf_{k \to \infty} E_{\varepsilon_k}(u_k).$$

Conversely, given a finite perimeter set $S \subseteq M$, there exists a family of smooth maps $(u_\varepsilon)_{\varepsilon > 0}$ with $|u_\varepsilon| \le 1$,

$$u_\varepsilon \to 1_S - 1_{M \setminus S}$$

pointwise a.e. and in $L^p(M)$ for all $p < \infty$, and

$$E_\varepsilon(u_\varepsilon) \to c_W \cdot \mathcal{F}(S).$$

Proof. Given a sequence (ε_k, u_k) as in the statement, we will write ε and u_ε in place of ε_k and u_k, with a slight abuse of notation. Up to a subsequence, we can assume that the lim inf is a limit. Also, letting $v_\varepsilon := \min\{\max\{u_\varepsilon, -2\}, 2\}$, note that by (2.6) we have

$$E_\varepsilon(v_\varepsilon) \leq E_\varepsilon(u_\varepsilon), \quad \int_M |u_\varepsilon - v_\varepsilon|^2 \leq \int_{\{|u_\varepsilon| > 2\}} |u_\varepsilon|^2 \leq C \int_M W(u_\varepsilon) \leq C\varepsilon.$$

Thus, we can assume without loss of generality that $|u_\varepsilon| \leq 2$, up to replacing u_ε with v_ε.

Since $|\nabla u_\varepsilon| \leq \lambda^{-1} F_x(\nabla u_\varepsilon)$, by Cauchy–Schwarz we have

$$\int_M \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| \leq \lambda^{-1} \int_M \sqrt{2W(u_\varepsilon)}F_x(\nabla u_\varepsilon) \leq \lambda^{-1}E_\varepsilon(u_\varepsilon) \leq C$$

for some constant $C > 0$ independent of ε. Thus, letting $H(t) := \int_0^t \sqrt{2W(s)}\,ds$ and $w_\varepsilon := H(u_\varepsilon)$, we have a uniform BV bound:

$$\int_M (|w_\varepsilon| + |\nabla w_\varepsilon|) \leq C.$$

By the compact embedding $BV(M) \hookrightarrow L^1(M)$, up to a subsequence we can find $w_0 \in BV(M)$ such that $w_\varepsilon \to w_0$ in L^1 and pointwise a.e., and thus also in $L^2(M)$, as $|w_\varepsilon| \leq 2$.

Since $H : \mathbb{R} \to \mathbb{R}$ is continuous and bijective and $w_\varepsilon \to w_0$ pointwise a.e., we have

$$u_\varepsilon = H^{-1}(w_\varepsilon) \to H^{-1}(w_0) =: u_0 \text{ pointwise a.e.,}$$

and the limit u_0 takes values in ± 1 a.e. since

$$\int_M W(u_0) \leq \liminf_{\varepsilon \to 0} W(u_\varepsilon) = \lim_{\varepsilon \to 0} O(\varepsilon) = 0.$$

Moreover, $S := \{u_0 = 1\} = \{w_0 = H(1)\}$ is a set of finite perimeter, as its indicator function is precisely $\frac{w_0 - H(-1)}{H(1) - H(-1)} \in BV(M)$. The desired bound on the $\mathcal{F}$-perimeter follows easily from the convexity of F at each $x \in M$: indeed, a straightforward adaptation of [10, Theorem 2.38] to the Riemannian case gives

$$\int_M F_x\left(\frac{\nabla w_0}{|\nabla w_0|}\right) d|\nabla w_0| \leq \liminf_{\varepsilon \to 0} \int_M F_x(\nabla w_\varepsilon)$$

$$= \liminf_{\varepsilon \to 0} \int_M H'(u_\varepsilon)F_x(\nabla u_\varepsilon)$$

$$= \liminf_{\varepsilon \to 0} \int_M \sqrt{2W(u_\varepsilon)}F_x(\nabla u_\varepsilon)$$

$$\leq \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon).$$

This gives the desired conclusion, since

$$\nabla 1_S = \nabla \frac{w_0 - H(-1)}{H(1) - H(-1)} = \frac{\nabla w_0}{H(1) - H(-1)} = \frac{\nabla w_0}{c_W}$$

and hence

$$\mathcal{F}(S) = \int_{\partial^* S} F_x(\nu_x)\,d\mathcal{H}^{n-1}(x) = \frac{1}{c_W} \int_M F_x\left(\frac{\nabla w_0}{|\nabla w_0|}\right) d|\nabla w_0|.$$

Conversely, given a set $S \subseteq M$ of finite perimeter, by [10, Theorem 3.42] we can find a sequence of smooth open sets S_k such that

$$\mathcal{H}^n(S_k \Delta S) \to 0, \quad \text{Per}(S_k) \to \text{Per}(S),$$

where Per denotes the isotropic perimeter. Thus, by Reshetnyak's continuity principle [10, Theorem 2.39], we have the convergence of $\mathcal{F}$-perimeters:

$$\int_{\partial S_k} F_x(\nu_x)\,d\mathcal{H}^{n-1}(x) \to \int_{\partial^* S} F_x(\nu_x)\,d\mathcal{H}^{n-1}(x).$$

Thus, to prove the existence of a recovery sequence, we can just consider the case where S is a smooth open set; the conclusion then follows by a standard diagonal argument. In this case, we consider the one-dimensional heteroclinic solution $U : \mathbb{R} \to \mathbb{R}$ such that

$$U' = \sqrt{2W(U)}, \quad U(0) = 0, \quad \lim_{t \to \pm\infty} U(t) = \pm 1$$

and we let

$$U_\gamma(t) := \begin{cases} -1 & \text{for } t \leq -2\gamma \\ U(\gamma t/(2\gamma + t)) & \text{for } t \in (-2\gamma, -\gamma] \\ U(t) & \text{for } t \in [-\gamma, \gamma] \\ U(\gamma t/(2\gamma - t)) & \text{for } t \in [\gamma, 2\gamma) \\ 1 & \text{for } t \geq 2\gamma. \end{cases}$$

Using the exponential decay of U' and of $W(U)$ at infinity, it is easy to check that, as $\gamma \to \infty$, we have

$$\int_{\mathbb{R} \setminus [-\gamma, \gamma]} [(U')^2/2 + W(U)] = O(e^{-c\gamma}), \quad \int_{-\gamma}^{\gamma} [(U')^2/2 + W(U)] = \int_{-\gamma}^{\gamma} \sqrt{2W(U)} U' \to c_W.$$

Then, taking $\delta > 0$ small and writing $B_\delta(\partial S)$ as a disjoint union of geodesics of the form

$$\{\ell_p(t) := \exp_p(t\nu_p) \mid t \in (-\delta, \delta)\}$$

as p ranges in ∂S, we can define

$$u_{\varepsilon,\gamma}(\ell_p(t)) := U_\gamma\left(\frac{t}{\varepsilon F_p(\nu_p)}\right).$$

As long as $2\gamma \cdot \varepsilon \max_{p \in \partial S} F_p(\nu_p) < \delta$, we can extend this to a smooth map $u_{\varepsilon,\gamma} : M \to [-1,1]$ by $u_{\varepsilon,\gamma} := -1$ on $S \setminus B_\delta(\partial S)$ and $u_{\varepsilon,\gamma} := 1$ on $(M \setminus S) \setminus B_\delta(\partial S)$. We observe that

$$\nabla u_{\varepsilon,\gamma}(\ell_p(t)) = U_\gamma'\left(\frac{t}{\varepsilon F_p(\nu_p)}\right)\frac{\ell_p'(t)}{\varepsilon F_p(\nu_p)} + O(|t|/\varepsilon) = U_\gamma'\left(\frac{t}{\varepsilon F_p(\nu_p)}\right)\frac{\ell_p'(t)}{\varepsilon F_p(\nu_p)} + O(\gamma),$$

where the error term comes from differentiation of $F \circ \nu$ at the nearest-point projection p; note that $|U_\gamma'| \leq C$ and that both sides vanish when $|t| > 2\gamma\varepsilon F_p(\nu_p)$. Moreover,

$$|\nabla u_{\varepsilon,\gamma}|(\ell_p(t)) \leq C\varepsilon^{-1}e^{-c\gamma} \quad \text{for } \gamma\varepsilon F_p(\nu_p) < |t| < \delta$$

and similarly

$$W(u_{\varepsilon,\gamma})(\ell_p(t)) \leq Ce^{-c\gamma} \quad \text{for } \gamma\varepsilon F_p(\nu_p) < |t| < \delta.$$

Thus, we have

$$\int_{\{\gamma\varepsilon F_p(\nu_p) < |t| < \delta\}} [\varepsilon F_{\ell_p(t)}(\nabla u_{\varepsilon,\gamma})^2/2 + \varepsilon^{-1}W(u_{\varepsilon,\gamma})](\ell_p(t))\, dt = O(e^{-c\gamma}),$$

for a possibly different $c > 0$, as the integrand is nonzero only on an interval of size $O(\gamma\varepsilon)$. Hence,

$$E_\varepsilon(u_{\varepsilon,\gamma}) = (1 + O(\delta)) \int_{\partial S} \int_{\{|t| < \gamma\varepsilon F_p(\nu_p)\}} [\varepsilon F_{\ell_p(t)}(\nabla u_{\varepsilon,\gamma})^2/2 + \varepsilon^{-1}W(u_{\varepsilon,\gamma})](\ell_p(t))\, dt\, d\mathcal{H}^{n-1}(p) + O(e^{-c\gamma}).$$

Since

$$F_{\ell_p(t)}(\nabla u_{\varepsilon,\gamma}(\ell_p(t))) = (1 + O(\delta))\varepsilon^{-1}U_\gamma'\left(\frac{t}{\varepsilon F_p(\nu_p)}\right) + O(\gamma)$$

by the previous expansion, we deduce that

$$E_\varepsilon(u_{\varepsilon,\gamma}) = (1 + O(\delta)) \int_{\partial S} \int_{\{|t| < \gamma\varepsilon F_p(\nu_p)\}} \left[\frac{1}{2\varepsilon}U_\gamma'\left(\frac{t}{\varepsilon F_p(\nu_p)}\right)^2 + \frac{1}{\varepsilon}W(U_\gamma)\left(\frac{t}{\varepsilon F_p(\nu_p)}\right)\right] dt\, d\mathcal{H}^{n-1}(p)$$
$$+ O(\gamma^2\varepsilon) + O(e^{-c\gamma}),$$

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as long as $\gamma\varepsilon < 1$. By a simple change of variables, we get

$$E_\varepsilon(u_{\varepsilon,\gamma}) = (1 + O(\delta)) \int_{\partial S} F_p(\nu_p)\, d\mathcal{H}^{n-1}(p) \cdot \int_{-\gamma}^{\gamma} [(U_\gamma')^2/2 + W(U_\gamma)] + O(\gamma^2\varepsilon) + O(e^{-c\gamma}).$$

This converges to $\mathcal{F}(S)$ as we let $\varepsilon \to 0$, then $\gamma \to \infty$ and $\delta \to 0$. By a diagonal argument, the conclusion follows. $\qquad\square$

5 A generalization of Modica's bound

The following pointwise bound for critical points u of the isotropic Allen–Cahn

$$\varepsilon \frac{|\nabla u|^2}{2} \leq \frac{W(u)}{\varepsilon}$$

was first proved by Modica [43] in the Euclidean setting. This is the fundamental tool used in deriving a sharp monotonicity formula in the isotropic case [34, 33].

In the anisotropic setting, we obtain an analogous bound; as expected, it no longer yields a monotonicity formula. Nonetheless, it will be a crucial ingredient in the proof of rectifiability of the limit of the energy densities.

From now on up to the end of the paper, given a map $u : M \to \mathbb{R}$, when evaluating $F(x, \nabla u(x))$ with a slight abuse of notation we will look at ∇u as a vector field in the tangent bundle $\nabla u : M \to TM$, and use the compact expression $F(\nabla u)$ in place of $F(\cdot, \nabla u(\cdot))$.

Theorem 5.1. *Letting* $u : M \to [-1, 1]$ *be a critical point of* E_ε, *we have*

$$F(\nabla u) \leq \varepsilon^{-1}\sqrt{2W(u)} + C,$$

where C *depends only on* M^n, g *and* F. *As a consequence,*

$$\frac{\varepsilon F^2(\nabla u)}{2} \leq \frac{W(u)}{\varepsilon} + C$$

for a possibly different $C = C(M^n, g, F)$.

Proof. Recall that, by Proposition 3.3 and Remark 3.4, away from $\{\nabla u = 0\}$, u is smooth and we can expand the Euler–Lagrange equation (3.4) as

$$A^{ij}\partial_{ij}u = \varepsilon^{-2}W'(u) + O(\nabla u),$$

where the coefficients $A^{ij}(x)$ are essentially the second derivatives of $F_x^2/2$ at ∇u; more precisely, in any coordinate chart we have

$$A^{ij} = g^{i\ell}g^{jm}\partial_{\ell m}(F^2/2)(\nabla u) = g^{i\ell}g^{jm}[F(\nabla u)\partial_{\ell m}F(\nabla u) + \partial_\ell F(\nabla u)\partial_m F(\nabla u)],$$

where when differentiating F and F^2 we use the convention that $\partial_k F$ denotes the partial derivative along $\partial_k = \frac{\partial}{\partial x^k} \in T_x M$ of $F_x = F|_{T_x M}$, at any given $x \in M$ (and similarly for higher-order derivatives and for F^2). The error term is a function $G(x, \nabla u)$, with $G(x, v)$ smooth away from $\{v = 0\}$ and 1-homogeneous in v, so that $|\frac{\partial^{\alpha+\beta}G}{\partial x^\alpha \partial v^\beta}| \leq C_{\alpha,\beta}|v|^{1-|\beta|}$.

We would like to show that

$$F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)} \leq \Lambda$$

everywhere, where $\Lambda > 0$ will be chosen later. To do this, consider a maximum point $\hat{x}$ for the difference

$$\zeta := F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)},$$

and assume by contradiction that $\zeta(\hat{x}) > \Lambda$, so that in particular $\nabla u(\hat{x}) \neq 0$, and thus u is smooth around $\hat{x}$, as well as $|u(\hat{x})| < 1$. We now choose a coordinate system centered at $\hat{x}$, with $g_{ij}(0) = \delta_{ij}$ and $\partial_k g_{ij}(0) = 0$, as well as

$$\nabla u(0) = |\nabla u(0)|e_n.$$

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Since A^{ij} is positive definite, we have

$$A^{ij}\partial_{ij}[F(\nabla u) - \varepsilon\sqrt{2W(u)}](\hat{x}) = A^{ij}\partial_{ij}\zeta(\hat{x}) \leq 0.$$

We now compute that at $\hat{x} = 0$ we have

$$\begin{aligned}
\partial_{ij}[F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)}] &= \partial_i[g^{k\ell}\partial_k F(\nabla u)\partial_{j\ell}u - \varepsilon^{-1}(\sqrt{2W})'(u)\partial_j u + O(\nabla u)] \\
&= \partial_{k\ell}F(\nabla u)\partial_{i\ell}u\partial_{jk}u + \partial_k F(\nabla u)\partial_{ijk}u - \varepsilon^{-1}(\sqrt{2W})'(u)\partial_{ij}u \\
&\quad - \varepsilon^{-1}(\sqrt{2W})''(u)\partial_i u\partial_j u + O(\nabla u) + O(D^2 u),
\end{aligned}$$

where the error terms come from differentiating F in the spatial variable at least once.

Once we multiply by A^{ij} and sum over i, j, we get

$$I + A^{ij}\partial_k F(\nabla u)\partial_{ijk}u - \varepsilon^{-3}\frac{W'(u)^2}{\sqrt{2W(u)}} - \varepsilon^{-1}(\sqrt{2W})''(u)F^2(\nabla u) \leq O(\varepsilon^{-1}\nabla u) + O(D^2 u), \qquad (5.1)$$

where we omit the sum over i, j and we set

$$I := A^{ij}\partial_{k\ell}F(\nabla u)\partial_{i\ell}u\partial_{jk}u$$

and we used the Euler–Lagrange equation and the fact that

$$A^{ij}\partial_i u\partial_j u = D^2(F^2/2)(\nabla u)[\nabla u, \nabla u] = F^2(\nabla u),$$

since F^2 is 2-homogeneous.

We rewrite the second term of (5.1) as

$$\begin{aligned}
&\partial_k(A^{ij}\partial_{ij}u)\partial_k F(\nabla u) - \partial_k A^{ij}\partial_k F(\nabla u)\partial_{ij}u \\
&= \varepsilon^{-2}W''(u)\partial_k u\partial_k F(\nabla u) - \partial_{ij\ell}(F^2/2)(\nabla u)\partial_{k\ell}u\partial_k F(\nabla u)\partial_{ij}(u) + O(\nabla u) + O(D^2 u),
\end{aligned}$$

where we used again the Euler–Lagrange equation and we expanded $\partial_k A^{ij}$ using the chain rule. Since F is 1-homogeneous, at $\hat{x}$ we have

$$\partial_k u\partial_k F(\nabla u) = F(\nabla u).$$

Moreover, since $\nabla\zeta(\hat{x}) = 0$, at $\hat{x}$ we have

$$\partial_{k\ell}u\partial_k F(\nabla u) = \partial_\ell[F(\nabla u)] + O(\nabla u) = \varepsilon^{-1}\partial_\ell\sqrt{2W(u)} + O(\nabla u) = \varepsilon^{-1}(\sqrt{2W})'(u)\partial_\ell u + O(\nabla u),$$

giving

$$\partial_{ij\ell}(F^2/2)(\nabla u)\partial_{k\ell}u\partial_k F(\nabla u)\partial_{ij}(u) = \varepsilon^{-1}(\sqrt{2W})'(u)\partial_{ij\ell}(F^2/2)(\nabla u)\partial_\ell u\partial_{ij}(u) + O(D^2 u),$$

thanks to the fact that $\partial_{ij\ell}(F^2/2)$ is (-1)-homogeneous, so that $|\partial_{ij\ell}(F^2/2)(\nabla u)| \leq C|\nabla u|^{-1}$. Also, since $\partial_{ij}(F^2/2)$ is 0-homogeneous, we have

$$\partial_{ij\ell}(F^2/2)(\nabla u)\partial_\ell u = 0.$$

In summary, at $\hat{x}$ we get

$$A^{ij}\partial_k F(\nabla u)\partial_{ijk}u = \varepsilon^{-2}W''(u)F(\nabla u) + O(D^2 u).$$

Thus, (5.1) becomes

$$I + \varepsilon^{-2}W''(u)F(\nabla u) - \varepsilon^{-3}\frac{W'(u)^2}{\sqrt{2W(u)}} - \varepsilon^{-1}(\sqrt{2W})''(u)F^2(\nabla u) \leq O(\varepsilon^{-1}\nabla u) + O(D^2 u).$$

Now I is nonnegative, since it is the trace of the product of two positive semidefinite matrices; in fact, by (2.1) and the fact that $D^2F(v)[v,\cdot] = 0$ for $v \neq 0$ (by 0-homogeneity of DF_x), we have

$$
\begin{aligned}
I &:= A^{ij}\partial_{k\ell}F(\nabla u)\partial_{i\ell}u\partial_{jk}u \\
&= \mathrm{tr}((D^2F)(D^2uAD^2u)) \\
&\geq \frac{\lambda}{|\nabla u|}\mathrm{tr}((I - e_n^* \otimes e_n^*)(D^2uAD^2u)) \\
&\geq \frac{\lambda^2}{|\nabla u|}\mathrm{tr}((I - e_n^* \otimes e_n^*)(D^2u)^2) \\
&\geq \frac{\lambda^2}{|\nabla u|}\sum_{i=1}^{n}\sum_{j=1}^{n-1}|\partial_{ij}u|^2,
\end{aligned}
$$

where we repeatedly used the fact that $\mathrm{tr}(AB) \geq \mathrm{tr}(A'B')$ if $A \geq A' \geq 0$ and $B \geq B' \geq 0$. Moreover, by the Euler–Lagrange equation, the lower bound $A^{nn} \geq \lambda$, and the estimate $|W'| \leq C\sqrt{W}$ (as $W''(\pm 1) > 0$), we have

$$
|\partial_{nn}u| \leq C\sum_{(i,j)\neq(n,n)}|\partial_{ij}u| + O(\nabla u) + C\varepsilon^{-2}\sqrt{W(u)}. \tag{5.2}
$$

By the assumption that $\varepsilon^{-1}\sqrt{W(u)} < F(\nabla u)$ at $\hat{x}$, the estimate (5.2) in turn implies that at $\hat{x}$

$$
|\partial_{nn}u| \leq C\sum_{(i,j)\neq(n,n)}|\partial_{ij}u| + O(\varepsilon^{-1}\nabla u).
$$

Thus, by Cauchy's inequality, we can absorb the term $O(D^2u)$, getting

$$
I + O(D^2u) \geq -C|\nabla u| - C\varepsilon^{-1}|\nabla u| \geq -C\varepsilon^{-1}F(\nabla u)
$$

at $\hat{x}$, where C depends on the implied constant in $O(D^2u)$. We deduce that

$$
\varepsilon^{-2}W''(u)F(\nabla u) - \varepsilon^{-3}\frac{W'(u)^2}{\sqrt{2W(u)}} - \varepsilon^{-1}(\sqrt{2W})''(u)F^2(\nabla u) \leq C\varepsilon^{-1}F(\nabla u). \tag{5.3}
$$

Since $W'' = (\sqrt{2W})'' \cdot \sqrt{2W} + \frac{(W')^2}{2W}$, we can rewrite (5.3) as

$$
-\varepsilon^{-1}(\sqrt{2W})''(u)F(\nabla u)[F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)}] + \varepsilon^{-2}\frac{(W')^2}{2W}[F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)}] \leq -I + O(\varepsilon^{-1}\nabla u)
$$

at $\hat{x}$; recalling that $-(\sqrt{2W})'' \geq c > 0$ by (2.5), we reach

$$
c\varepsilon^{-1}F(\nabla u)[F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)}] + \varepsilon^{-2}\frac{(W')^2}{2W}[F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)}] \leq C\varepsilon^{-1}F(\nabla u).
$$

This contradicts the fact that $F(\nabla u) - \varepsilon^{-1}\sqrt{2W(u)} > \Lambda$, once we take Λ large enough that $c\Lambda \geq C$. $\quad\square$

6 Uniform bounds for stable solutions

In this section we derive bounds on the second fundamental form of level sets for stable solutions $u_{\varepsilon,\delta}$ with respect to $E_{\varepsilon,\delta}$ and consequently for limits u_ε thereof. Moreover we obtain lower density bounds for the energy. Since F^2 is not of class C^2, in many statements we will replace it with the perturbed integrands F_δ^2 already considered earlier, in order to make sense of stability. Let us start with a simple observation.

Proposition 6.1. *There exists $\rho = \rho(M^n, g, F) > 0$ large enough such that the following holds. Given $x_0 \in M$, we identify $T_{x_0}M \cong \mathbb{R}^n$ isometrically, so that the restriction $(F_\delta)_{x_0} = F_\delta|_{T_{x_0}M}$ gives an autonomous integrand $\overline{F}_\delta : \mathbb{R}^n \to [0, \infty)$. If a constant $\overline{u} \in [-1, 1]$ is a stable critical point for*

$$\overline{E}_{1,\delta}(u) := \int_{B_\rho(0)} [\overline{F}_\delta(\nabla u)^2/2 + W(u)]$$

on the Euclidean ball $B_\rho(0) \subset \mathbb{R}^n$, then $\overline{u} \in \{-1, 1\}$.

Proof. Indeed, assume by contradiction that $u_0 \in (-1, 1)$. For any $\eta \in C_c^1(B_\rho(0))$, stability gives

$$\int_{B_\rho(0)} [D^2(\overline{F}_\delta^2/2)(0)[\nabla\eta, \nabla\eta] + W''(\overline{u})\eta^2] \geq 0.$$

Since $W'(\overline{u}) = W'(\pm 1) = 0$ and $W''(\pm 1) > 0$, we must have $|\overline{u}| \leq 1 - c$ for some $c > 0$. Also, using again the fact that $W'(\overline{u}) = 0$, we have

$$W''(\overline{u}) = 2(\sqrt{W})''(\overline{u})\sqrt{W(\overline{u})}.$$

Using (2.5), we see that $-W''(\overline{u}) \geq c$ for a possibly different $c > 0$. We deduce that

$$c\int_{B_\rho} \eta^2 \leq \int_{B_\rho} D^2(\overline{F}_\delta^2/2)(0)[\nabla\eta, \nabla\eta] \leq C\int_{B_\rho} |\nabla\eta|^2 \tag{6.1}$$

for all $\eta \in C_c^1(B_\rho)$. This is impossible once we take $\rho > 0$ large enough. $\qquad\square$

The following are useful consequences.

Lemma 6.2. *Given $\gamma > 0$, there exist constants $c > 0$ and $\varepsilon_0 > 0$, depending on γ and (M^n, g, F), such that*

$$\int_{B_{\rho\varepsilon}(p)} \sqrt{2W(u)}|\nabla u| \geq c\varepsilon^{n-1}$$

whenever $|u(p)| \leq 1 - \gamma$, for any stable critical point $u : B_{\rho\varepsilon}(p) \to \mathbb{R}$ of $E_{\varepsilon,\delta}$, provided that $\varepsilon \in (0, \varepsilon_0)$.

Proof. Arguing by contradiction, dilating the domain by a factor ε^{-1}, assume that

$$\int_{B_\rho^{(\varepsilon)}} \sqrt{2W(\tilde{u}_\varepsilon)}|\nabla\tilde{u}_\varepsilon| \to 0$$

for a sequence of rescaled functions $\tilde{u}_\varepsilon$, defined on rescaled geodesic balls $B_\rho^{(\varepsilon)}$ and stably critical for E_{1,δ_ε}, for some $\delta_\varepsilon \in (0, 1)$. Since $\varepsilon \to 0$, the rescaled metrics converge to the Euclidean one.

Since u_ε is uniformly $C^{1,\alpha}$ on $B_{\rho'}^{(\varepsilon)}$, for any $\rho' \in (0, \rho)$ (see Proposition 3.3 and its proof), once we identify each $B_\rho^{(\varepsilon)}$ with the Euclidean ball $B_\rho(0)$, these solutions converge in C_{loc}^1, along a subsequence, to a critical point $\tilde{u}_0 : B_\rho(0) \to [-1, 1]$ for E_{1,δ_0}, where $\delta_0 := \lim_{\varepsilon\to 0} \delta_\varepsilon \in [0, 1]$. Here the limit energy E_{1,δ_0} involves the autonomous integrand F_{p_0}, where $p_0 = \lim_{\varepsilon\to 0} p_\varepsilon$. Also, $|\tilde{u}_0(0)| = \lim_{\varepsilon\to 0} |u_\varepsilon(p_\varepsilon)| \leq 1 - \gamma$ and

$$\int_{B_\rho(0)} \sqrt{2W(\tilde{u}_0)}|\nabla\tilde{u}_0| = 0,$$

so that $\nabla\tilde{u}_0$ vanishes on the open set $\{\tilde{u}_0 \in (-1, 1)\}$. We deduce that $\tilde{u}_0$ is a constant value in $[-1 + \gamma, 1 - \gamma]$. If $\delta_0 > 0$, then $\tilde{u}_0$ is also stable and we can immediately apply Proposition 6.1 to reach a contradiction.

If $\delta_0 = 0$, we can still derive an inequality like (6.1): namely, for ε small enough, depending on $\text{spt}(\eta)$, we can write

$$c\int_{B_\rho^{(\varepsilon)}} \eta^2 \leq C\int_{B_\rho^{(\varepsilon)}} |\nabla\eta|^2,$$

by exploiting the stability of $\tilde{u}_\varepsilon$ and using the fact that $W''(\tilde{u}_\varepsilon) \to W''(\tilde{u}_0) \leq -c < 0$ locally uniformly. In the limit we get $c\int_{B_\rho} \eta^2 \leq C\int_{B_\rho} |\nabla\eta|^2$, for all $\eta \in C_c^1(B_\rho)$, which is again impossible by our choice of ρ in the proof of Proposition 6.1, making (6.1) fail. $\qquad\square$

Lemma 6.3. *There exist constants $c > 0$ and $\varepsilon_0 > 0$, depending on (M^n, g, F), such that*

$$\int_{B_{2r}(p)} \varepsilon F_\delta(\nabla u)^2 \geq c \int_{B_r(p)} \frac{W(u)}{\varepsilon}$$

for any stable critical point $u : B_{2r}(p) \to \mathbb{R}$ of $E_{\varepsilon,\delta}$, provided that $\rho\varepsilon < \min\{r, \varepsilon_0\}$.

Proof. Arguing by contradiction, assume that a stable solution u_ε satisfies the reverse inequality for arbitrarily small $c, \varepsilon > 0$. Since we can cover $B_r(p)$ by a family of balls $B_{\rho\varepsilon}(x) \subseteq B_{2r}(p)$ with bounded overlap, we can find a sequence of stable solutions u_ε (with $\varepsilon \to 0$) and points $x_\varepsilon \in M$ such that

$$\int_{B_{2\rho\varepsilon}(x_\varepsilon)} \varepsilon F(\nabla u_\varepsilon)^2 < c_\varepsilon \int_{B_{\rho\varepsilon}(x_\varepsilon)} \frac{W(u_\varepsilon)}{\varepsilon}, \tag{6.2}$$

with $c_\varepsilon \to 0$.

We can rescale u_ε to functions $\tilde{u}_\varepsilon$, defined on rescaled geodesic balls $B_{2\rho}^{(\varepsilon)}$ converging to the Euclidean ball $B_{2\rho}(0)$. As in the previous proof, we have $\tilde{u}_\varepsilon \to \tilde{u}_0$ in C^1_{loc} up to a subsequence. Since

$$\int_{B_{2\rho}^{(\varepsilon)}} |\nabla \tilde{u}_\varepsilon|^2 \leq C \int_{B_{2\rho}^{(\varepsilon)}} F(\nabla \tilde{u}_\varepsilon)^2 < C \cdot c_\varepsilon \int_{B_\rho^{(\varepsilon)}} W(\tilde{u}_\varepsilon) \to 0,$$

in the limit we deduce that $\tilde{u}_0$ is constant on $B_{2\rho}(0)$.

If u_ε is critical and stable for $E_{\varepsilon,\delta_\varepsilon}$ with $\delta_0 := \lim_{\varepsilon \to 0} \delta_\varepsilon > 0$, then $\tilde{u}_0$ is critical and stable for E_{1,δ_0}, so that Proposition 6.1 gives $\tilde{u}_0 \in \{\pm 1\}$; if instead $\delta_0 = 0$, we can reach the same conclusion by arguing exactly as in the previous proof.

Now, recalling that $W''(\pm 1) > 0$, we can fix $\gamma \in (0, 1)$ such that

$$-\operatorname{sgn}(s)W'(s)(1 - |s|) \geq 4c(1 - |s|)^2 \geq (1 - s^2)^2 \text{ for } |s| \in [1 - \gamma, 1],$$

while, for a possibly different $c > 0$, we also have

$$c(1 - s^2)^2 \leq W(s) \leq C(1 - s^2)^2 \quad \text{for } |s| \leq 1.$$

Since $|\tilde{u}_0| = 1$, we have $|\tilde{u}_\varepsilon| \in [1 - \gamma, 1]$ on $B_{3\rho/2}^{(\varepsilon)}$ eventually.

Let us take a cut-off function $\varphi_\varepsilon \in C^1_c(B_{3\rho/2}^{(\varepsilon)})$, equal to 1 on $B_\rho^{(\varepsilon)}$ and with $|\nabla \varphi_\varepsilon| \leq 4\rho^{-1}$. Assuming for instance that $\tilde{u}_\varepsilon \geq 1 - \gamma$ on the ball $B_{3\rho/2}^{(\varepsilon)}$ and testing the Euler–Lagrange equation (3.4) with $\varphi_\varepsilon^2(1 - \tilde{u}_\varepsilon)$, we find

$$-\int_{B_{2\rho}^{(\varepsilon)}} \varphi_\varepsilon^2 W'(\tilde{u}_\varepsilon)(1 - \tilde{u}_\varepsilon) = \int_{B_{2\rho}^{(\varepsilon)}} \langle a_\varepsilon(\nabla \tilde{u}_\varepsilon), \nabla(\varphi_\varepsilon^2(1 - \tilde{u}_\varepsilon)) \rangle,$$

for suitable $a_\varepsilon : TM \to TM$ with $|a_\varepsilon(v)| \leq C|v|$. Applying Young's inequality, we deduce that

$$-\int_{B_{2\rho}^{(\varepsilon)}} \varphi_\varepsilon^2 W'(\tilde{u}_\varepsilon)(1 - \tilde{u}_\varepsilon) \leq C(\sigma) \int_{B_{2\rho}^{(\varepsilon)}} |\nabla \tilde{u}_\varepsilon|^2 + \sigma \int_{B_{2\rho}^{(\varepsilon)}} \varphi_\varepsilon^2 (1 - \tilde{u}_\varepsilon)^2$$

for an arbitrarily small $\sigma > 0$. Using the fact that $\tilde{u}_\varepsilon \in [1 - \gamma, 1]$ on the support of φ_ε, we deduce that

$$c \int_{B_{2\rho}^{(\varepsilon)}} \varphi_\varepsilon^2 (1 - \tilde{u}_\varepsilon^2)^2 \leq C(\sigma) \int_{B_{2\rho}^{(\varepsilon)}} |\nabla \tilde{u}_\varepsilon|^2 + \sigma \int_{B_{2\rho}^{(\varepsilon)}} \varphi_\varepsilon^2 (1 - \tilde{u}_\varepsilon)^2,$$

and thus, taking $\sigma := c/2$, we get

$$\int_{B_{2\rho}^{(\varepsilon)}} \varphi_\varepsilon^2 (1 - \tilde{u}_\varepsilon^2)^2 \leq C \int_{B_{2\rho}^{(\varepsilon)}} |\nabla \tilde{u}_\varepsilon|^2,$$

or alternatively

$$\int_{B_{2\rho}^{(\varepsilon)}} \varphi_\varepsilon^2 W(\tilde{u}_\varepsilon)^2 \le C \int_{B_{2\rho}^{(\varepsilon)}} F(\nabla \tilde{u}_\varepsilon)^2.$$

Thus, we have

$$\int_{B_{\rho}^{(\varepsilon)}} W(\tilde{u}_\varepsilon)^2 \le C \int_{B_{2\rho}^{(\varepsilon)}} F(\nabla \tilde{u}_\varepsilon)^2,$$

which contradicts (6.2). The case $\tilde{u}_\varepsilon \in [-1, -1+\gamma]$ is analogous. $\qquad\square$

We record here a diffuse version of the *stability inequality* for $\mathcal{F}$-stationary hypersurfaces, similar to the one obtained for the isotropic Allen–Cahn [47], first for stable critical points of $E_{\varepsilon,\delta}$. We will then let $\delta \to 0$ to derive a consequence for critical points u_ε of E_ε which are limits $u_\varepsilon = \lim_{\delta \to 0} u_{\varepsilon,\delta}$ of stable critical points of $E_{\varepsilon,\delta}$.

Theorem 6.4. *Assume that u is a stable critical point for $E_{\varepsilon,\delta}$ on an open set $U \subseteq M$. Then we have*

$$\int_U \varphi^2 |\,\mathrm{II}_u|^2 \cdot \varepsilon F_\delta(\nabla u)^2 \le C(\varphi) \int_U e_{\varepsilon,\delta}(u),$$

where $\mathrm{II}_u(x)$ denotes the second fundamental form of the level set $\{u = u(x)\}$ if $\nabla u(x) \ne 0$, and it is set to be zero on $\{\nabla u = 0\}$.

Proof. Since u_δ is stable on U, for any $\varphi \in C_c^2(U)$ a straightforward computation shows that

$$\int_U \left[A_\delta^{ij} \partial_i \varphi \partial_j \varphi + \frac{W''(u_\delta)}{\varepsilon^2} \varphi^2 \right] \ge 0,$$

where as above $A_\delta^{ij} = g^{i\ell} g^{jm} \partial_{\ell m}(F_\delta^2/2)(\nabla u)$, and $\partial_{\ell m}(F_\delta^2/2)$ denotes the second derivative of $(F_\delta^2/2)|_{T_x M}$ along $\partial_\ell, \partial_m \in T_x M$, at any given $x \in M$.

Since $|\nabla u|$ is Lipschitz, by a standard approximation argument we can plug $\varphi|du_\delta|$ in place of φ, obtaining

$$\int_U \left[\varphi^2 \left(A_\delta^{ij} \partial_i |\nabla u| \partial_j |\nabla u| + \frac{W''(u)}{\varepsilon^2} |\nabla u|^2 \right) + A_\delta^{ij} \partial_i \varphi \partial_j \varphi |\nabla u|^2 + 2 A_\delta^{ij} \varphi |\nabla u| \partial_i \varphi \partial_j |\nabla u| \right] \ge 0. \qquad (6.3)$$

Next, testing criticality with $\varphi^2 \Delta u$, we have

$$\int_U \left[g^{ij} \partial_i (F_\delta^2/2)(\nabla u) \partial_j(\varphi^2 \Delta u) + \frac{W'(u)}{\varepsilon^2} \varphi^2 \Delta u \right] = 0.$$

Writing $\partial_j(\varphi^2 \Delta u) = \varphi^2 g^{k\ell} \partial_{kj\ell} u + \varphi[O(\nabla u) + O(D^2 u)]$, with implied constants depending on φ, and noting that

$$\partial_k[\partial_i(F_\delta^2/2)(\nabla u)] = g^{pq} \partial_{ip}(F_\delta^2/2)(\nabla u) \partial_{qk} u,$$

after an integration by parts and relabeling of indices we obtain

$$\int_U \left[\varphi^2 A_\delta^{ij} g^{k\ell} \partial_{ik} u \partial_{j\ell} u + \varphi^2 \frac{W''(u)}{\varepsilon^2} |\nabla u|^2 + \varepsilon^{-2} \langle \nabla(\varphi^2), \nabla(W(u)) \rangle \right]$$
$$= \int_U \varphi[O(|\nabla u|^2) + O(\nabla u)O(D^2 u)],$$

and hence

$$\int_U \left[\varphi^2 A_\delta^{ij} g^{k\ell} \partial_{ik} u \partial_{j\ell} u + \varphi^2 \frac{W''(u)}{\varepsilon^2} |\nabla u|^2 \right] = \int_U [\varepsilon^{-1} \cdot O(e_\varepsilon(u)) + \varphi \cdot O(\nabla u)O(D^2 u)]. \qquad (6.4)$$

Subtracting (6.3) from (6.4), we get

$$\int_U \varphi^2 A_\delta^{ij} [g^{k\ell} \partial_{ik} u \partial_{j\ell} u - \partial_i |\nabla u| \partial_j |\nabla u|] \le C(\varphi) \int_U [\varepsilon^{-1} e_\varepsilon(u) + \varphi |\nabla u| |D^2 u|].$$

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We now absorb the last error term: fixing a point $p \in M$ where $\nabla u(p) \neq 0$ and choosing a chart centered at p, with $g_{ij}(p) = \delta_{ij}$, $\partial_k g_{ij}(p) = 0$, and $\nabla u(p) = |\nabla u(p)|\partial_n$, as in (5.2) we observe that

$$|\partial_{nn} u(p)| \leq C \sum_{(i,j) \neq (n,n)} |\partial_{ij} u|,$$

while the integrand in the left-hand side is equal to φ^2 times

$$A_\delta^{ij}[\partial_{ik} u \partial_{jk} u - \partial_{in} u \partial_{jn} u] \geq \lambda' \sum_{i=1}^n \sum_{k=1}^{n-1} |\partial_{ik} u|^2.$$

On the other hand, the same term equals $A_\delta^{ij} \partial_{ik} u \partial_{jk} u \geq \lambda' \sum_{i,k=1}^n |\partial_{ik} u|^2$ at a.e. point where $|\nabla u| = 0$. Thus, by Cauchy's inequality, we reach the bound

$$\int \varphi^2 A_\delta^{ij}[g^{k\ell} \partial_{ik} u \partial_{j\ell} u - \partial_i |\nabla u| \partial_j |\nabla u|] \leq C(\varphi) \int_U \varepsilon^{-1} e_\varepsilon(u), \tag{6.5}$$

for a possibly larger $C(\varphi)$.

Let A_δ denote the section of $TM \otimes TM$ with components A_δ^{ij} (in a coordinate chart), which (using the metric g) we can recast as a positive definite $(1,1)$-tensor $A_\delta \geq \lambda' I$. On $\{\nabla u \neq 0\}$, letting $\nu := \frac{\nabla u}{|\nabla u|}$, we observe that the integrand on the left-hand side can be written more compactly as

$$\varphi^2 \operatorname{tr}((HA_\delta H)(g - \nu^* \otimes \nu^*)) \geq \lambda' \varphi^2 \operatorname{tr}(H^2(g - \nu^* \otimes \nu^*)), \quad H := D^2 u.$$

Given a point p and selecting a coordinate chart as above, writing $e_i := \partial_i$, for $i = 1, \ldots, n-1$ we have

$$|\operatorname{II}_u(e_i)|^2 = \sum_{j=1}^{n-1} |\langle \partial_i \nu, e_j \rangle|^2 = |\partial_i \nu|^2$$

at p, as the tangent space of the level set is spanned by $\{e_1, \ldots, e_{n-1}\}$, while $e_n = \nu$. Moreover

$$\partial_i \nu = \frac{H(e_i) - H(e_i, \nu)\nu}{|\nabla u|},$$

so that for all $j = 1, \ldots, n-1$ we have

$$\langle \partial_i \nu, e_j \rangle = \frac{H(e_i, e_j)}{|\nabla u|}.$$

Thus,

$$|\operatorname{II}_u|^2 = \sum_{i,j=1}^{n-1} \left| \frac{H(e_i, e_j)}{|\nabla u|} \right|^2 \leq \frac{\operatorname{tr}(H^2(g - \nu^* \otimes \nu^*))}{|\nabla u|^2}.$$

We deduce that

$$\int_U \varphi^2 |\operatorname{II}_u|^2 |\nabla u|^2 \leq C\varphi^2 \operatorname{tr}((HA_\delta H)(g - \nu^* \otimes \nu^*)).$$

The claim now follows from (6.5). $\qquad\square$

Now, given a critical point u_ε for the energy E_ε, we define the $(n-1)$-dimensional varifold $\tilde{V}_\varepsilon$ to be the measure on the Grassmannian bundle $G := G_{n-1}(M)$ given by

$$\tilde{V}_\varepsilon(f) := \int_{\{\nabla u_\varepsilon \neq 0\}} \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \cdot f(P_{u_\varepsilon}) \, d\operatorname{vol}_g, \tag{6.6}$$

for any $f \in C^0(G)$, where $P_{u_\varepsilon}(x) := (\nabla u_\varepsilon(x))^\perp \in G$ is the tangent plane to the level set $\{u_\varepsilon = u_\varepsilon(x)\}$ at x. Here we make a slight abuse of notation: we write $f(P_{u_\varepsilon}(x))$ in place of $f(x, P_{u_\varepsilon}(x))$.

Remark 6.5. If almost all level sets $\{u_\varepsilon = \lambda\}$ are regular then, viewing them as the $(n-1)$-varifolds $[\![\{u_\varepsilon = \lambda\}]\!]$, we have

$$\tilde{V}_\varepsilon = \int_{-1}^1 \sqrt{2W(\lambda)} \cdot [\![\{u_\varepsilon = \lambda\}]\!] \, d\lambda,$$

by the coarea formula. In other words, in this case $\tilde{V}_\varepsilon$ is simply a weighted average of the level sets of u_ε.

Further, given a sequence of critical points (u_ε) with $\varepsilon \to 0$ and

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) < \infty, \tag{6.7}$$

we can assume that the lim inf is a limit (up to a subsequence) and define the measure

$$d\mu = \lim_{\varepsilon \to 0} e_\varepsilon(u_\varepsilon) \, d\,\mathrm{vol}_g,$$

namely μ is the limit of the energy densities in duality with $C^0(M)$, up to a subsequence.

Corollary 6.6. *Assuming* (6.7), *up to a subsequence we can extract a limit varifold* $\tilde{V}_0$, *with weight*

$$\|\tilde{V}_0\| \leq C\mu.$$

If moreover each $u_\varepsilon = \lim_{\delta \to 0} u_{\varepsilon,\delta}$, *for a suitable sequence* $(u_{\varepsilon,\delta})_\delta$ *of critical points for* $E_{\varepsilon,\delta}$ *with Morse index* $\leq m$ *independently of* ε, δ, *then there exists a finite set of points* $\mathcal{S}$, *with* $\#\mathcal{S} \leq m$, *such that any* $p \notin \mathcal{S}$ *admits a neighborhood* U *where the* isotropic first variation $\delta\tilde{V}_0$ *satisfies*

$$|\delta\tilde{V}_0|(U) \leq C(U).$$

The sequence $\delta = \delta_k \to 0$ used in the limit $u_\varepsilon = \lim_{\delta \to 0} u_{\varepsilon,\delta}$ is allowed to depend on ε. In the last part of the statement, $C(U)$ denotes a finite constant which may depend on all data (in particular, on U).

Proof. The total weight of $\tilde{V}_\varepsilon$ on an open set $U \subseteq M$ equals

$$\|\tilde{V}_\varepsilon\| = \int_U \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| \leq C \int_U e_\varepsilon(u_\varepsilon),$$

by Cauchy–Schwarz and the bound $|\nabla u_\varepsilon|^2 \leq \lambda^{-2}F(\nabla u_\varepsilon)^2$, showing the first claim.

Now assume that $u_\varepsilon = \lim_{\delta \to 0} u_{\varepsilon,\delta}$ and that each $u_{\varepsilon,\delta}$ is stable on U. Then, recalling that the convergence is in $C^{1,\alpha}(M)$ and that the integrand defining $\tilde{V}_\varepsilon$ contains the weight $|\nabla u|$, which vanishes on the complement of $\{\nabla u \neq 0\}$, it is straightforward to check that in the sense of varifolds

$$\tilde{V}_\varepsilon = \lim_{\delta \to 0} \tilde{V}_{\varepsilon,\delta}.$$

Now, since $u_{\varepsilon,\delta}$ is smooth, we can apply Remark 6.5 to say that $\tilde{V}_{\varepsilon,\delta}$ is a weighted average of its level sets. By subadditivity of the first variation, we have

$$|\delta\tilde{V}_{\varepsilon,\delta}|(U) \leq \int_{-1}^1 \sqrt{2W(\lambda)} \cdot |\delta[\![\{u_{\varepsilon,\delta} = \lambda\}]\!]|(U) \, d\lambda$$

$$= \int_{-1}^1 \sqrt{2W(\lambda)} \int_{\{u_{\varepsilon,\delta}=\lambda\} \cap U} |H_{u_{\varepsilon,\delta}}| \, d\mathcal{H}^{n-1} \, d\lambda$$

$$\leq C \int_U |\mathrm{II}_{u_{\varepsilon,\delta}}| \sqrt{2W(u_{\varepsilon,\delta})}|\nabla u_{\varepsilon,\delta}|,$$

where we denoted by $H_{u_{\varepsilon,\delta}}$ the mean curvature of the level set and used again the coarea formula. Since trivially

$$\int_M \varepsilon^{-1} W(u_{\varepsilon,\delta}) \leq E_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \leq C,$$

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from Theorem 6.4 and Cauchy–Schwarz we get

$$|\delta\tilde{V}_{\varepsilon,\delta}|(U) \leq C(U).$$

By lower semicontinuity of the first variation, we deduce that

$$|\delta\tilde{V}_{\varepsilon}|(U) \leq \liminf_{\delta\to 0}|\delta\tilde{V}_{\varepsilon,\delta}|(U) \leq C(U), \tag{6.8}$$

and thus

$$|\delta\tilde{V}_0|(U) \leq \liminf_{\varepsilon\to 0}|\delta\tilde{V}_{\varepsilon}|(U) \leq C(U).$$

The conclusion is standard: for every ε, δ, let $S_{\varepsilon,\delta,r}$ be the set of points p such that $u_{\varepsilon,\delta}$ is unstable on $B_r(p)$. By Vitali's covering lemma, there exists a subcollection $S'_{\varepsilon,\delta,r}$ such that the balls $B_r(p)$ with $p \in S'_{\varepsilon,\delta,r}$ are disjoint, giving in particular $\#S'_{\varepsilon,\delta,r} \leq m$, and such that $u_{\varepsilon,\delta}$ is stable on $B_r(q)$ unless $q \in \bigcup_{p\in S'_{\varepsilon,\delta,r}} B_{5r}(p)$. The desired set is obtained by taking a limit of $S'_{\varepsilon,\delta,r}$ in the Hausdorff topology as $\delta \to 0$, then as $\varepsilon \to 0$, and finally as $r \to 0$ (along subsequences). $\qquad\square$

We are now in a position to prove the following. In fact, later on we will just use the consequence (6.10), even if the rectifiability of $\tilde{V}_0$ will simplify the argument. Later, $\tilde{V}_0$ will be replaced by a more appropriate varifold V, directly tied to the anisotropic first variation; the latter will be shown to be (c_W times) integer rectifiable on all of M.

Theorem 6.7. *On $M \setminus \mathcal{S}$ we have*

$$c\mu \leq \|\tilde{V}_0\| \leq C\mu$$

and the density of $\tilde{V}_0$ satisfies

$$\Theta^{n-1}(\|\tilde{V}_0\|, x) := \lim_{r\to 0}\frac{\|\tilde{V}_0\|(B_r(x))}{r^{n-1}} \in (0, \infty) \tag{6.9}$$

at $\|\tilde{V}_0\|$-almost every $x \in M \setminus \mathcal{S}$ (or equivalently at μ-a.e. $x \in M \setminus \mathcal{S}$), and hence $\tilde{V}_0$ is rectifiable on $M \setminus \mathcal{S}$. As a consequence, μ is a rectifiable measure on $M \setminus \mathcal{S}$ and

$$0 < \Theta^{n-1}_*(\mu, x) \leq \Theta^{n-1,*}(\mu, x) < \infty \quad \text{for } \mu\text{-a.e. } x \in M \setminus \mathcal{S}, \tag{6.10}$$

where $\Theta^{n-1}_*(\mu, x) := \liminf_{r\to 0}\frac{\mu(B_r(x))}{r^{n-1}}$ and $\Theta^{n-1,*}(\mu, x) := \limsup_{r\to 0}\frac{\mu(B_r(x))}{r^{n-1}}$.

Proof. While the upper bound $\|\tilde{V}_0\| \leq C\mu$ was already obtained (on all of M), we claim that the lower bound $\|\tilde{V}_0\| \geq c\mu$ on $M \setminus \mathcal{S}$ follows from (6.9). Indeed, given $p \notin \mathcal{S}$, let us fix a ball $B_{2r}(p) \subseteq U$, where U is the neighborhood given by the previous result. First of all, by Theorem 5.1, we have

$$\|\tilde{V}_{\varepsilon}\|(B_{2r}(p)) \geq c\int_{B_{2r}(p)}\sqrt{2W(u_{\varepsilon})}F(\nabla u_{\varepsilon}) \geq \int_{B_{2r}(p)}[c\varepsilon F(\nabla u)^2 - C\varepsilon F(\nabla u)] \geq c\int_{B_{2r}(p)}\varepsilon F(\nabla u)^2 - Cr^n,$$

thanks to the uniform C^1 bound $F(\nabla u_{\varepsilon}) \leq C\varepsilon^{-1}$, which again follows from Theorem 5.1.

By Lemma 6.3, which applies since $u_{\varepsilon,\delta}$ is stable on U by construction (for ε, and in turn δ, small enough), we deduce that

$$\|\tilde{V}_{\varepsilon}\|(B_{2r}(p)) \geq c\int_{B_r(p)}e_{\varepsilon}(u_{\varepsilon}) - Cr^n,$$

and hence in the limit $\varepsilon \to 0$ we get $\|\tilde{V}_0\|(\overline{B}_{2r}(p)) \geq c\mu(B_r(p)) - Cr^n$. Approximating r from below, we deduce

$$\|\tilde{V}_0\|(B_{2r}(p)) \geq c\mu(B_r(p)) - Cr^n.$$

Assuming that $\|\tilde{V}_0\|(B_r(p)) \geq c(p)r^{n-1}$ for any $r > 0$ small enough, we can then find $r > 0$ as small as we want and such that

$$\|\tilde{V}_0\|(B_{2r}(p)) \geq c(p)r^{n-1}, \quad \|\tilde{V}_0\|(B_r(p)) \geq 4^{-n}\|\tilde{V}_0\|(B_{2r}(p)).$$

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We then have $r^n = o(\|\tilde{V}_0\|(B_r(p)))$, and thus

$$\|\tilde{V}_0\|(B_r(p)) \geq 4^{-n}\|\tilde{V}_0\|(B_{2r}(p)) \geq c\mu(B_r(p)),$$

from which the bound $\|\tilde{V}_0\| \geq c\mu$ follows, thanks to Besicovitch's differentiation theorem.

To show the second part of the statement, it suffices to show that $\Theta^{n-1,*}(\|\tilde{V}_0\|, x) > 0$ for $\|\tilde{V}_0\|$-a.e. x: indeed, we already proved in Corollary 6.6 that the varifold $\tilde{V}_0$ has locally bounded (isotropic) first variation in $M \setminus \mathcal{S}$, so that here its rectifiability follows by Allard's classical rectifiability criterion [2, Section 5], which gives (6.9). Finally, (6.10) is now clear on $M \setminus \mathcal{S}$.

To check that $\Theta^{n-1,*}(\|\tilde{V}_0\|, x) > 0$ for $\|\tilde{V}_0\|$-a.e. x, let us fix U as above, and fix $U' \subset U$ and $r_0 > 0$ such that $B_{2r_0}(x) \subseteq U$ for all $x \in U'$. Given $\gamma > 0$, we let $\mathcal{G}_{\varepsilon,\gamma}$ denote the set of points $x \in U'$ such that

$$|u_\varepsilon(x)| \leq 1 - 2\gamma, \quad |\delta\tilde{V}_\varepsilon|(B_r(x)) \leq \frac{\|\tilde{V}_\varepsilon\|(B_r(x))}{\gamma} \tag{6.11}$$

for all $r \in (0, r_0)$. Since $|\delta\tilde{V}_\varepsilon|(U) \leq C(U)$ by (6.8), a simple application of Besicovitch's covering lemma gives

$$\|\tilde{V}_\varepsilon\|(U' \setminus \mathcal{G}_{\varepsilon,\gamma}) \leq C(U)\gamma + \int_{U' \cap \{|u_\varepsilon|>1-2\gamma\}} \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|.$$

We claim that the last term vanishes in the limit $\varepsilon, \gamma \to 0$. Indeed, testing the Euler–Lagrange equation (3.4) with $\lambda - u_\varepsilon$ we have

$$\int_M \varepsilon^{-1}W'(u_\varepsilon)(\lambda - u_\varepsilon) \leq C \int_M \varepsilon|\nabla u_\varepsilon|^2 \leq CE_\varepsilon(u_\varepsilon) \leq C;$$

choosing $\lambda \in (-1, 1)$ to be the maximum point of $W|_{[-1,1]}$ (recall that $\sqrt{W}$ is assumed to be strictly concave on $[-1, 1]$), we have $(\lambda - s)W'(s) \geq 0$ for all $s \in [-1, 1]$, and actually

$$(\lambda - s)W'(s) \geq -c\,\mathrm{sgn}(s)W'(s) \geq c\sqrt{W(s)} \quad \text{for } |s| \in [1 - 2\gamma, 1]$$

for $\gamma > 0$ small enough, as $W''(\pm 1) > 0$. We infer that

$$\int_{\{|u_\varepsilon|>1-2\gamma\}} \varepsilon^{-1}\sqrt{W(u_\varepsilon)} \leq C,$$

and thus, since $\sqrt{W(s)} \leq C(1 - |s|)$ on $[-1, 1]$ (again as $W''(\pm 1) > 0$),

$$\int_{\{|u_\varepsilon|>1-2\gamma\}} \varepsilon^{-1}W(u_\varepsilon) \leq C\gamma. \tag{6.12}$$

By Theorem 5.1 and the last two bounds, we have shown that

$$\int_{\{|u_\varepsilon|>1-2\gamma\}} \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| \leq C \int_{\{|u_\varepsilon|>1-2\gamma\}} [\varepsilon^{-1}W(u_\varepsilon) + \sqrt{W(u_\varepsilon)}] \leq C\gamma + C\varepsilon.$$

In summary, we deduce that there exists a small $\gamma_0 > 0$ such that

$$\|\tilde{V}_\varepsilon\|(U' \setminus \mathcal{G}_{\varepsilon,\gamma}) \leq C(U)(\gamma + \varepsilon)$$

for all $\gamma \in (0, \gamma_0)$ and $\varepsilon > 0$ small.

Up to a subsequence, let $\mathcal{G}_{0,\gamma}$ denote the Hausdorff limit of the closures $\overline{\mathcal{G}}_{\varepsilon,\gamma}$, along a subsequence $\varepsilon \to 0$ depending on $\gamma \in (0, \gamma_0)$. We claim that

$$\Theta^{n-1}(\|\tilde{V}_0\|, x) > 0$$

at each $x \in \mathcal{G}_{0,\gamma}$. Indeed, thanks to (6.11), we can apply [53, Theorem 17.6] to the varifold $\tilde{V}_\varepsilon$ with $\alpha = 1$; note that here we use geodesic balls rather than Euclidean ones, but the proof still carries through. Hence we obtain

$$\frac{\|\tilde{V}_\varepsilon\|(B_r(x))}{r^{n-1}} \geq c(\gamma)\frac{\|\tilde{V}_\varepsilon\|(B_{\rho\varepsilon}(x))}{(\rho\varepsilon)^{n-1}} \geq c(\gamma) > 0$$

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for any $x \in \mathcal{G}_{\varepsilon,\gamma}$ and $r \in (\rho\varepsilon, r_0)$, thanks to Lemma 6.2, which applies as $B_{\rho\varepsilon}(x) \subseteq U$ and $|u_{\varepsilon,\delta}(x)| \leq 1 - \gamma$ for δ small enough, by (6.11). Thus, we also have

$$\frac{\|\tilde{V}_0\|(\overline{B}_r(x))}{r^{n-1}} \geq c(\gamma) > 0$$

for any $x \in \mathcal{G}_{0,\gamma}$. Finally, we have

$$\|\tilde{V}_0\|(U' \setminus \mathcal{G}_{0,\gamma}) \leq \liminf_{\varepsilon \to 0} \|\tilde{V}_\varepsilon\|(U' \setminus \overline{\mathcal{G}}_{\varepsilon,\gamma}) \leq C(U)\gamma.$$

Since $\|\tilde{V}_0\|$ has positive density at all points in $\bigcup_{\gamma \in (0,\gamma_0)} \mathcal{G}_{0,\gamma}$ and we can cover $M \setminus \mathcal{S}$ with countably many such U', the statement follows. Note that the subsequence $\varepsilon \to 0$ defining $\mathcal{G}_{0,\gamma}$ depends on γ, but this is irrelevant. $\square$

7   Stress-energy tensor and integrality of the limit varifold

Given $u_\varepsilon : M \to \mathbb{R}$ critical for E_ε, a straightforward computation using inner variations shows that, for any vector field $X \in C^1(M, TM)$, denoting by DX the $(1,1)$-tensor given by the Levi-Civita connection, we have

$$\int_M \langle T_\varepsilon, DX \rangle = \int_M O(\varepsilon|\nabla u|^2 |X|), \tag{7.1}$$

where T_ε is the stress-energy tensor, namely the $(1,1)$-tensor given by

$$T_\varepsilon := e_\varepsilon(u_\varepsilon)I - \varepsilon \nabla u_\varepsilon \otimes D(F^2/2)(\nabla u_\varepsilon), \tag{7.2}$$

with $D(F^2/2)(\nabla u_\varepsilon) \in T_x^* M$ denoting the differential of $F_x^2/2 = (F^2/2)|_{T_x M}$ (viewed as a function $T_x M \to \mathbb{R}$) at $\nabla u(x)$, for any given $x \in M$; the error term comes from the fact that F_x^2 depends on x.

In the sequel, it is useful to consider the map

$$C_F(\nu) := I - \frac{\nu \otimes DF(\nu)}{F(\nu)},$$

where $\nu \in TM$ is a unit vector. We observe that

$$T_\varepsilon = [e_\varepsilon(u_\varepsilon) - \varepsilon F(\nabla u_\varepsilon)^2]I + \varepsilon F(\nabla u_\varepsilon)^2 C_F(\nu_\varepsilon), \quad \nu_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \quad \text{on } \{\nabla u_\varepsilon \neq 0\},$$

while we let $\nu_\varepsilon := 0$ and $C_F(\nu_\varepsilon) := 0$ on $\{\nabla u_\varepsilon = 0\}$. We have the trivial bound

$$|T_\varepsilon| \leq C e_\varepsilon(u_\varepsilon).$$

Assuming (6.7), let T_0 be a limit of the measures $T_\varepsilon \, d\,\mathrm{vol}_g$, up to a subsequence.

Proposition 7.1. *The limit T_0 has the form*

$$dT_0 = ((1 - \lambda)I + \lambda A) \, d\mu + d\xi,$$

where $\lambda : M \to [0,1]$ and the measure ξ satisfies $|\xi| \leq C \,\mathrm{vol}_g$, while for μ-a.e. x the tensor $A(x)$ belongs to the (compact) convex hull

$$\mathcal{C}_x := \mathrm{co}(\{C_F(\nu) : \nu \in T_x M, \ |\nu| = 1\}).$$

Proof. Indeed, we have

$$T_\varepsilon = [(1 - \lambda_\varepsilon)I + \lambda_\varepsilon C_F(\nu_\varepsilon)]e_\varepsilon(u_\varepsilon) + \xi_\varepsilon,$$

where

$$\lambda_\varepsilon := \min\left\{\frac{\varepsilon F(\nabla u_\varepsilon)^2}{e_\varepsilon(u_\varepsilon)}, 1\right\} \in [0,1] \tag{7.3}$$

and defined to be zero on $\{e_\varepsilon(u_\varepsilon) = 0\}$, and

$$\xi_\varepsilon = -(\varepsilon F(\nabla u_\varepsilon)^2 - e_\varepsilon(u_\varepsilon))^+ I + (\varepsilon F(\nabla u_\varepsilon)^2 - e_\varepsilon(u_\varepsilon))^+ C_F(\nu_\varepsilon).$$

Crucially, by the generalization of Modica's bound, namely Theorem 5.1, we have $\varepsilon F(\nabla u_\varepsilon)^2 - e_\varepsilon(u_\varepsilon) \leq C$ pointwise, and thus

$$|\xi_\varepsilon| \leq C.$$

Since $(1 - \lambda_\varepsilon)I + \lambda_\varepsilon C_F(\nu_\varepsilon) \in \mathrm{co}(\{I\} \cup \mathcal{C}_x)$ at any given $x \in M$, it follows that

$$dT_0 = B\,d\mu + d\xi,$$

where ξ is the limit of $\xi_\varepsilon\,d\mathrm{vol}_g$ and $B(x) \in \mathrm{co}(\{I\} \cup \mathcal{C}_x)$ a.e., as desired. $\qquad\square$

Theorem 7.2. *Assuming that $u_\varepsilon = \lim_{\delta \to 0} u_{\varepsilon,\delta}$ is a limit of critical points for $E_{\varepsilon,\delta}$ with Morse index $\leq m$, then the measures T_0 and μ are rectifiable. In fact, we have*

$$\lambda = 0, \quad A(x) = C_F(\nu_x)$$

for a suitable unit vector $\nu_x \in T_x M$, at μ-a.e. $x \in M$. Furthermore, writing $d\mu = \theta\,d(\mathcal{H}^{n-1} \llcorner \Sigma)$ for a suitable rectifiable Borel set $\Sigma \subset M$ with σ-finite $\mathcal{H}^{n-1}$ measure and $\theta : \Sigma \to (0, \infty)$, the $(n-1)$-dimensional varifold

$$dV(x, \nu^\perp) := \frac{\theta(x)}{F(\nu_x)} \delta_{\nu_x}(\nu) \otimes d(\mathcal{H}^{n-1} \llcorner \Sigma)(x)$$

is rectifiable and F-stationary, and ν_x is $\mathcal{H}^{n-1}$-a.e. the unit normal to Σ (unique up to sign).

It follows from the formula for V that its anisotropic energy is

$$\mathcal{F}(V) = \mu(M), \quad \mathcal{F}(V; U) = \mu(U) \text{ for all } U \subseteq M \text{ Borel.}$$

Proof. We observe that μ-a.e. $x \in M$ is an approximate continuity point of λ and A, and moreover by Theorem 6.7 satisfies

$$0 < \Theta^{n-1}_*(\mu, x) \leq \Theta^{n-1,*}(\mu, x) < \infty, \tag{7.4}$$

provided that $x \notin \mathcal{S}$. In this case, thanks to Proposition 7.1, any blow-up $\tilde{T}_0$ of T_0 at any such point $x_0 \in M$ will be of the form

$$d\tilde{T}_0 = B\,d\tilde{\mu},$$

where

$$B = (1 - \lambda_0)I + \lambda_0 \int_{\mathbb{S}^{n-1}} C_{F_{x_0}}(\nu)\,d\alpha(\nu) \tag{7.5}$$

is a constant matrix given by a suitable $\lambda_0 \in [0, 1]$ and a probability measure α on $\mathbb{S}^{n-1}$, and $\tilde{\mu}$ is a blow-up of μ at x_0. Indeed, ξ disappears in the blow-up, as $|\xi|(B_r(x_0)) = o(r^{n-1}) = o(\mu(B_r(x_0)))$, thanks to the fact that $\mu(B_r(x_0)) \geq cr^{n-1}$ for r small. Note that a blow-up defined on $\mathbb{R}^n$ exists also when $x_0 \in \mathcal{S}$, in which case $\tilde{\mu} = \delta_0$.

As guaranteed by (7.4) (or by $\tilde{\mu} = \delta_0$), the measure $\tilde{\mu}$ cannot be a constant multiple of the Lebesgue measure on $T_{x_0} M \cong \mathbb{R}^n$, as $\Theta^{n-1}_*(\tilde{\mu}, 0) > 0$. On the other hand, (7.1) implies

$$|\langle T_0, DX \rangle| \leq C \int_M |X|\,d\mu;$$

given $Y \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, we can plug $X(p) := \sum_{i=1}^n Y^i(r^{-1} \exp_{x_0}^{-1}(p))e_i(p)$, extended to zero outside the domain of $\exp_{x_0}^{-1}$, for a fixed orthonormal frame $\{e_i\}_{i=1}^n$ defined near x_0 inducing the chosen identification $T_{x_0} M \cong \mathbb{R}^n$. Letting $r \to 0$, we deduce that

$$\langle \tilde{T}_0, DY \rangle = 0 \quad \text{for all } Y \in C^1_c(\mathbb{R}^n, \mathbb{R}^n).$$

Taking Y of the form $Y = \varphi v$, we then see that $\tilde{\mu}$ is invariant along $B^\top v$. Since $\tilde{\mu}$ is not a multiple of the Lebesgue measure, we deduce that

$$\ker(B) \neq \{0\}.$$

Let then $\nu_0 \in \ker(B)$ be a unit vector. Recalling (7.5), we see that

$$0 = \langle DF_{x_0}(\nu_0), B\nu_0 \rangle = (1 - \lambda_0) F_{x_0}(\nu_0) + \lambda_0 \int_{\mathbb{S}^{n-1}} \langle DF_{x_0}(\nu_0), C_{F_{x_0}}(\nu)\nu_0 \rangle \, d\alpha(\nu). \tag{7.6}$$

By 1-homogeneity and convexity of F_{x_0}, we have that

$$DF_{x_0}(\nu)[\nu] = F(\nu), \quad DF_{x_0}(\nu)[\nu'] = F_{x_0}(\nu) + DF_{x_0}(\nu)[\nu' - \nu] \le F_{x_0}(\nu')$$

and, since F is even, it also holds that $|DF_{x_0}(\nu)[\nu_0]| \le F_{x_0}(\nu_0)$ and $|DF_{x_0}(\nu_0)[\nu]| \le F_{x_0}(\nu)$. Hence, we obtain that

$$\langle DF_{x_0}(\nu_0), C_{F_{x_0}}(\nu)\nu_0 \rangle = F_{x_0}(\nu_0) - \frac{DF_{x_0}(\nu)[\nu_0]}{F(\nu)} DF_{x_0}(\nu_0)[\nu] \ge 0. \tag{7.7}$$

Plugging (7.7) in (7.6), we deduce that $\lambda_0 = 1$. Moreover, since F_{x_0} is strictly convex along non-radial directions, (7.7) can be an equality only when $\nu = \pm\nu_0$.

Thus, α is concentrated on $\{\pm\nu_0\}$ so that, using again that F is even, we obtain

$$B = A(x_0) = I - \frac{\nu_0 \otimes DF_{x_0}(\nu_0)}{F(x_0)}.$$

For the same reason, $\ker(B(x_0)) = \operatorname{span}\{\nu_0\}$ and thus the image of $B^\top$ is $\nu_0^\perp$. Hence, $\tilde\mu$ is invariant along the hyperplane $\nu_0^\perp$.

We cannot have $x_0 \in \mathcal{S}$, since otherwise we would have $\tilde\mu = \delta_0$, a contradiction with the invariance of $\tilde\mu$ along $\nu_0^\perp$. Hence, $\mu(\mathcal{S}) = 0$ and the conclusion follows from the rectifiability of μ proved in Theorem 6.7, since for generic x_0 any blow-up must be a constant multiple of $\mathcal{H}^{n-1} \llcorner \nu_\Sigma(x_0)^\perp$, yielding $\nu_0 = \pm\nu_\Sigma(x_0)$. The F-stationarity of V is equivalent to the fact that T_0 is divergence-free. $\qquad\square$

Remark 7.3. The previous proof generalizes the *atomic condition* found in [20, Definition 1.1], which in codimension one characterizes convex integrands F that are strictly convex along non-radial directions: see [20, Theorem 1.3]. In fact, we could avoid appealing to the rectifiability of $\|\tilde V_0\|$ and rely just on (6.10): once we reach $\lambda_0 = 0$, the rectifiability of μ follows from [20, Lemma 2.2].

Remark 7.4. We observe that

$$\lim_{\varepsilon \to 0} \int_M (\varepsilon F(\nabla u_\varepsilon)^2 - e_\varepsilon(u_\varepsilon))^+ = \lim_{\varepsilon \to 0} \int_M \left(\frac{\varepsilon}{2} F(\nabla u_\varepsilon)^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \right)^+ = 0. \tag{7.8}$$

Indeed, any weak limit υ of the integrand (as a measure) satisfies $\upsilon \le C\mu$, as well as $\upsilon \le C \operatorname{vol}_g$ by Theorem 5.1. Since the measures μ and vol_g are mutually singular, we deduce that $\upsilon = 0$ and hence (7.8). In particular, we also have

$$\xi = 0.$$

Remark 7.5. Recalling the definition of λ_ε from (7.3), on the compact set

$$\hat G := \{\operatorname{co}(\{I\} \cup \mathcal{C}_x) \mid x \in M\}$$

we can consider the positive measures

$$dV_\varepsilon(x, Z) := \delta_{B_\varepsilon(x)}(Z) e_\varepsilon(u_\varepsilon)(x) \, d\operatorname{vol}_g(x),$$

where

$$B_\varepsilon(x) := (1 - \lambda_\varepsilon(x)) I + \lambda_\varepsilon(x) C_F(\nu_\varepsilon(x)) = \frac{T_\varepsilon(x) - \xi_\varepsilon(x)}{e_\varepsilon(u_\varepsilon)(x)}$$

(the last equality holding on $\{e_\varepsilon(u_\varepsilon) > 0\}$), and a subsequential limit V_0. Letting $\pi : \hat G \to M$ denote the canonical projection and considering the disintegration of $dV_0(x, Z) = \alpha_x(Z) \otimes d\mu(x)$ with respect to π, let V_0' denote the $\hat G$-valued measure on M given by replacing each probability measure α_x with its center of mass, namely

$$dV_0'(x) := Z_x \, d\mu(x), \quad Z_x := \int_{\pi^{-1}(x)} Z \, d\alpha_x(Z).$$

It is straightforward to check that, if we have a sequence of measures $W_k \rightharpoonup W_\infty$ in $\hat{G}$, then $W_k' \rightharpoonup W_\infty'$. In particular, since $V_\varepsilon \rightharpoonup V_0$ and $V_\varepsilon' = (T_\varepsilon - \xi_\varepsilon) \, d\,\mathrm{vol}_g \rightharpoonup T_0$ as $\xi_\varepsilon \to 0$ by Remark 7.4, we deduce that $V_0' = T_0$. Exactly the same argument used in the previous proof then shows that the disintegration α_x of V_0 consists of a Dirac mass at μ-a.e. $x \in M$, i.e., $V_0 = V$ up to identifying $C_F(\nu)$ with $\nu^\perp$.

We are now left to prove that $c_W^{-1} V$ is actually an integral varifold, or equivalently

$$\frac{\theta(x)}{c_W F(\nu_x)} \in \mathbb{N}, \quad \text{for } \|V\| - a.e.\ x,$$

where we recall that $c_W = \int_{-1}^1 \sqrt{2W(t)} \, dt$.

We can fix a μ-generic point x_0, so that we can assume $x_0 \notin \mathcal{S}$ as $\mu(\mathcal{S}) = 0$, (6.10) holds at x_0, and μ has an approximate tangent plane at x_0; as seen in the previous proof, we can also assume that μ and T_0 blow up to

$$\theta \cdot \mathcal{H}^{n-1} \llcorner \nu^\perp, \quad \theta C_{F_{x_0}}(\nu) \cdot \mathcal{H}^{n-1} \llcorner \nu^\perp,$$

respectively, where $\nu = \nu_{x_0}$ and $\theta = \theta(x_0)$. Further, by (7.8) and (6.12) we have

$$\limsup_{\varepsilon \to 0} \int_{\{|u_\varepsilon| > 1 - 2\gamma\}} e_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon \to 0} \int_{\{|u_\varepsilon| > 1 - 2\gamma\}} 2W(u_\varepsilon) + \limsup_{\varepsilon \to 0} \int_M \left(\frac{\varepsilon}{2} F(\nabla u_\varepsilon)^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \right)^+ \leq C\gamma.$$

Hence, an application of Besicovitch's covering lemma shows that, for a constant C which may depend on the point x_0, we can also assume that

$$\limsup_{\varepsilon \to 0} \int_{B_r(x_0) \cap \{|u_\varepsilon| \geq 1 - \gamma\}} e_\varepsilon(u_\varepsilon) \leq C\gamma\mu(\overline{B}_r(x_0)) \leq C\gamma r^{n-1} \quad \text{for all } r \in (0, 1). \tag{7.9}$$

We choose coordinates so that $x_0 = 0$ and the weak tangent plane is (θ times)

$$P := \{x \in \mathbb{R}^n \ : \ x^n = 0\}.$$

By a diagonal argument, we can replace (u_ε) with a sequence of critical points, defined on larger and larger balls $B_{R_\varepsilon}(0)$ endowed with metrics g_ε converging to the Euclidean one and integrands $F^\varepsilon \to F_{x_0}$, such that

$$e_\varepsilon(u_\varepsilon) \, dx \rightharpoonup \theta \, d\mathcal{H}^{n-1} \llcorner P,$$

as well as

$$T_\varepsilon \, dx \rightharpoonup \theta C_{F_{x_0}}(e_n) \, d\mathcal{H}^{n-1} \llcorner P.$$

Remark 7.6. Since the new sequence u_ε is obtained by rescaling a sequence of critical points on a closed manifold, the pointwise bound (1.2) still holds. Actually, we have

$$\varepsilon F(\nabla u_\varepsilon)^2/2 \leq \varepsilon^{-1} W(u_\varepsilon) + c_\varepsilon, \quad c_\varepsilon \to 0, \tag{7.10}$$

since if the new u_ε is obtained by rescaling a function $u_{\varepsilon'}$ from the original sequence then the constant C in (1.2) becomes $C\varepsilon'/\varepsilon = C/\rho$, where ρ is the (larger and larger) dilation factor.

Moreover, using (7.9) and another diagonal argument, we can even assume that

$$\limsup_{\varepsilon \to 0} \int_{B_{10}(0) \cap \{|u_\varepsilon| \geq 1 - \gamma\}} e_\varepsilon(u_\varepsilon) \leq C\gamma \tag{7.11}$$

for any $\gamma \in \{2^{-k} \mid k \in \mathbb{N}\}$, and hence for all $\gamma \in (0, 1)$.

Let $\eta > 0$ and $S_\varepsilon := \{|\nu_\varepsilon - e_n| > \eta\}$, where $\nu_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ is defined using the metric g_ε, while $|\nu_\varepsilon - e_n|$ is computed using the Euclidean metric. Then Remark 7.5, namely the fact that

$$\delta_{B_\varepsilon(x)}(Z) e_\varepsilon(u_\varepsilon)(x) \, d\,\mathrm{vol}_g(x) = dV_\varepsilon(x, Z) \rightharpoonup dV_0(x, Z) = \delta_{C_{F_{x_0}}(e_n)}(Z) \, d(\theta \mathcal{H}^{n-1} \llcorner P)(x),$$

gives

$$\int_{B_{10}(0)\cap S_\varepsilon} e_\varepsilon(u_\varepsilon) = V_\varepsilon(\pi^{-1}(B_{10}(0)\cap S_\varepsilon)) \to 0.$$

On the other hand, on $B_{10}(0) \setminus S_\varepsilon$ we have $|\partial_i u_\varepsilon| \leq C\eta|\nabla u_\varepsilon| \leq C\eta F(\nabla u_\varepsilon)$ for $i < n$. Hence, for all $i = 1, \ldots, n-1$ we have

$$\limsup_{\varepsilon\to 0} \int_{B_{10}(0)} \varepsilon|\partial_i u_\varepsilon|^2 \leq \limsup_{\varepsilon\to 0} \int_{B_{10}(0)\cap S_\varepsilon} \varepsilon|\partial_i u_\varepsilon|^2 + \limsup_{\varepsilon\to 0} \int_{B_{10}(0)\setminus S_\varepsilon} \varepsilon|\partial_i u_\varepsilon|^2 \leq C\eta^2 \int \varepsilon F^2(\nabla u_\varepsilon),$$

so that

$$\lim_{\varepsilon\to 0} \int_{B_{10}(0)} \varepsilon|\partial_i u_\varepsilon|^2 = 0. \tag{7.12}$$

Let us fix a nonnegative cut-off function $\chi \in C_c^1((-1,1))$, equal to 1 on $(-1/2, 1/2)$.

Lemma 7.7. *There exist Borel sets $E_\varepsilon \subset B_1^{n-1}(0)$ such that*

$$\mathcal{L}^{n-1}(E_\varepsilon) \to 0$$

and, for any sequence of points $p_\varepsilon \in B_1^{n-1}(0) \setminus E_\varepsilon$,

$$\int_{\{p_\varepsilon\}\times[-1,1]} \chi(x^n)e_\varepsilon(u_\varepsilon(p_\varepsilon, x^n))\, dx^n \to \theta.$$

Proof. We test (7.1) with a vector field of the form $\chi\varphi e_j$, where $\varphi \in C_c^1(B_1^{n-1}(0))$ and $j = 1, \ldots, n-1$. We get

$$\int_{B_1^{n-1}(0)} e_\varepsilon(u_i)\partial_j(\chi\varphi) = \int_{B_1^{n-1}(0)} \varepsilon\partial_j u_\varepsilon D(F^2/2)(\nabla u_\varepsilon)[\nabla\varphi] + o(\|\nabla\varphi\|_{C^0}),$$

where the error term comes from the ambient metric g_ε, which converges to the Euclidean one as $\varepsilon \to 0$, and the error term in (7.1), which is also negligible in the limit $\varepsilon \to 0$, as F^ε converges to the autonomous integrand F_{x_0}. By (7.12), letting

$$f_\varepsilon(x^1, \ldots, x^{n-1}) := \int_{-1}^1 \chi(x^n)e_\varepsilon(u_\varepsilon(x^1, \ldots, x^n))\, dx^n$$

be the integral on the slice, we then have

$$\left|\int_{B_1^{n-1}(0)} f_\varepsilon\partial_j\varphi\right| \leq \sigma_\varepsilon\|\nabla\varphi\|_{C^0}$$

for a vanishing sequence $\sigma_\varepsilon \to 0$, as $\varepsilon \to 0$. The conclusion then follows from Allard's strong constancy lemma [5, Theorem 1.(4)], once we recall that

$$f_\varepsilon\, dx \rightharpoonup \theta\, dx$$

as measures on $B_1^{n-1}(0)$. $\qquad\square$

We have the following simple result in $\mathbb{R}^n$, for autonomous F.

Lemma 7.8. *Given $\Lambda > 0$ and $\gamma \in (0,1)$, there exists $\delta(\Lambda, \gamma, F) > 0$ such that either*

$$|u| \geq 1 - C\gamma \quad on\ \{0\} \times (-R, R)$$

or the energy on the slice

$$\frac{1}{F(e_n)} \int_{\{0\}\times(-R,R)} e_1(u) \in \bigcup_{k\in\mathbb{N}\setminus\{0\}} (kc_W - Ck\gamma^2, kc_W + Ck\gamma^2) \tag{7.13}$$

for any critical point

$$u : B_1^{n-1}(0) \times (-R, R) \to \mathbb{R}$$

of the energy E_1, provided that $R \in (0, \Lambda]$, $|u| \le 1$, $F(\nabla u)^2 \le W(u) + \gamma^2$,

$$\lim_{t \to \pm R} |u(0, t)| \in [1 - \gamma, 1],$$

and

$$\int_{B_1^{n-1}(0) \times (-R, R)} \sum_{j=1}^{n-1} |\partial_j u|^2 \le \delta.$$

In other words, if u is almost constant along the first $n - 1$ coordinate directions, then its energy is close to ($F(e_n)$ times) a multiple of the energy $c_W = \int_{-1}^{1} \sqrt{2W}$ of the heteroclinical solution, unless u is essentially constant. In the statement, C is a universal constant depending just on W, on which the other constants tacitly depend as well.

Proof. Assuming for every $\delta > 0$ we have a counterexample $u^\delta : B_1^{n-1}(0) \times (-R_\delta, R_\delta) \to \mathbb{R}$, in the limit $\delta \to 0$ (up to a subsequence) we get a one-dimensional function $u(x) = U(x^n)$ critical for E_1, defined on $B_1^{n-1}(0) \times (-R, R)$ for a limiting $R = \lim_{\delta \to 0} R_\delta \in [0, \Lambda]$; we can assume that $R > 0$ since otherwise the first conclusion for u^δ trivially holds, thanks to the Lipschitz bound on u^δ (implied by the assumption $F(\nabla u^\delta)^2 \le W(u^\delta) + \gamma^2$).

Note that we can pass to the limit the Euler–Lagrange equation (3.4) and also

$$\lim_{\delta \to 0} \int_{\{0\} \times (-R_\delta, R_\delta)} e_1(u^\delta) = \int_{\{0\} \times (-R, R)} e_1(u),$$

thanks to the same Lipschitz bound and Proposition 3.3, which upgrades it to convergence in $C_{loc}^{1, \alpha}$. Since u is one-dimensional, it is smooth and satisfies

$$U : (-R, R) \to \mathbb{R}, \quad F^2(e_n) U'' = W'(U).$$

Thus, $\xi := F^2(e_n)(U')^2/2 - W(U)$ is constant; actually, it is bounded by $C\gamma^2$ in absolute value: this follows from the uniform convergence $u^\delta \to u$ and from equicontinuity, which give

$$\lim_{t \to \pm R} |u(0, t)| = \lim_{\delta \to 0} \lim_{t \to \pm R_\delta} |u^\delta(0, t)| \in [1 - \gamma, 1]$$

and thus $\lim_{t \to \pm R} W(u)(0, t) \le C\gamma^2$, as well as the fact that $F(\nabla u)^2 \le W(u) + \gamma^2$.

The solution U can be extended to all of $\mathbb{R}$ and, by classical theory of Newtonian systems (viewing $-W(U)$ as the potential energy), its extension is periodic if $\xi < 0$, constant or heteroclinic if $\xi = 0$ (i.e., either $U \equiv \pm 1$ or it is monotone, with limits ± 1 at infinity), or divergent if $\xi > 0$ (i.e., goes to $\pm\infty$ at infinity, since $U' = \pm\frac{\sqrt{2W(U) + 2\xi}}{F(e_n)}$ with constant sign; recall that the right-hand side is $\simeq \pm|U|$ for $|U| \ge 2$). Here we are assuming that γ is small enough, so that the steady state in $(-1, 1)$, corresponding to the maximum point for $W|_{(-1,1)}$, is not one of the possibilities.

In the second and third cases, U is monotone; applying the elementary inequality $0 \le a + b - 2\sqrt{ab} \le |a - b|$ with $a := F^2(e_n)(U')^2/2$ and $b := W(U)$, we see that

$$\int_{\{0\} \times (-R, R)} e_1(u) = \int_{-R}^{R} F(e_n) \sqrt{2W(U)} |U'| + O(\gamma^2), \tag{7.14}$$

from which the claim follows: either $|U| \ge 1 - \gamma$ or (7.13) holds with $k = 1$, thanks to a simple change of variables and the boundary condition $|U(\pm R)| \in [1 - \gamma, 1]$.

If instead we are in the first case where $\xi < 0$, then we can divide $[-R, R]$ into $N \ge 1$ consecutive intervals $[t_j, t_{j+1}]$ where U is monotone, for $j = 0, \ldots, N - 1$, such that $t_0 = -R$, $t_N = R$, and

$$U(t_j) \in W^{-1}(-\xi) \quad \text{for all } j = 1, \ldots, N - 1$$

(as $U'(t_j) = 0$ and hence $\xi = -W(U(t_j))$ for these j). Using again (7.14) and writing

$$\int_{-R}^{R} F(e_n)\sqrt{2W(U)}|U'| = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} F(e_n)\sqrt{2W(U)}|U'|,$$

the claim follows again from the monotonicity of U on each interval, together with the fact that $U(t_j) \in W^{-1}(-\xi)$ forces $|U(t_j)| \in [1 - C\gamma, 1]$, as $-\xi \in [0, C\gamma^2]$ and W is nearly quadratic around ± 1. Thus, the claim must also hold for u^δ eventually, a contradiction. $\quad\square$

Remark 7.9. Clearly, the same also holds on a small geodesic ball in M, rescaled to have unit size. Indeed, on such rescaled ball, the metric is close to the Euclidean one and F is almost autonomous.

We are finally ready to prove integrality of the limit varifold V built in Theorem 7.2, which reduces to the following statement thanks to the blow-up reduction.

Theorem 7.10. We have $\frac{\theta}{F_{x_0}(e_n)} \in c_W \mathbb{N}$.

Proof. Let us fix $\gamma \in (0, 1)$ small and recall that, since by assumption the limit of the energy densities is supported on P, we have

$$\int_{B_{10}(0)} (1 - \chi(x^n))e_\varepsilon(u_\varepsilon)(x)\, dx \to 0. \tag{7.15}$$

Using the previous bounds, we can find $p_\varepsilon \in B_1^{n-1}(0)$ such that

$$\int_{-1}^{1} (1 - \chi(t))e_\varepsilon(u_\varepsilon)(p_\varepsilon, t)\, dt \to 0,$$

thanks to (7.15), as well as

$$\int_{-1}^{1} \chi(t)e_\varepsilon(u_\varepsilon)(p_\varepsilon, t)\, dt \to \theta$$

by virtue of Lemma 7.7,

$$\frac{1}{\varepsilon^{n-1}} \int_{B_\varepsilon^{n-1}(p_\varepsilon) \times [-1,1]} \sum_{j=1}^{n-1} |\partial_j u_\varepsilon|^2 \to 0 \tag{7.16}$$

by (7.12) and the standard weak-$(1,1)$ maximal bound, and finally

$$\limsup_{\varepsilon \to 0} \int_{-1}^{1} 1_{[1-C_0\gamma,1]}(|u_\varepsilon|(p_\varepsilon, t))e_\varepsilon(u_\varepsilon)(p_\varepsilon, t)\, dt \leq C\gamma \tag{7.17}$$

thanks to (7.11), where $C_0 := C$ is the constant appearing in Lemma 7.8.

We write $\{t \in (-1, 1) : |u_\varepsilon|(p_\varepsilon, t) < 1 - \gamma\} = \bigcup_{j=1}^{N_\varepsilon} I_j$ as a union of disjoint open intervals. Since $W(u_\varepsilon) \geq c(\gamma) > 0$ on $\{|u_\varepsilon| < 1 - \gamma\}$, we have

$$\sum_j |I_j| \leq C(\gamma) \int_{-1}^{1} W(u_\varepsilon)(p_\varepsilon, t)\, dt \leq C(\gamma) \int_{-1}^{1} \varepsilon e_1(p_\varepsilon, t)\, dt \leq C(\gamma)\varepsilon =: \Lambda\varepsilon.$$

This implies that, for ε small enough, $\chi(t)$ vanishes for $t \in I_0 \cup I_N$; in order to study

$$\theta = \lim_{\varepsilon \to 0} \int_{-1}^{1} \chi(t)e_\varepsilon(u_\varepsilon)(p_\varepsilon, t)\, dt = \lim_{\varepsilon \to 0} \int_{-1}^{1} e_\varepsilon(u_\varepsilon)(p_\varepsilon, t)\, dt,$$

we can then look at the intermediate intervals I_j for $j = 1, \ldots, N-1$. We can now apply Lemma 7.8 (and Remark 7.9) to each of the cylinders

$$B_\varepsilon^{n-1}(p_\varepsilon) \times I_j, \quad j \in \{1, \ldots, N-1\},$$

dilated by a factor ε^{-1}, so that the dilated function becomes critical for E_1. The assumptions of this lemma hold eventually, thanks to the bound $\varepsilon^{-1}|I_j| \leq \Lambda$, the bound (7.10), the fact that

$$|u_\varepsilon(p_\varepsilon, t)| \in [1 - \gamma, 1] \quad \text{for } t \in \partial I_j,$$

and (7.16).

We deduce that either $|u_\varepsilon(p_\varepsilon, t)| \in [1 - C_0\gamma, 1]$ for all $t \in I_j$ or

$$\frac{1}{F_{x_0}(e_n)} \int_{I_j} e_\varepsilon(u_\varepsilon)(p_\varepsilon, t) \in \bigcup_{k \in \mathbb{N} \setminus \{0\}} (kc_W - Ck\gamma^2, kc_W + Ck\gamma^2).$$

In particular, in the second case we have $\frac{1}{F_{x_0}(e_n)} \int_{I_j} e_\varepsilon(u_\varepsilon)(p_\varepsilon, t) \geq c_W/2$ (provided γ is small enough). Since the energy on the full slice $\{p_\varepsilon\} \times [-1, 1]$ is at most 2θ, we deduce that there is an upper bound C for the number of such intervals of the second kind, independent of γ.

Recalling (7.17), we arrive at

$$\frac{1}{F_{x_0}(e_n)} \int_{-1}^{1} e_\varepsilon(u_\varepsilon)(p_\varepsilon, t) \, dt = \frac{1}{F_{x_0}(e_n)} \int_{[-1,1] \setminus (I_0 \cup I_N)} e_\varepsilon(u_\varepsilon)(p_\varepsilon, t) \, dt = n_\varepsilon c_W + O(\gamma),$$

for some $n_\varepsilon \in \mathbb{N}$, with an implied constant independent of ε, γ. Since

$$\frac{\theta}{F_{x_0}(e_n)} = \lim_{\varepsilon \to 0} \frac{1}{F_{x_0}(e_n)} \int_{-1}^{1} e_\varepsilon(u_\varepsilon)(p_\varepsilon, t) \, dt = \lim_{\varepsilon \to 0} n_\varepsilon c_W + O(\gamma),$$

we obtain that the distance of $\frac{\theta}{F_{x_0}(e_n)}$ from $c_W \mathbb{N}$ is bounded by $O(\gamma)$. Since γ was arbitrary, we reach the conclusion. $\qquad \square$

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Antonio De Rosa,
Department of Decision Sciences and BIDSA, Bocconi University, Milan, Italy,
Email address: antonio.derosa@unibocconi.it,

Alessandro Pigati,
Department of Decision Sciences and BIDSA, Bocconi University, Milan, Italy,
Email address: alessandro.pigati@unibocconi.it