FREE BOUNDARY MINIMAL SURFACES: A NONLOCAL APPROACH

FRANCESCA DA LIO AND ALESSANDRO PIGATI

Abstract. Given a C^k-smooth closed embedded manifold $\mathcal{N} \subset \mathbb{R}^m$, with $k \geq 2$, and a compact connected C^∞-smooth Riemannian surface (S, g) with $\partial S \neq \emptyset$, we consider $\frac{1}{2}$-*harmonic maps* $u \in H^{1/2}(\partial S, \mathcal{N})$. These maps are critical points of the nonlocal energy

$$E(f; g) := \int_S |\nabla \widetilde{u}|^2 \, d\mathrm{vol}_g, \tag{1}$$

where $\widetilde{u}$ is the harmonic extension of u in S. We express the energy (1) as a sum of the $\frac{1}{2}$-energies at each boundary component of ∂S (suitably identified with the circle $\mathcal{S}^1$), plus a quadratic term which is continuous in the $H^s(\mathcal{S}^1)$ topology, for any $s \in \mathbb{R}$. We show the $C^{k-1,\delta}$ regularity of $\frac{1}{2}$-harmonic maps. We also establish a connection between free boundary minimal surfaces and critical points of E with respect to variations of the pair (f, g), in terms of the Teichmüller space of S.

Contents

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1. Introduction

Let (S,g) be a connected C^∞-smooth surface with nonempty boundary ∂S, equipped with a smooth metric g (S is not necessarily oriented) and let $\mathcal{N}\subset\mathbb{R}^m$ be an embedded closed (i.e. compact without boundary) C^2-smooth submanifold.

We set

$$H^{1/2}(\partial S,\mathcal{N}):=\left\{f\in H^{1/2}(\partial S,\mathbb{R}^m):f(x)\in\mathcal{N}\text{ for a.e. }x\right\}.$$

Given a map $f\in H^{1/2}(\partial S,\mathcal{N})$, we define the $\frac{1}{2}$-*energy* of f to be

$$E(f;g):=\int_S\left|\nabla\widetilde{f}\right|^2 d\mathrm{vol}_g. \tag{2}$$

Here $\widetilde{f}$ denotes the harmonic extension of f, i.e. the unique harmonic map $\widetilde{f}\in H^1(S,\mathbb{R}^m)$ such that $\widetilde{f}\Big|_{\partial S}=f$. We observe that $E(f;g)$ depends only on the conformal class of g.

Definition 1.1. A map $u\in H^{1/2}(\partial S,\mathcal{N})$ is called $\frac{1}{2}$-*harmonic* if u is a critical point for the $\frac{1}{2}$-energy $E=E(\cdot;g)$, in the following sense: for any $\phi\in C^\infty(\partial S,\mathbb{R}^m)$ we have

$$\frac{d}{dt}E(\Pi(u+t\phi))\Big|_{t=0}=0, \tag{3}$$

where $\Pi:\mathcal{U}\to\mathcal{N}$ is any fixed C^2 projection, defined on some tubular neighborhood $\mathcal{U}$ of $\mathcal{N}$.

Definition 1.1 extends the one introduced for the first time in [**DLR09**] in the case $S=\mathbb{D}$ or in the noncompact case $S=\mathbb{H}$ ($\mathbb{D}$ and $\mathbb{H}$ being the unit disk and the upper half-plane, respectively). One can check that $\Pi(u+t\phi)=u+tv+o(t)$ in $H^{1/2}(\partial S,\mathbb{R}^m)$ as $t\to0$, where $v:=d\Pi(u)[\phi]$, and therefore[1]

$$\frac{d}{dt}E(\Pi(u+t\phi))\Big|_{t=0}=2\int_S\langle\nabla\widetilde{u};\nabla\widetilde{v}\rangle\,d\mathrm{vol}_g=2\int_{\partial S}d\Pi(u)[\phi]\cdot\frac{\partial\widetilde{u}}{\partial\nu}\,d\mathrm{vol}_g.$$

By a standard density argument, u is $\frac{1}{2}$-harmonic if and only if

$$\int_{\partial S}d\Pi(u)[\phi]\cdot\frac{\partial\widetilde{u}}{\partial\nu}\,d\mathrm{vol}_g=0, \tag{4}$$

for any $\phi\in L^\infty\cap H^{1/2}(\partial S,\mathbb{R}^m)$ (which is a Banach algebra), which is in turn equivalent to ask

$$\int_{\partial S}\frac{\partial\widetilde{u}}{\partial\nu}\cdot v\,d\mathrm{vol}_g=0 \tag{5}$$

for any $v\in L^\infty\cap H^{1/2}(\partial S,\mathbb{R}^m)$ satisfying $v\in T_u\mathcal{N}$ a.e. In particular, the definition is independent of the choice of Π.

Let $P^T(\xi)$ denote the orthogonal projection onto the tangent space $T_\xi\mathcal{N}$, for $\xi\in\mathcal{N}$, and observe that $P^T\in C^1(\mathcal{N},\mathbb{R}^{m\times m})$. In the paper we will also call $P^N:=I-P^T$ the projection onto the normal space. The same argument showing the equivalence of (4) and (5) proves that one can replace $d\Pi$

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[1] The normal derivative $\frac{\partial\widetilde{u}}{\partial\nu}\in H^{-1/2}(\partial S,\mathbb{R}^m)$ is defined precisely by asking that, for any $v\in H^{1/2}(\partial S,\mathbb{R}^m)$,

$$\int_S\langle\nabla\widetilde{u};\nabla\widetilde{v}\rangle\,d\mathrm{vol}_g=\int_{\partial S}\frac{\partial\widetilde{u}}{\partial\nu}\cdot v\,d\mathrm{vol}_g.$$

with P^T in (4) (notice that, on $\mathcal{N}$, P^T is the differential of the nearest point projection, canonically defined near $\mathcal{N}$, but we cannot use this projection in (3) as it is merely C^1). Hence,

$$\begin{aligned} u \text{ is } \frac{1}{2}\text{-harmonic} \quad &\Leftrightarrow \quad \int_{\partial S} \frac{\partial \widetilde{u}}{\partial \nu} \cdot P^T(u) v \, d\mathrm{vol}_g = 0, \quad \forall v \in L^\infty \cap H^{1/2}(\partial S, \mathbb{R}^m) \\ &\Leftrightarrow \quad P^T(u) \frac{\partial \widetilde{u}}{\partial \nu} = 0 \quad \text{in } \mathcal{D}'(\partial S). \end{aligned} \tag{6}$$

Solutions to the last equation are of special geometric interest because they are strictly connected to the so-called *free boundary minimal surfaces*, in the following sense.

Definition 1.2. We say that a map $\widetilde{u} \in C^2(S, \mathbb{R}^m)$ is a *free boundary (branched) minimal immersion* with supporting manifold $\mathcal{N}$ if it is a harmonic map which is also conformal (with the possible exception of finitely many points where $d\widetilde{u}$ vanishes) and meets $\mathcal{N}$ orthogonally, i.e.

$$P^T(u) \frac{\partial \widetilde{u}}{\partial \nu} = 0 \quad \text{on } \partial S.$$

In the case $S = \mathbb{D}$ the following connection between $\frac{1}{2}$-harmonic maps $u : \mathcal{S}^1 \to \mathcal{N}$ and free boundary minimal disks is now a well-known fact (see e.g. [**DaL15**, **MS15**, **DaL17**] and Remark 3.3).

Proposition 1.3. The harmonic extension $\widetilde{u}$ of a $\frac{1}{2}$-harmonic map $u \in H^{1/2}(\mathcal{S}^1, \mathcal{N})$ is conformal. Geometrically, this means that u is the boundary of a free boundary (branched) minimal disk.

We point out that Proposition 1.3 has been at the origin of the study of $\frac{1}{2}$-harmonic maps.

In this paper we are going to investigate the regularity of $\frac{1}{2}$-harmonic maps $u \in H^{1/2}(\partial S, \mathcal{N})$. Besides showing the Hölder continuity of such maps, we will illustrate how to bootstrap to higher regularity. Precisely we will show the following.

Theorem 1.4. Let $\mathcal{N} \subset \mathbb{R}^m$ be a C^k-smooth closed embedded manifold, with $k \geq 2$, and let $u \in H^{1/2}(\partial S, \mathcal{N})$ be $\frac{1}{2}$-harmonic. Then

$$u \in \bigcap_{0<\delta<1} C^{k-1,\delta}(\partial S, \mathcal{N}).$$

In particular, if $\mathcal{N}$ is C^∞ then $u \in C^\infty(\partial S, \mathcal{N})$.

The proof of Theorem 1.4 is rather technical and we defer it to the appendix.

We point out that one of the key steps to prove the regularity of $\frac{1}{2}$-harmonic maps is the representation of the energy $E(f; g)$ as a sum of the fractional $\frac{1}{2}$-energies at each boundary component (according to a suitable identification with $\mathcal{S}^1$), plus a quadratic term which is continuous in the H^s-topology, for any $s \in \mathbb{R}$. The identification of the energy of $\widetilde{f}$ with a fractional energy on the boundary in the case of the flat disk $\mathbb{D}$ is a well-known fact.

In the model case where $S = A_t := \overline{B}_t \setminus B_1$, $t > 1$, we have the following decomposition.

Lemma 1.5. Let $a, b \in H^{1/2}(\mathcal{S}^1, \mathbb{R}^m)$ and define $f \in H^{1/2}(\partial A_t, \mathbb{R}^m)$ by setting $f(e^{i\theta}) := a(e^{i\theta})$, $f(te^{i\theta}) := b(e^{i\theta})$. Then the Dirichlet energy of the harmonic extension $\widetilde{f} \in H^1(A_t, \mathbb{R}^m)$ is given by

$$\begin{aligned}\frac{1}{2\pi}\int_{A_t}|\nabla\widetilde{f}|^2 =&\sum_n |n|\,(|a_n|^2+|b_n|^2)+\frac{|b_0-a_0|^2}{\log t}\\ &+\sum_{n>0} n\left(\frac{4}{t^{2n}-1}(|a_n|^2+|b_n|^2)-\frac{8t^n}{t^{2n}-1}\Re(a_n\cdot\overline{b_n})\right)\\ =&\frac{1}{2\pi}\left(\|(-\Delta)^{1/4}a\|^2_{L^2(\mathcal{S}^1)}+\|(-\Delta)^{1/4}b\|^2_{L^2(\mathcal{S}^1)}+\mathcal{B}_t((a,b),(a,b))\right),\end{aligned} \tag{7}$$

where $\mathcal{B}_t : \mathcal{D}'(\mathcal{S}^1, \mathbb{R}^m)^2 \times \mathcal{D}'(\mathcal{S}^1, \mathbb{R}^m)^2 \to \mathbb{R}$ is a symmetric bilinear functional.

By using the decomposition (7) we succeed in rewriting condition (6) in the form of a nonlocal linear Schrödinger system with an antisymmetric potential, as it has been done in [**DLR11**, **DLS17**, **MS17**] in the case of the flat disk.

We will also show that the *conformality*[2] of the harmonic extension $\widetilde{u}$ is equivalent to criticality of E with respect to variations of the *conformal class* of S. For instance, if S is diffeomorphic to an annulus, then up to a conformal diffeomorphism we can assume that $(S, g) = (A_t, g_{\mathbb{R}^2})$ for some $t > 0$ (see Theorem A.1). In this case a variation of the conformal class corresponds to a variation of the parameter t.

Theorem 1.6. Let $a, b \in H^{1/2}(\mathcal{S}^1, \mathcal{N})$ and define $u[t] \in H^{1/2}(\partial A_t, \mathcal{N})$ by $u[t](e^{i\theta}) := a(e^{i\theta})$, $u[t](te^{i\theta}) := b(e^{i\theta})$. Assume that $u[\bar{t}]$ is $\frac{1}{2}$-harmonic for the annulus $(A_{\bar{t}}, g_{\mathbb{R}^2})$. Then its harmonic extension is conformal if and only if

$$\frac{d}{dt}E_t(u[t])\Big|_{t=\bar{t}} = 0,$$

where E_t is the $\frac{1}{2}$-energy for A_t.

We will extend Theorem 1.6 to the hyperbolic case where S is neither a disk nor an annulus (see Theorem 3.4).

In the interesting special case where $\mathcal{N}$ is the boundary of a *convex* C^∞-smooth domain Ω, we also prove the following result.

Corollary 1.7. The harmonic extension $\widetilde{u}$ defines a conformal (branched) free boundary minimal immersion $\widetilde{u}\colon (\mathring{S}, \partial S) \to (\Omega, \partial\Omega)$, with branch points *only in* $\mathring{S}$, if and only if u is a nontrivial critical point of $E(f; g)$ with respect to the pair (f, g).[3]

In view of the results in this paper, it would be interesting to study the flow version of the energy $E(f; g)$, where the evolution of the conformal class of g would be given by the lack of conformality of $\widetilde{u}$, in a similar way as for the Teichmüller harmonic map flow studied in [**RT16**]. This would correspond to a Teichmüller $\frac{1}{2}$-harmonic flow.

This paper is organized as follows.

- Section 2 provides the decomposition of the $\frac{1}{2}$-energy (2) in terms of nonlocal operators defined on ∂S; we also obtain a similar decomposition for the related Dirichlet-to-Neumann operator.

[2] *Conformality* will mean *weak conformality*, i.e. at every point $d\widetilde{u}$ either is a linear conformal map or vanishes.

[3] The meaning of criticality with respect to g will be specified in Section 3.

- Section 3 establishes the criterion for the conformality of the harmonic extension $\widetilde{f}$, as well as Corollary 1.7.
- In Section A we show a well-known uniformization theorem for compact annuli, exhibiting a conformal equivalence which is smooth up to the boundary; this is needed for the construction made in Section 2.
- Section B collects the definitions and basic facts concerning all the functional spaces involved in the paper; in particular we show some useful results about the space $\dot{H}^{1/2}(\mathbb{R})$.
- In Section C we recall some fundamental three-term commutator estimates, which were first obtained in [**DLR09**], as well as a two-term commutator estimate due to Coifman–Rochberg–Weiss from [**CRW76**].
- Section D details the proof of the Hölder continuity of a $\frac{1}{2}$-harmonic map u and uses localized versions of the integrability by compensation effects recalled in Section C.
- In Section E we bootstrap the results of Section D to obtain higher regularity of u, i.e. Theorem 1.4, exploiting another two-term commutator.

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2. Decomposition of the energy

The purpose of this section is to obtain the decomposition of the $\frac{1}{2}$-energy (2) in terms of nonlocal operators defined on ∂S.

We will also show that the so-called Dirichlet-to-Neumann operator

$$H^{1/2}(\partial S, \mathbb{R}^m) \to H^{-1/2}(\partial S, \mathbb{R}^m), \quad f \mapsto \frac{\partial \widetilde{f}}{\partial \nu}$$

can be represented as the sum of the usual fractional Laplacian at each boundary component and a remainder $\mathcal{B}\colon \mathcal{D}'(\partial S, \mathbb{R}^m) \to C^\infty(\partial S, \mathbb{R}^m)$, which represents a sort of interaction between the boundary data.

We will start from the model case of the flat annulus, where this decomposition is explicit.

2.1. The case of an annular domain.

For a fixed $t > 1$, let $A_t := \overline{B}_t \setminus B_1 \subset \mathbb{C}$ be the standard annulus with the Euclidean metric.

Given $f \in H^{1/2}(\partial A_t, \mathbb{R}^m)$, we denote

$$a(e^{i\theta}) := f(e^{i\theta}),\ b(e^{i\theta}) := f(te^{i\theta}) \in H^{1/2}(\mathcal{S}^1, \mathbb{R}^m).$$

We use the notation $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ for the Fourier coefficients of the two functions, namely

$$a_n := \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-in\theta}\,d\theta, \quad b_n := \frac{1}{2\pi}\int_0^{2\pi} b(e^{i\theta})e^{-in\theta}\,d\theta.$$

We observe that $\sum_{n\in\mathbb{Z}} 2\pi\,|n|\,|a_n|^2 = \|(-\Delta)^{1/4}a\|_{L^2}^2$ and similarly for b.

Given $(a,b),(c,d)\in\mathcal{D}'(\mathcal{S}^1,\mathbb{R}^m)^2\times\mathcal{D}'(\mathcal{S}^1,\mathbb{R}^m)^2$ we define the following symmetric bilinear operator:

$$
\begin{aligned}
\mathcal{B}_t((a,b),(c,d)) :=\ & 2\pi\frac{(b_0-a_0)\cdot(d_0-c_0)}{\log t} \\
& +\sum_{n>0}\frac{8\pi n}{t^{2n}-1}\Re(a_n\cdot\overline{c_n}+b_n\cdot\overline{d_n}-t^n a_n\cdot\overline{d_n}-t^n b_n\cdot\overline{c_n}).
\end{aligned}
\tag{8}
$$

Lemma 2.1. $\mathcal{B}_t$ is a sequentially continuous bilinear functional on $\mathcal{D}'(\mathcal{S}^1,\mathbb{R}^m)^2\times\mathcal{D}'(\mathcal{S}^1,\mathbb{R}^m)^2$.

Proof. Assume $a,b,c,d\in H^s(\mathcal{S}^1,\mathbb{R}^m)$. Since $t>1$ we have

$$
\begin{aligned}
\frac{|\mathcal{B}_t((a,b),(c,d))|}{2\pi} \le & \frac{|b_0-a_0|\,|d_0-c_0|}{\log t}+\sum_{n>0}\frac{4n}{t^{2n}-1}(|a_n|\,|c_n|+|b_n|\,|d_n|) \\
& +\sum_{n>0}\frac{4nt^n}{t^{2n}-1}(|a_n|\,|d_n|+|b_n|\,|c_n|) \\
\lesssim & \sum_{n\ge0}(1+n^2)^s(|a_n|\,|c_n|+|b_n|\,|d_n|+|a_n|\,|d_n|+|b_n|\,|c_n|) \\
\le & \|a\|_{H^s}\|c\|_{H^s}+\|b\|_{H^s}\|d\|_{H^s}+\|a\|_{H^s}\|d\|_{H^s}+\|b\|_{H^s}\|c\|_{H^s},
\end{aligned}
\tag{9}
$$

thanks to the elementary estimate $\frac{nt^n}{t^{2n}-1}\lesssim n^{2s}$ and the Cauchy–Schwarz inequality (the implied constants depend of course on s,t). Since $\mathcal{D}'(\mathcal{S}^1)=\bigcup_{s\in\mathbb{R}}H^s(\mathcal{S}^1)$, we get in particular that $\mathcal{B}_t$ is a linear functional on $\mathcal{D}'(\mathcal{S}^1,\mathbb{R}^m)^2\times\mathcal{D}'(\mathcal{S}^1,\mathbb{R}^m)^2$.

If $(((a_i,b_i),(c_i,d_i)))_{i\in\mathbb{N}}$ is a sequence converging to $((a,b),(c,d))$ in this space, by the uniform boundedness principle (applied to the Fréchet space $\mathcal{D}(\mathcal{S}^1,\mathbb{R}^m)$) we deduce that the set $\{a_i,b_i,c_i,d_i\mid i\in\mathbb{N}\}$ is bounded in $H^{s+1}(\mathcal{S}^1)$, for some real s.

By the compact embedding $H^{s+1}(\mathcal{S}^1)\hookrightarrow H^s(\mathcal{S}^1)$, any subsequence admits a further subsequence converging in $H^s(\mathcal{S}^1,\mathbb{R}^m)^2\times H^s(\mathcal{S}^1,\mathbb{R}^m)^2$, where we have already shown the continuity of $\mathcal{B}_t$. This shows that $\mathcal{B}_t((a_i,b_i),(c_i,d_i))\to\mathcal{B}_t((a,b),(c,d))$. $\square$

Lemma 2.2. For any $f\in H^{1/2}(\partial A_t,\mathbb{R}^m)$, the Dirichlet energy of its harmonic extension $\widetilde{f}\in H^1(A_t,\mathbb{R}^m)$ is given by

$$
\begin{aligned}
\int_{A_t}|\nabla\widetilde{f}|^2 =& 2\pi\sum_n|n|\,(|a_n|^2+|b_n|^2)+2\pi\frac{|b_0-a_0|^2}{\log t} \\
& +2\pi\sum_{n>0}n\left(\frac{4}{t^{2n}-1}(|a_n|^2+|b_n|^2)-\frac{8t^n}{t^{2n}-1}\Re(a_n\cdot\overline{b_n})\right) \\
=& \left\|(-\Delta)^{1/4}a\right\|_{L^2(\mathcal{S}^1)}^2+\left\|(-\Delta)^{1/4}b\right\|_{L^2(\mathcal{S}^1)}^2+\mathcal{B}_t((a,b),(a,b)).
\end{aligned}
\tag{10}
$$

Proof. One can check, e.g. by a density argument involving trigonometric polynomials, that the harmonic extension $\widetilde{f}$ is given by

$$
\widetilde{f}(re^{i\theta})=a_0+\frac{b_0-a_0}{\log t}\log r+\sum_{n\ne0}\frac{t^nb_n-a_n}{t^{2n}-1}r^ne^{in\theta}+\sum_{n\ne0}\frac{t^{2n}a_n-t^nb_n}{t^{2n}-1}r^{-n}e^{in\theta}.
\tag{11}
$$

Calling

$$
\widetilde{c}=\frac{b_0-a_0}{\log t},\quad c_n=\frac{t^nb_n-a_n}{t^{2n}-1},\quad c_n'=\frac{t^{2n}a_n-t^nb_n}{t^{2n}-1},
$$

we have

$$
\begin{aligned}
\frac{\partial \widetilde{f}}{\partial r}(r,\theta) &= \widetilde{c}r^{-1} + \sum_{n\neq 0} n(c_n r^{n-1} - c_n' r^{-n-1})e^{in\theta}, \\
\frac{1}{r}\frac{\partial \widetilde{f}}{\partial \theta} &= \sum_{n\neq 0} in(c_n r^{n-1} + c_n' r^{-n-1})e^{in\theta}.
\end{aligned} \tag{12}
$$

Thus the Dirichlet energy of $\widetilde{f}$ equals

$$
\begin{aligned}
\int_{A_t} \left|\nabla \widetilde{f}\right|^2 =& 2\pi \left|\widetilde{c}\right|^2 \log t + 2\pi \sum_{n\neq 0} n \left(|c_n|^2 (t^{2n}-1) - \left|c_n'\right|^2 (t^{-2n}-1) \right) \\
=& 2\pi \frac{|b_0 - a_0|^2}{\log t} + 2\pi \sum_{n\neq 0} \frac{n}{t^{2n}-1} \left(|t^n b_n - a_n|^2 + |t^n a_n - b_n|^2 \right).
\end{aligned}
$$

Since $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$, we deduce

$$
\begin{aligned}
\frac{\int_{A_t} \left|\nabla \widetilde{f}\right|^2}{2\pi} =& \frac{|b_0 - a_0|^2}{\log t} + \sum_{n\neq 0} \frac{n}{t^{2n}-1} \left((t^{2n}+1)(|a_n|^2 + |b_n|^2) - 4t^n \Re(a_n \cdot \overline{b_n}) \right) \\
=& \frac{|b_0 - a_0|^2}{\log t} + \sum_{n\neq 0} |n| \left(|a_n|^2 + |b_n|^2\right) \\
&+ \sum_{n>0} n \left(\frac{4}{t^{2n}-1}(|a_n|^2 + |b_n|^2) - \frac{8t^n}{t^{2n}-1} \Re(a_n \cdot \overline{b_n}) \right). \qquad \square
\end{aligned}
$$

Lemma 2.3. The normal derivatives on ∂B_1 and ∂B_t are given by

$$
\frac{\partial \widetilde{f}}{\partial \nu}(e^{i\theta}) = (-\Delta)^{1/2} a + \mathcal{R}_t[a,b], \quad \frac{\partial \widetilde{f}}{\partial \nu}(te^{i\theta}) = t^{-1}(-\Delta)^{1/2} b + t^{-1} R_t[b,a], \tag{13}
$$

where $R_t : \mathcal{D}'(\mathcal{S}^1, \mathbb{R}^m)^2 \to C^\infty(\mathcal{S}^1, \mathbb{R}^m)$ is a sequentially continuous linear operator defined by

$$
\mathcal{R}_t[a,b](e^{i\theta}) := -\frac{b_0 - a_0}{\log t} + \sum_{n>0} \frac{2n}{t^{2n}-1}(a_n - t^n b_n)e^{in\theta} + \sum_{n<0} \frac{2n}{t^{2n}-1}(t^{2n} a_n - t^n b_n)e^{in\theta}. \tag{14}
$$

Proof. Let $\alpha(e^{i\theta}) := \frac{\partial \widetilde{f}}{\partial \nu}(e^{i\theta})$ and $\beta(e^{i\theta}) := \frac{\partial \widetilde{f}}{\partial \nu}(te^{i\theta})$. Given any $h \in C^\infty(\partial A_t, \mathbb{R}^m)$, let $c(e^{i\theta}) := h(e^{i\theta})$ and $d(e^{i\theta}) := h(te^{i\theta})$. Since $\widetilde{f}$ is harmonic we get

$$
\begin{aligned}
2\pi \sum_n \alpha_n \cdot \overline{c_n} + 2\pi t \sum_n \beta_n \cdot \overline{d_n} &= \int_{\partial A_t} \frac{\partial \widetilde{f}}{\partial \nu} h = \int_{A_t} \nabla \widetilde{f} \cdot \nabla \widetilde{h} \\
&= \int_{\mathcal{S}^1} (-\Delta)^{1/4} a (-\Delta)^{1/4} c + \int_{\mathcal{S}^1} (-\Delta)^{1/4} b (-\Delta)^{1/4} d + \mathcal{B}_t((a,b),(c,d)).
\end{aligned}
$$

From this equation we easily get (13), with $\mathcal{R}_t[a,b]$ given by (14). We observe that the formula (14) can also be obtained directly from (12). The continuity of $\mathcal{R}_t$ is proved by arguing as in the proof of Lemma 2.1. $\square$

Remark 2.4. The symmetry $\mathcal{B}_t((a,b),(c,d)) = \mathcal{B}_t((b,a),(d,c))$, as well as the fact that the formulas for $t\frac{\partial \widetilde{f}}{\partial \nu}(te^{i\theta})$ and $\frac{\partial \widetilde{f}}{\partial \nu}(e^{i\theta})$ can be obtained from each other by exchanging a and b, are not surprising in view of the existence of the conformal map

$$A_t \to A_t, \quad z \mapsto \frac{tz}{|z|^2},$$

which exchanges the two boundary components.

2.2. **General compact surfaces with boundary.** The boundary ∂S is the disjoint union of finitely many circles diffeomorphic to $\mathcal{S}^1$:

$$\partial S = \bigsqcup_{j=1}^{k} C^{(j)}.$$

We can find, for each j, a smooth map

$$\phi_j : [0,1] \times \mathcal{S}^1 \to S$$

with the following properties:

- ϕ_j is a diffeomorphism onto its image;
- $\phi_j(\{0\} \times \mathcal{S}^1) = C^{(j)}$;
- $\phi_j([0,1] \times \mathcal{S}^1) \cap \phi_{j'}([0,1] \times \mathcal{S}^1) = \emptyset$ for any $j \neq j'$.

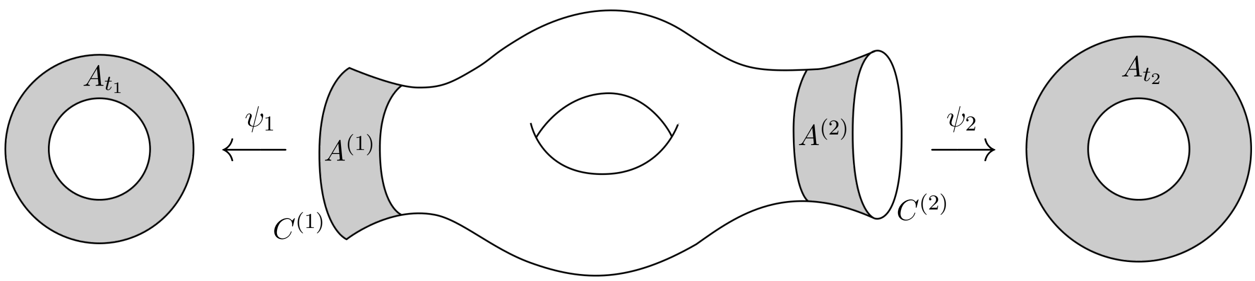
Applying Theorem A.1 to the annulus

$$A^{(j)} := \phi_j([0,1] \times \mathcal{S}^1),$$

we can find a conformal transformation $\psi_j : A^{(j)} \to A_{t_j}$ (where $A_{t_j} := \overline{B}_{t_j} \setminus B_1 \subset \mathbb{C}$, equipped with the flat metric) such that $\psi_j(C^{(j)}) = \partial B_1$. Finally, we call

$$S' := S \setminus \bigsqcup_{j=1}^{k} \phi_j([0,1) \times \mathcal{S}^1).$$

The picture illustrates our decomposition of S.



We notice that S' is still a smooth surface with boundary

$$\partial S' = \bigsqcup_{j=1}^{k} \phi_j(\{1\} \times \mathcal{S}^1) = \bigsqcup_{j=1}^{k} \psi_j^{-1}(\partial B_{t_j}).$$

Lemma 2.5. For any $f \in H^{1/2}(\partial S, \mathcal{N})$ the $\frac{1}{2}$-energy $E(f; g)$ admits the decomposition

$$E(f; g) = \sum_j \|f_j\|^2_{H^{1/2}} + \mathcal{B}_S((f_j)_{j=1}^k, (f_j)_{j=1}^k),$$

where $f_j(e^{i\theta}) := f \circ \psi_j^{-1}(e^{i\theta})$ and $\mathcal{B}_S : \mathcal{D}'(\mathcal{S}^1, \mathbb{R}^m)^k \times \mathcal{D}'(\mathcal{S}^1, \mathbb{R}^m)^k \to \mathbb{R}$ is a sequentially continuous symmetric bilinear functional.

Proof. Let $G \in C^\infty((\mathring{S} \times S) \setminus \Delta)$ be the Green function for the Dirichlet problem (see e.g. [**Aub98**]), satisfying for each $x \in \mathring{S}$

$$\begin{cases} -\Delta_g G(x, \cdot) = \delta_x & \text{on } S \\ G(x, \cdot) = 0 & \text{on } \partial S \end{cases}$$

and let $H \in C^\infty(\mathring{S} \times \partial S)$ which is defined, for any fixed $x \in \mathring{S}$, by the formula $H(x, \cdot) := -\frac{\partial}{\partial \nu} G(x, \cdot)$. For any $f \in H^{1/2}(\partial S, \mathbb{R}^m)$ and any $x \in \mathring{S}$, the harmonic extension is given by the formula

$$\widetilde{f}(x) = \int_{\partial S} H(x, y) f(y) \, d\mathrm{vol}_g(y).$$

Now $\{H(x, \cdot) \mid x \in S'\}$ is a compact subset of $C^\infty(\partial S)$ and in particular is bounded in $C^k(\partial S)$ for all $k \geq 0$. The same holds for the derivatives of any order in x. Therefore the map

$$\mathcal{D}'(\partial S, \mathbb{R}^m) \to C^\infty(S', \mathbb{R}^m), \quad f \mapsto \widetilde{f}\Big|_{S'}$$

given by the above formula is sequentially continuous. In particular, $(f, h) \mapsto \int_{S'} \langle \nabla \widetilde{f}; \nabla \widetilde{h} \rangle \, d\mathrm{vol}_g$ defines a sequentially continuous symmetric bilinear operator on $\mathcal{D}'(\partial S, \mathbb{R}^m) \times \mathcal{D}'(\partial S, \mathbb{R}^m)$.

Moreover, for any $j \in \{1, \dots, k\}$, let

$$\kappa_j(e^{i\theta}) := \widetilde{f} \circ \psi_j^{-1}(t_j e^{i\theta}) \in H^{1/2}(\mathcal{S}^1, \mathbb{R}^m).$$

By conformal invariance we have $\Delta(\widetilde{f} \circ \psi_j^{-1}) = 0$ on A_{t_j} and

$$\int_{A^{(j)}} |\nabla \widetilde{f}|^2 \, d\mathrm{vol}_g = \int_{A_{t_j}} |\nabla(\widetilde{f} \circ \psi_j^{-1})|^2 = \left\|(-\Delta)^{1/4} f_j\right\|_{L^2}^2 + \left\|(-\Delta)^{1/4} \chi_j\right\|_{L^2}^2 + \mathcal{B}_{t_j}((f_j, \kappa_j), (f_j, \kappa_j))$$

by Lemma 2.2. We remark that $f \mapsto \kappa_j$ is sequentially continuous as a linear map $\mathcal{D}'(\partial S, \mathbb{R}^m) \to C^\infty(\mathcal{S}^1, \mathbb{R}^m)$. Finally, we can write

$$E(f; g) = \sum_{j=1}^k \int_{A^{(j)}} |\nabla \widetilde{f}|^2 \, d\mathrm{vol}_g + \int_{S'} |\nabla \widetilde{f}|^2 \, d\mathrm{vol}_g = \sum_{j=1}^k \left\|(-\Delta)^{1/4} f_j\right\|_{L^2}^2 + \mathcal{B}_S((f_j), (f_j)),$$

where for any $f, h \in H^{1/2}(\partial S, \mathbb{R}^m)$ we let

$$\mathcal{B}_S((f_j), (h_j)) := \sum_{j=1}^k \int_{\mathcal{S}^1} (-\Delta)^{1/4} \kappa_j (-\Delta)^{1/4} \xi_j + \sum_{j=1}^k B_{t_j}((f_j, \kappa_j), (h_j, \xi_j)) + \int_{S'} \langle \nabla \widetilde{f}; \nabla \widetilde{h} \rangle \, d\mathrm{vol}_g,$$

with $h_j(e^{i\theta}) := h \circ \psi_j^{-1}(e^{i\theta})$ and $\xi_j(e^{i\theta}) := \widetilde{h} \circ \psi_j^{-1}(t_j e^{i\theta})$. □

Lemma 2.6. For any $\ell = 1, \dots, k$, the normal derivative on $C^{(\ell)}$ is given by

$$\frac{\partial \widetilde{f}}{\partial \nu} = e^{\lambda_\ell}((-\Delta)^{1/2} f_j) \circ \psi_\ell + e^{\lambda_\ell} R_\ell((f_j)_{j=1}^k) \circ \psi_\ell,$$

where $e^{2\lambda_\ell}$ is defined by $g = e^{2\lambda_\ell} \psi_\ell^*(g_{\mathbb{R}^2})$ and $R_\ell : \mathcal{D}'(\mathcal{S}^1, \mathbb{R}^m)^k \to C^\infty(\mathcal{S}^1, \mathbb{R}^m)$ is a sequentially continuous linear operator.

Proof. Indeed, for any $\varphi \in C^\infty(S, \mathbb{R}^m)$ supported in $\phi_\ell([0,1) \times \mathcal{S}^1)$,

$$\begin{aligned}\int_{C^{(\ell)}} e^{\lambda_\ell} \varphi \cdot ((-\Delta)^{1/2} f_\ell) \circ \psi_\ell \, d\mathrm{vol}_g &= \int_{\mathcal{S}^1} \varphi \circ \psi_\ell^{-1} \cdot (-\Delta)^{1/2} f_\ell \\ &= \int_{\partial A_{t_j}} \varphi \circ \psi_\ell^{-1} \cdot \frac{\partial (\widetilde{f} \circ \psi_\ell^{-1})}{\partial \nu} - \int_{\mathcal{S}^1} \varphi \circ \psi_\ell^{-1} \cdot R_{t_\ell}[f_\ell, \kappa_\ell] \\ &= \int_{A_{t_\ell}} \langle \nabla(\varphi \circ \psi_\ell^{-1}); \nabla(\widetilde{f} \circ \psi_\ell^{-1}) \rangle - \int_{C^{(\ell)}} e^{\lambda_\ell} \varphi \cdot R_{t_\ell}[f_\ell, \kappa_\ell] \circ \psi_\ell \, d\mathrm{vol}_g,\end{aligned}$$

where the operator R_{t_ℓ} is provided by (13). But, by conformality of ψ_ℓ,

$$\int_{A_{t_\ell}} \langle \nabla(\varphi \circ \psi_\ell^{-1}); \nabla(\widetilde{f} \circ \psi_\ell^{-1}) \rangle = \int_{A^{(\ell)}} \langle \nabla\varphi; \nabla\widetilde{f} \rangle \, d\mathrm{vol}_g = \int_S \langle \nabla\varphi; \nabla\widetilde{f} \rangle \, d\mathrm{vol}_g = \int_{\partial S} \varphi \cdot \frac{\partial \widetilde{f}}{\partial \nu} \, d\mathrm{vol}_g$$

and thus we can let $R_\ell((f_j)_{j=1}^k) := R_{t_\ell}[f_\ell, \kappa_\ell]$. $\square$

3. Conformality of the harmonic extension

This section is devoted to show that, if the energy of the harmonic extension $\widetilde{u}$ is also critical with respect to variations of the conformal class, then $\widetilde{u}$ is conformal. We will use the Teichmüller space $\mathcal{T}(S)$ of S to describe such variations. Throughout the section we will assume that S is orientable (actually this hypothesis can be dropped: one can repeat the same theory on the two-sheeted oriented cover $\widetilde{S}$, restricting to equivariant metrics and variations).

We will start from the easier case of the annulus, which can be treated in an elementary fashion (due to the simple explicit form of its Teichmüller space).

Remark 3.1. In the disk case, i.e. $S = \mathbb{D}$, conformality holds automatically for $\frac{1}{2}$-harmonic maps (and indeed in this case $\mathcal{T}(S)$ is trivial): see Remark 3.3 below.

Recall that the disk and the annulus have Euler characteristic 1 and 0, respectively. If the surface S has a different topology, then its Euler characteristic is

$$\chi(S) = 2 - 2g - k < 0$$

(with an abuse of notation, we denote by g also the genus of S, while $k \geq 1$ is the number of boundary components). Thus S is intrinsically *hyperbolic*, namely by Gauss–Bonnet theorem any constant curvature metric such that ∂S is totally geodesic must have negative curvature. In this case $\mathcal{T}(S)$ does not possess an immediate presentation as for the annulus, although it is well known that it is diffeomorphic to $\mathbb{R}^{6g+3k-6}$ (and can be parametrized by means of the so-called Fenchel–Nielsen coordinates). The precise definition of $\mathcal{T}(S)$ is given below.

3.1. **The annular case.**

If S is diffeomorphic to an annulus, then up to a conformal diffeomorphism we can assume that $(S, g) = (A_t, g_{\mathbb{R}^2})$ for some $t > 0$, thanks to Theorem A.1. A variation of the conformal class (or, more precisely, the conformal class up to diffeomorphisms isotopic to the identity) corresponds to a variation of the parameter t.

For any $a, b \in H^{1/2}(\mathcal{S}^1, \mathbb{R}^m)$, we define $u[t] \in H^{1/2}(\partial A_t, \mathbb{R}^m)$ by $u[t](e^{i\theta}) := a(e^{i\theta})$, $u[t](te^{i\theta}) := b(e^{i\theta})$. We will denote by $\widetilde{u}[t]$ the harmonic extension of $u[t]$.

Lemma 3.2. For any $a, b \in H^1(\mathcal{S}^1, \mathbb{R}^m)$ we have

$$\frac{d}{dt}E_t(u[t]) = \int_{\partial B_t} \left(\frac{1}{t^2} \left| \frac{\partial \widetilde{u}}{\partial \theta} \right|^2 - \left| \frac{\partial \widetilde{u}}{\partial r} \right|^2 \right).$$

Proof. We can assume $a, b \in C^\infty(\mathcal{S}^1, \mathbb{R}^m)$ (by a density argument, using the fact that $E_t(u[t])$ depends smoothly on $(t, a, b) \in (1, \infty) \times H^{1/2}(\mathcal{S}^1) \times H^{1/2}(\mathcal{S}^1)$, as can be seen from the explicit formula (7)). So $\widetilde{u}[t](z)$ defines a smooth function on the set

$$\{(t, z) \in (1, \infty) \times \mathbb{C} : 1 \le |z| \le t\}.$$

By the divergence theorem we have

$$\begin{aligned}
\frac{d}{dt}\left(\int_{A_t} |\nabla \widetilde{u}[t]|^2 \right) &= \int_{\partial B_t} |\nabla \widetilde{u}[t]|^2 + 2\int_{A_t} \langle \nabla \widetilde{u}[t]; \nabla \left(\frac{d\widetilde{u}[s]}{ds}\Big|_{s=t} \right) \rangle \\
&= \int_{\partial B_t} \left(\left| \frac{\partial \widetilde{u}[t]}{\partial r} \right|^2 + \frac{1}{t^2} \left| \frac{\partial \widetilde{u}[t]}{\partial \theta} \right|^2 \right) + 2\int_{\partial B_t} \frac{\partial \widetilde{u}[t]}{\partial r} \cdot \left(\frac{d\widetilde{u}[s]}{ds}\Big|_{s=t} \right)
\end{aligned}$$

(as $\frac{d\widetilde{u}}{ds} = 0$ on ∂B_1). Differentiating the identity $\widetilde{u}[s](se^{i\theta}) = b(e^{i\theta})$ in s we get

$$\left(\frac{d\widetilde{u}[s]}{ds} \right)(se^{i\theta})\Big|_{s=t} = -\frac{\partial \widetilde{u}[t]}{\partial r}(te^{i\theta}).$$

Hence, combining these identities,

$$\frac{d}{dt}E_t(u[t]) = \int_{\partial B_t} \left(\left| \frac{\partial \widetilde{u}[t]}{\partial r} \right|^2 + \frac{1}{t^2} \left| \frac{\partial \widetilde{u}[t]}{\partial \theta} \right|^2 \right) - 2\int_{\partial B_t} \left| \frac{\partial \widetilde{u}[t]}{\partial r} \right|^2. \qquad \square$$

Proof of Theorem 1.6. We introduce the Hopf differential

$$h(z) := H(z)\, dz \otimes dz, \quad H(z) := \frac{\partial \widetilde{u}}{\partial z} \cdot \frac{\partial \widetilde{u}}{\partial z} = \frac{1}{4}\left(\left| \frac{\partial \widetilde{u}}{\partial x} \right|^2 - \left| \frac{\partial \widetilde{u}}{\partial y} \right|^2 - 2i \frac{\partial \widetilde{u}}{\partial x} \cdot \frac{\partial \widetilde{u}}{\partial y} \right).$$

A well-known straightforward computation shows that h is a holomorphic quadratic differential, i.e. H is holomorphic, vanishing identically if and only if $\widetilde{u}$ is (weakly) conformal. From Theorem 1.4 it follows in particular that $u \in C^1(\partial A_t)$. Since $0 = P^T(u)\, \frac{\partial \widetilde{u}}{\partial r}$ and $0 = P^N(u)\, \frac{\partial u}{\partial \theta} = P^N(u)\, \frac{\partial \widetilde{u}}{\partial \theta}$ on ∂A_t, we have that

$$\frac{\partial \widetilde{u}}{\partial r} \cdot \frac{\partial \widetilde{u}}{\partial \theta} = 0 \text{ on } \partial A_{\bar{t}}.$$

By the maximum principle we deduce that, for any $z = re^{i\theta} \in \mathring{A}_{\bar{t}}$,

$$-2\Im((re^{i\theta})^2 H(re^{i\theta})) = r\frac{\partial \widetilde{u}}{\partial r}(re^{i\theta}) \cdot \frac{\partial \widetilde{u}}{\partial \theta}(re^{i\theta}) = 0,$$

i.e. the harmonic function $\Im(z^2 H(z))$ vanishes identically. Since $z^2 H(z)$ is holomorphic, it must coincide with a real constant c. By Lemma 3.2, $c = 0$ precisely when $\frac{d}{dt}E_t(u[t])\big|_{t=\bar{t}} = 0$, since

$$-4c = -4\Re((re^{i\theta})^2 H(re^{i\theta})) = \left| \frac{\partial \widetilde{u}}{\partial \theta}(re^{i\theta}) \right|^2 - r^2 \left| \frac{\partial \widetilde{u}}{\partial r}(re^{i\theta}) \right|^2. \qquad \square$$

Remark 3.3. In the disk case we get $z^2 H(z) = c$ for some real c, hence $c = 0$ (being H bounded near the origin) and $H(z) = 0$.

3.2. **The hyperbolic case.** Assume now that $\chi(S) < 0$ (i.e. S is not a disk nor an annulus) and $\mathcal{N}$ is C^∞-smooth. Let $\mathcal{M}(S)$ be the space of all Riemannian metrics on S and $\mathcal{P}(S)$ the space of all smooth positive functions $S \to \mathbb{R}$. $\mathcal{M}(S)$ is an open subset of the Fréchet space $\Gamma(S^2 S)$ (smooth symmetric covariant 2-tensors on S). The quotient

$$\mathcal{C}(S) := \mathcal{M}(S)/\mathcal{P}(S)$$

is the set of conformal classes on S. Moreover, let $\mathcal{M}_{-1}(S) \subseteq \mathcal{M}(S)$ be the subset of metrics having constant curvature -1 and making ∂S totally geodesic. Every equivalence class $[g] \in \mathcal{C}(S)$ has exactly one representative $e^{2\lambda} g \in \mathcal{M}_{-1}(S)$, $\lambda \in C^\infty(S)$ being a solution of Liouville's equation

$$\begin{cases} \Delta\lambda = K + e^{2\lambda} & \text{on } S \\ \frac{\partial\lambda}{\partial\nu} = \kappa & \text{on } \partial S, \end{cases}$$

where K is the Gaussian curvature of g and κ is the geodesic curvature of the boundary (i.e. $\kappa = \langle \nabla_{\dot\gamma}\dot\gamma, \nu\rangle$ if ∂S is locally parametrized by a unit-speed curve γ). The map

$$\upsilon : \mathcal{M}(S) \to \mathcal{M}_{-1}(S), \quad \upsilon(g) := e^{2\lambda} g$$

is C^∞-smooth (as a map from $\mathcal{M}(S)$ into itself).

In order to have a finite-dimensional space, we quotient $\mathcal{C}(S)$ by the (right) action of the group $\mathcal{D}_0(S)$ of diffeomorphisms isotopic to the identity. The set

$$\mathcal{T}(S) := \mathcal{C}(S)/\mathcal{D}_0(S)$$

is the Teichmüller space of S. It can be given a canonical structure of $(6g + 3k - 6)$-dimensional differentiable manifold. The resulting map $\pi : \mathcal{M}(S) \to \mathcal{T}(S)$ is smooth and admits locally a smooth section taking values into $\mathcal{M}_{-1}(S)$.

For the proofs of these facts, we refer the reader to [**FT84**], where the Teichmüller theory for closed surfaces is developed. See also [**DHT92**], which illustrates the necessary modifications for surfaces with boundary (using the convenient device of the Schottky double).

Theorem 3.4. Let (S, g) be a Riemannian surface with $\partial S \neq \emptyset$, $\chi(S) < 0$ and let $\phi : U \to \mathcal{M}(S)$ be a local smooth section of π through g (i.e. $\pi(g) \in U$ and $\phi(\pi(g)) = g$). If $u \in H^{1/2}(\partial S, \mathcal{N})$ is $\frac{1}{2}$-harmonic with respect to g, then $\widetilde{u} : (S, g) \to \mathbb{R}^m$ is conformal if and only if $\pi(g)$ is a critical point for the map

$$p \mapsto E(u; \phi(p)).$$

We remark that the harmonic extension $\widetilde{u}_p \in H^1(S)$ of $u \in H^{1/2}(\partial S)$ with respect to $\phi(p)$ depends on the couple $(u, p) \in H^{1/2}(\partial S, \mathbb{R}^m) \times U$ in a smooth fashion: this follows from the inverse function theorem applied to the map

$$H^1(S) \times U \to H^{-1}(S) \times H^{1/2}(\partial S) \times U, \quad (w, p) \mapsto (-\Delta_{\phi(p)} w, w\big|_{\partial S}, p).$$

In particular, the function $(u, p) \mapsto E(u; \phi(p))$ is smooth as well.

Proof. ($\Leftarrow$) Replacing g with $\upsilon(g)$ and ϕ with $\upsilon \circ \phi$, we can assume that $g \in \mathcal{M}_{-1}(S)$ and $\phi(U) \subseteq \mathcal{M}_{-1}(S)$: indeed, thanks to the conformal invariance of the Dirichlet energy, $E(w; g') = E(w; \upsilon(g'))$ for any $w \in H^{1/2}(\partial S, \mathbb{R}^m)$ and any metric g', so u is still $\frac{1}{2}$-harmonic with respect to $\upsilon(g)$ and $\pi(\upsilon(g)) = \pi(g)$ is still critical for $p \mapsto E(u; \upsilon \circ \phi(p))$.

The Hopf differential h of the map $\widetilde{u}$, defined in any local conformal chart (for g) by the formula

$$h(z) := H(z)\, dz \otimes dz, \quad H(z) := \frac{\partial \widetilde{u}}{\partial z} \cdot \frac{\partial \widetilde{u}}{\partial z} = \frac{1}{4}\left(\left|\frac{\partial \widetilde{u}}{\partial x}\right|^2 - \left|\frac{\partial \widetilde{u}}{\partial y}\right|^2 - 2i \frac{\partial \widetilde{u}}{\partial x} \cdot \frac{\partial \widetilde{u}}{\partial y} \right),$$

is a globally defined holomorphic quadratic differential (i.e. H is holomorphic in any conformal chart), as a consequence of the fact that $\Delta_g \widetilde{u} = 0$. The conformality of $\widetilde{u}$ is equivalent to $h = 0$.

Moreover, h is real at the boundary ∂S, meaning that in any local conformal chart (V, z) mapping $V \cap \partial S$ into the real line $\{\Im(z) = 0\}$ we have

$$\frac{\partial \widetilde{u}}{\partial x} \cdot \frac{\partial \widetilde{u}}{\partial y} = 0$$

on the real line. Indeed, at such points $\frac{\partial \widetilde{u}}{\partial x} \in T_{u(z)}\mathcal{N}$, while $\frac{\partial \widetilde{u}}{\partial y} \perp T_{u(z)}\mathcal{N}$ by $\frac{1}{2}$-harmonicity (observe that by the preceding regularity result we have $\widetilde{u} \in C^\infty$ up to the boundary).

Let now $v := d\pi_g[\Re(h)]$. Since $g \in \mathcal{M}_{-1}(S)$, the symmetric tensor $d\phi_{\pi(g)}[v]$ can be decomposed as

$$d\phi_{\pi(g)}[v] = \Re(q) + \mathcal{L}_X g, \tag{15}$$

where q is a holomorphic quadratic differential which is real at ∂S, while $\mathcal{L}_X g$ is the Lie derivative of g with respect to a vector field X satisfying $X\Big|_{\partial S} \parallel \partial S$ (see [**FT84**] for the corresponding statement for closed surfaces). The tensor $\mathcal{L}_X g$ belongs to the kernel of $d\pi_g$, as X generates a one-parameter subgroup of $\mathcal{D}_0$. Thus, using $\pi \circ \phi = \mathrm{id}_U$,

$$d\pi_g[\Re(h)] = v = d\pi_g[d\phi_{\pi(g)}[v]] = d\pi_g[\Re(q) + \mathcal{L}_X g] = d\pi_g[\Re(q)].$$

But $\mathcal{T}(S)$ is built precisely in such a way that the map $d\pi_g$ restricts to a bijection from the space of such real quadratic differentials to $T_{\pi(g)}\mathcal{T}(S)$. We deduce that $\Re(h) = \Re(q)$.

We also remark that $dE(u; \cdot)_g[\mathcal{L}_X g] = 0$: indeed, calling Φ_t^X the flow generated by X, we have

$$E(u; (\Phi_t^X)^* g) = E(u \circ \Phi_{-t}^X; g)$$

and differentiation at $t = 0$ gives

$$dE(u; \cdot)_g[\mathcal{L}_X g] = -2 \int_{\partial S} \frac{\partial \widetilde{u}}{\partial \nu} \cdot du[X] \, d\mathrm{vol}_g = 0,$$

by characterization (6) of $\frac{1}{2}$-harmonicity. From (15) we finally deduce that

$$\begin{aligned} 0 = & dE(u; \phi(\cdot))_{\pi(g)}[v] = dE(u; \cdot)_g[\Re(q) + \mathcal{L}_X g] = dE(u; \cdot)_g[\Re(h)] \\ = & -\int_S \Re(h)[\nabla \widetilde{u}; \nabla \widetilde{u}] \, d\mathrm{vol}_g + \frac{1}{2} \int_S |\nabla \widetilde{u}|_g^2 \, \mathrm{tr}_g(\Re(h)) \, d\mathrm{vol}_g \end{aligned}$$

(using the fact that the variation of $\widetilde{u}$ gives no contribution, thanks to harmonicity). But, as is readily seen in conformal coordinates,

$$\mathrm{tr}_g(\Re(h)) = 0, \quad \Re(h)[\nabla \widetilde{u}; \nabla \widetilde{u}] = 2 \, |\Re(h)|_g^2 \, .$$

We infer that $\Re(h) = 0$, which implies $h = 0$.

($\Rightarrow$) Conversely, for any $v \in T_{\pi(g)}U$, we can write (assuming again $\phi = v \circ \phi$)

$$d\phi_{\pi(g)}[v] = \Re(q) + \mathcal{L}_X g$$

for suitable q and X as before. We have

$$\begin{aligned} & dE(u; \phi(\cdot))_{\pi(g)}[v] = dE(u; \cdot)_g[\Re(q) + \mathcal{L}_X g] = dE(u; \cdot)_g[\Re(q)] \\ & = -\int_S \Re(q)[\nabla \widetilde{u}; \nabla \widetilde{u}] \, d\mathrm{vol}_g + \frac{1}{2} \int_S |\nabla \widetilde{u}|_g^2 \, \mathrm{tr}_g(\Re(q)) \, d\mathrm{vol}_g. \end{aligned}$$

Again we have $\mathrm{tr}_g(\Re(q)) = 0$, while the conformality of $\widetilde{u}$ gives $\frac{\partial \widetilde{u}}{\partial z} \cdot \frac{\partial \widetilde{u}}{\partial z} = 0$ in conformal coordinates, hence $\Re(q)[\nabla \widetilde{u}; \nabla \widetilde{u}] = 0$ as well. $\square$

Proof of Corollary 1.7. In view of the preceding results, it suffices to show that, for a nontrivial $\frac{1}{2}$-harmonic map u with conformal $\widetilde{u}$, we have $\widetilde{u}(\mathring{S}) \subseteq \Omega$ and $\nabla\widetilde{u} \neq 0$ at the boundary ∂S. Recall that $\widetilde{u}$ is C^∞-smooth up to the boundary of S.

Since $\widetilde{u}$ is nontrivial we have $\frac{\partial\widetilde{u}}{\partial\nu} \neq 0$ at some $x' \in \partial S$ (being $\int_S |\nabla\widetilde{u}|^2 \, d\mathrm{vol}_g = \int_{\partial S} \widetilde{u} \cdot \frac{\partial\widetilde{u}}{\partial\nu} \, d\mathrm{vol}_g$). Combining this with the condition $P^T(u)\frac{\partial\widetilde{u}}{\partial\nu} = 0$, we get $\widetilde{u}(x'') \in \Omega$ for at least an $x'' \in \mathring{S}$.

Fix now any point $p \notin \Omega$. By convexity of Ω, there exists an affine map $F : \mathbb{R}^m \to \mathbb{R}$ such that $F(p) \leq 0$ and $F(\Omega) \subseteq (0, \infty)$. Since $F \circ u$ takes nonnegative values (as u takes values in $\partial\Omega$) and $F \circ \widetilde{u}(x'') > 0$, by the strong maximum principle we get $F \circ \widetilde{u} > 0$ on $\mathring{S}$. Hence, $\widetilde{u}(\mathring{S}) \subseteq \Omega$.

Finally, if $x \in \partial S$ we can let $p := u(x)$: then Hopf's lemma gives $\frac{\partial(F\circ\widetilde{u})}{\partial\nu}(x) < 0$. In particular, $\nabla\widetilde{u}$ never vanishes at ∂S. □

Appendix A. Uniformization theorem for annuli with boundary

Theorem A.1. Let (A, g) be a compact Riemannian two-dimensional manifold with boundary, diffeomorphic to $[0,1] \times \mathcal{S}^1$. Then there exists some $t > 1$ such that (A, g) is conformally equivalent to the standard annulus $A_t := \overline{B}_t \setminus B_1 \subset \mathbb{C}$.

Proof. We fix a diffeomorphism $\phi : [0,1] \times \mathcal{S}^1 \to A$ and we orient A by declaring that ϕ is orientation-preserving. We call $\gamma_j : \mathcal{S}^1 \to \partial A$ the restrictions $\gamma_j := \phi(j, \cdot)$, for $j = 0, 1$, so that γ_1 preserves the orientation while γ_0 reverses it. Let $u \in C^\infty(A)$ be the unique harmonic function which equals j on $\gamma_j(\mathcal{S}^1)$ (for $j = 0, 1$). Denoting by $\mathring{A}$ the interior of A, we remark that by the maximum principle $0 < u < 1$ on $\mathring{A}$ and by Hopf's lemma $*du[\dot{\gamma}_j] > 0$ for $j = 0, 1$. Recall that, in local conformal coordinates (x, y), $*du = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$.

Let $\kappa := \int_{\gamma_0} *du > 0$. Since $*du$ is closed, $\int_\gamma *du \in \kappa\mathbb{Z}$ for any closed, piecewise smooth curve γ taking values in A. Thus, we can define $v \in C^\infty(A, \mathbb{R}/\kappa\mathbb{Z})$ by the formula $v(p) := \int_\alpha *du$, where α is any piecewise smooth curve joining $\gamma_0(0)$ to p. Now the map

$$\psi : A \to \mathbb{C}, \quad \psi := \exp\left(\frac{2\pi}{\kappa}(u + iv)\right)$$

is well defined and smooth.

The metric g, together with the orientation, induces a complex structure on A. As v locally lifts to a primitive of $*du$, we have $dv = *du$. Hence, in local conformal coordinates, the map $u + iv : A \to \mathbb{C}/i\kappa\mathbb{Z}$ satisfies the Cauchy–Riemann equations and is thus holomorphic; so ψ is holomorphic as well. We now prove that ψ is a diffeomorphism onto its image. Since $*du[\dot{\gamma}_i] > 0$, the compact set $F := \{p \in A : d\psi(p) = 0\}$ is contained in $\mathring{A}$. As ψ is holomorphic, F is finite. We have $F' := \psi^{-1}(\psi(F)) \subseteq \mathring{A}$ (as $\psi(\partial A) \cap \psi(\mathring{A}) = \emptyset$), so by holomorphicity F' is finite as well.

It is easy to check that $\psi\Big|_{A\setminus F'} : A \setminus F' \to \psi(A) \setminus \psi(F)$ is a covering (indeed, any $z \in \psi(A) \setminus \psi(F)$ has finitely many preimages $p_1, \dots, p_k \in A \setminus F'$; we can find open disjoint neighborhoods $U_j \subseteq A \setminus F'$ of p_j which are all mapped diffeomorphically onto some neighborhood V of z; up to replacing V with $V \setminus \psi(A \setminus \bigsqcup_j U_j)$ and shrinking each U_j accordingly, V is evenly covered by $\bigsqcup_j U_j$). But ψ is injective on $\partial A \subseteq A \setminus F'$, so $\psi\Big|_{A\setminus F'}$ is injective and hence a diffeomorphism onto its image.

As ψ is holomorphic, ψ cannot be injective in any punctured neighborhood of any point in F. It follows that $F = \emptyset$, thus also $F' = \emptyset$. Finally, calling $t := \exp\left(\frac{2\pi}{\kappa}\right)$, we have $\psi(A) \subseteq A_t$ and

$\psi(\partial A) = \partial A_t$. As $\psi(\mathring{A}) = \psi(A) \cap \mathring{A}_t$, $\psi(\mathring{A})$ is both open and closed in $\mathring{A}_t$, so by connectedness it follows that $\psi : A \to A_t$ is surjective. The map ψ provides the desired conformal equivalence. $\square$

Appendix B. Functional spaces

In this section we recall the definition of the functional spaces used in the paper, as well as some of their main properties and some key facts concerning the so-called Littlewood–Paley dyadic decomposition.

We denote respectively by $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ the spaces of (real or complex) Schwartz functions and tempered distributions. All the functional spaces used in this paper should be understood as subspaces of $\mathcal{S}'(\mathbb{R})$. Given a function $\varphi \in \mathcal{S}(\mathbb{R})$, we denote either by $\widehat{\varphi}$ or by $\mathcal{F}\varphi$ the Fourier transform of φ, i.e.

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} v(x) e^{-2\pi i \xi x}\, dx,$$

while if $v \in \mathcal{S}'(\mathbb{R})$ we define $\widehat{v} = \mathcal{F}v \in \mathcal{S}'(\mathbb{R})$ by $\langle \widehat{v}, \varphi \rangle := \langle v, \widehat{\varphi} \rangle$.

We recall the definition of the inhomogeneous fractional Sobolev (Bessel potential) spaces: for a real s and $1 < p < \infty$ we let

$$H^{s,p}(\mathbb{R}) := \left\{ v \in \mathcal{S}'(\mathbb{R}) : \|v\|_{H^{s,p}} := \left\| \mathcal{F}^{-1}[(1 + 4\pi^2 |\xi|^2)^{s/2} \mathcal{F}v] \right\|_{L^p} < \infty \right\}.$$

Observe that $H^{s,p}(\mathbb{R})$ is stable under multiplication by Schwartz functions, i.e. if $v \in H^{s,p}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$ then $v\psi \in H^{s,p}(\mathbb{R})$ (see e.g. the proof of [**GS12**]).

We also recall the definition of the homogeneous fractional Sobolev spaces that will be used in the paper, namely $\dot{H}^{1/2}(\mathbb{R})$ and $\dot{H}^{-1/2}(\mathbb{R})$:

$$\dot{H}^{1/2}(\mathbb{R}) := \left\{ v \in L^2_{loc}(\mathbb{R}) : \|v\|^2_{\dot{H}^{1/2}} := \iint_{\mathbb{R}^2} \frac{|v(x) - v(y)|^2}{|x - y|^2}\, dx\, dy < \infty \right\},$$

$$\dot{H}^{-1/2}(\mathbb{R}) := \left\{ v \in \mathcal{S}'(R) : \widehat{v} \in L^2_{loc}(\mathbb{R}) \text{ and } \int_{\mathbb{R}} |\xi|^{-1} |\widehat{v}(\xi)|^2\, d\xi < \infty \right\}.$$

We remark that $\dot{H}^{1/2}(\mathbb{R})$ is naturally a subspace of $\mathcal{S}'(\mathbb{R})$, although $\|\cdot\|_{\dot{H}^{1/2}}$ is only a seminorm (which vanishes on constant functions).

We recall that, given $v \in \dot{H}^{1/2}(\mathbb{R})$, we always have $\widehat{v} \in L^2_{loc}(\mathbb{R} \setminus \{0\})$ and moreover in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ we can identify the distribution $|\xi|^{1/2}\widehat{v}$ with an $L^2(\mathbb{R})$ function (which we continue to denote, by abuse of notation, with $|\xi|^{1/2}\widehat{v}$) with

$$\int_{\mathbb{R}} |\xi|\, |\widehat{v}(\xi)|^2\, d\xi = c\, \|v\|^2_{\dot{H}^{1/2}} \tag{16}$$

for some constant $c > 0$ (see e.g. the proof of [**DNPV12**]).

We list below some useful elementary results concerning $\dot{H}^{1/2}(\mathbb{R})$.

Lemma B.1. Given $v \in \dot{H}^{1/2}(\mathbb{R})$, for any $j \geq 0$ it holds

$$\|v\|_{L^2(B(0,2^j))} \lesssim 2^{j/2} \left| (v)_{B(0,1)} \right| + (j+1) 2^{j/2} \|v\|_{\dot{H}^{1/2}}.$$

Proof. We notice that, for $j \geq 0$,

$$\left\| v - (v)_{B(0,2^j)} \right\|^2_{L^2(B(0,2^j))} \lesssim 2^{-j} \iint_{B(0,2^j)^2} |v(x) - v(y)|^2 \lesssim 2^j \iint_{B(0,2^j)^2} \frac{|v(x) - v(y)|^2}{|x - y|^2},$$

therefore $\left\|v-(v)_{B(0,2^j)}\right\|_{L^2(B(0,2^j))} \lesssim 2^{j/2}\|v\|_{\dot H^{1/2}}$.

Similarly for $j \geq 1$

$$\left|(v)_{B(0,2^{j-1})}-(v)_{B(0,2^j)}\right| \lesssim 2^{-j}\int_{B(0,2^j)}\left|v-(v)_{B(0,2^j)}\right| \lesssim \|v\|_{\dot H^{1/2}}\ .$$

The desired inequality follows from these estimates and

$$\|v\|_{L^2(B(0,2^j))} \leq \left\|v-(v)_{B(0,2^j)}\right\|_{L^2(B(0,2^j))}+2^{j/2}\sum_{\ell=1}^{j}\left|(v)_{B(0,2^{\ell-1})}-(v)_{B(0,2^\ell)}\right|+2^{j/2}\left|(v)_{B(0,1)}\right|. \quad \square$$

Lemma B.2. Given $v \in \dot H^{1/2}(\mathbb{R})$, there exists a sequence $v_k \in \mathcal{S}(\mathbb{R})$, with $\widehat{v}_k \in C_c^\infty(\mathbb{R}\setminus\{0\})$,[4] and a sequence $c_k \in \mathbb{R}$ such that

$$\|v-(v_k+c_k)\|_{\dot H^{1/2}} = \|v-v_k\|_{\dot H^{1/2}} \to 0, \qquad v_k+c_k \overset{*}{\rightharpoonup} v \quad \text{in } \mathcal{S}'(\mathbb{R}),$$
$$\|v-(v_k+c_k)\|_{L^2(B(0,2^j))} \lesssim (j+1)2^{j/2}\|v-(v_k+c_k)\|_{\dot H^{1/2}}\ .$$

Proof. Fix $\chi \in C_c^\infty(\mathbb{R})$ with $\mathbf{1}_{B(0,1/2)} \leq \chi \leq \mathbf{1}_{B(0,1)}$. As observed above, the function

$$w_k := (\chi(2^{-k}\cdot)-\chi(2^k\cdot))\widehat{v}$$

belongs to $L^2(\mathbb{R})$, is supported in the annulus $\{2^{-k-1} \leq |\xi| \leq 2^k\}$ and verifies $\int |\xi|\,|w_k(\xi)|^2\,d\xi < \infty$. We can find $v_k \in \mathcal{S}(\mathbb{R})$ with $\widehat{v}_k \in C_c^\infty(\mathbb{R}\setminus\{0\})$ and $\int |\xi|\,|w_k-\widehat{v}_k|^2(\xi)\,d\xi \leq 2^{-k}$. Since $\int_{\mathbb{R}\setminus\{0\}} |\xi|\,|\widehat{v}-w_k|^2(\xi)\,d\xi \to 0$, we get

$$\|v-v_k\|^2_{\dot H^{1/2}} \simeq \int_{\mathbb{R}\setminus\{0\}} |\xi|\,|\widehat{v}-\widehat{v}_k|^2(\xi)\,d\xi \to 0.$$

We now choose c_k in such a way that $(v_k+c_k)_{B(0,1)} = (v)_{B(0,1)}$. The last part of the claim follows from Lemma B.1 and the convergence $v_k+c_k \overset{*}{\rightharpoonup} v$ is an immediate consequence. $\square$

Remark B.3. If v lies also in $L^\infty(\mathbb{R})$, we can also ensure that $\|v_k\|_{L^\infty}, |c_k| \lesssim \|v\|_{L^\infty}$, with $v_k+c_k \to v$ a.e. Indeed, $\mathcal{F}^{-1}(\chi(2^{-k}\cdot)-\chi(2^k\cdot))$ is bounded in $L^1(\mathbb{R})$, so $\|\check{w}_k\|_{L^\infty} \lesssim \|v\|_{L^\infty}$; moreover, v_k can be chosen arbitrarily close to $\check{w}_k$ in $L^\infty(\mathbb{R})$. Since $v_k+c_k \to v$ in $L^2_{loc}(\mathbb{R})$, we can ensure a.e. convergence by passing to a subsequence.

We also define the Hardy space $\mathcal{H}^1(\mathbb{R})$ as

$$\mathcal{H}^1(\mathbb{R}) := \left\{v \in L^1(\mathbb{R}) : \sup_{t>0}|\varphi_t * v|(x) \in L^1(\mathbb{R})\right\},$$

where $\varphi \in \mathcal{S}(\mathbb{R})$ is an arbitrary function such that $\int \varphi \neq 0$ and $\varphi_t(y) := t^{-1}\varphi(t^{-1}y)$. This definition does not depend on the choice of φ (for this and many useful characterizations of $\mathcal{H}^1(\mathbb{R})$, we refer the reader to [**Gra14M**] and [**Ste93**]).

Finally we define the Lorentz spaces $L^{2,1}(\mathbb{R})$ and $L^{2,\infty}(\mathbb{R})$:

$$L^{2,1}(\mathbb{R}) := \left\{v \in L^1_{loc}(\mathbb{R}) : \int_0^\infty \mathcal{L}^1(\{|f|>t\})^{1/2}\,dt < \infty\right\},$$
$$L^{2,\infty}(\mathbb{R}) := \left\{v \in L^1_{loc}(\mathbb{R}) : \sup_{t>0} t\mathcal{L}^1(\{|f|>t\})^{1/2} < \infty\right\},$$

[4]With abuse of notation, we denote by $C_c^\infty(\mathbb{R}\setminus\{0\})$ the space of those functions in $C_c^\infty(\mathbb{R})$ which are supported in $\mathbb{R}\setminus\{0\}$.

where $\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$. These are Banach spaces with the norms

$$\|v\|_{L^{2,1}} := \int_0^\infty t^{-1/2} \left(\sup_{t \le \mathcal{L}^1(E) < \infty} \fint_E |v| \right) dt, \quad \|v\|_{L^{2,\infty}} := \sup_{0 < \mathcal{L}^1(E) < \infty} \mathcal{L}^1(E)^{-1/2} \int_E |v|$$

and $L^{2,\infty}(\mathbb{R})$ is the dual of $L^{2,1}(\mathbb{R})$: see e.g. [**Gra14C**] and [**Gra14C**].

B.1. **Products, fractional Laplacian and Hilbert–Riesz transform.** We fix a nonnegative bump function $\rho \in C_c^\infty(\mathbb{R})$ with $\int \rho = 1$. Given $v, w \in \mathcal{S}'(\mathbb{R})$, we define their product

$$vw := \lim_{\epsilon \to 0} (\rho_\epsilon * v)(\rho_\epsilon * w)$$

as a limit in $\mathcal{S}'(\mathbb{R})$, provided that it exists. Notice that $(\rho_\epsilon * v)(\rho_\epsilon * w) \in C^\infty \cap \mathcal{S}'(\mathbb{R})$. In general, this limit could fail to exist or could depend on ρ. In all the instances appearing in this paper, we are implicitly claiming that the product is defined, is associative and is independent of ρ.

From the definition of $\dot{H}^{1/2}(\mathbb{R})$ it easily follows that $\dot{H}^{1/2} \cap L^\infty(\mathbb{R})$ is an algebra, i.e. it is closed under the product: more precisely,

$$\|vw\|_{\dot{H}^{1/2}} \lesssim \|v\|_{\dot{H}^{1/2}} \|w\|_{L^\infty} + \|v\|_{L^\infty} \|w\|_{\dot{H}^{1/2}}, \quad \|vw\|_{L^\infty} \le \|v\|_{L^\infty} \|w\|_{L^\infty}$$

whenever $v, w \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$. Using this and the obvious inclusion $\mathcal{S}(\mathbb{R}) \subseteq \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$, as well as (16), one checks that the product vw can always be formed when $v \in \dot{H}^{-1/2}(\mathbb{R})$ and $w \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$.

Moreover, for any real s, we define the fractional Laplacian $(-\Delta)^{s/2}$ as

$$(-\Delta)^{s/2} v := \lim_{\epsilon \to 0} \mathcal{F}^{-1}[(\epsilon^2 + 4\pi^2 |\xi|^2)^{s/2} \mathcal{F} v],$$

provided that the limit exists in $\mathcal{S}'(\mathbb{R})$; in other words, we approximate the fractional Laplacian by means of Bessel potentials. We recall some properties of the fractional Laplacian for the values of s mostly used in the paper, namely $s = \pm\frac{1}{4}$.

Clearly, $(-\Delta)^{1/4}$ maps $L^2(\mathbb{R})$ isomorphically onto $\dot{H}^{-1/2}(\mathbb{R})$, with inverse $(-\Delta)^{-1/4}$. The following statement is less obvious.

Lemma B.4. If $v \in \dot{H}^{1/2}(\mathbb{R})$, then $(-\Delta)^{1/4} v$ exists, lies in $L^2(\mathbb{R})$ and is given by

$$(-\Delta)^{1/4} v = \mathcal{F}^{-1}\left((2\pi |\xi|)^{1/2} \widehat{v} \right),$$

where we denote by $(2\pi|\xi|)^{1/2}\widehat{v}$ the function in $L^2(\mathbb{R})$ agreeing with the corresponding distribution on $\mathbb{R} \setminus \{0\}$.

Proof. We denote by w the function in $L^2(\mathbb{R})$ which coincides with $(2\pi|\xi|)^{1/2}\widehat{v}$ in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$. We observe that $(\epsilon^2 + 4\pi^2\xi^2)^{1/4} = \epsilon^{1/2} + \frac{\xi^2}{2} \int_0^1 4\pi^2 t (\epsilon^2 + 4\pi^2 t^2 \xi^2)^{-3/4}\, dt$, with the second term vanishing for $\xi = 0$. Using Lemma B.2 and (16) we get

$$\begin{aligned}(\epsilon^2 + 4\pi^2\xi^2)^{1/4} \widehat{v} &= \epsilon^{1/2} \widehat{v} + \lim_{k \to \infty} \left(\frac{\xi^2}{2} \int_0^1 4\pi^2 t (\epsilon^2 + 4\pi^2 t^2 \xi^2)^{-3/4}\, dt \right) (\widehat{v}_k + \widehat{c}_k) \\ &= \epsilon^{1/2} \widehat{v} + \left(\int_0^1 \frac{t}{2} (2\pi|\xi|)^{3/2} (\epsilon^2 + 4\pi^2 t^2 \xi^2)^{-3/4}\, dt \right) w.\end{aligned}$$

Finally, $\epsilon^{1/2}\widehat{v} \overset{*}{\rightharpoonup} 0$ in $\mathcal{S}'(\mathbb{R})$ and the nonnegative integral converges to $\mathbf{1}_{\mathbb{R} \setminus \{0\}}$ from below. $\square$

A similar proof shows that $(-\Delta)^{1/2}v = \mathcal{F}^{-1}(2\pi\,|\xi|\,\widehat{v})$, so $(-\Delta)^{1/2}v = (-\Delta)^{1/4}(-\Delta)^{1/4}v$.

One has also the following integral representation for $(-\Delta)^{1/4}v$.

Lemma B.5. For all $v \in \dot{H}^{1/2}(\mathbb{R})$ and some constant $c > 0$ (independent of v) we have

$$(-\Delta)^{1/4}v(x) = c \lim_{\epsilon\to 0} \int_{\mathbb{R}\setminus B(x,\epsilon)} \frac{v(x)-v(y)}{|x-y|^{3/2}}\,dy \quad \text{in } L^2(\mathbb{R}).$$

Proof. Let $w_\epsilon(x) := \int_{\mathbb{R}\setminus B(x,\epsilon)} \frac{v(x)-v(y)}{|x-y|^{3/2}}\,dy$ (which lies in $\mathcal{S}'(\mathbb{R})$ by Lemma B.1) and take $\varphi \in \mathcal{S}(\mathbb{R})$. Fubini's theorem gives

$$\begin{aligned}\langle \widehat{w}_\epsilon, \varphi\rangle = \langle w_\epsilon, \widehat{\varphi}\rangle &= \iint_{\mathbb{R}\times(\mathbb{R}\setminus B(0,\epsilon))} \frac{v(x)-v(x+h)}{|h|^{3/2}}\widehat{\varphi}(x)\,dx\,dh \\ &= \iint_{\mathbb{R}\times(\mathbb{R}\setminus B(0,\epsilon))} \frac{v(x)(\widehat{\varphi}(x)-\widehat{\varphi}(x-h))}{|h|^{3/2}}\,dx\,dh = \int_{\mathbb{R}\setminus B(0,\epsilon)} \frac{\left\langle \widehat{v}, (1-e^{2\pi i h x})\varphi(x)\right\rangle}{|h|^{3/2}}\,dh.\end{aligned}$$

Since $(1-e^{2\pi i h x})\varphi(x)$ vanishes at 0, Lemmas B.2 and B.4 show that

$$\left\langle \widehat{v}, (1-e^{2\pi i h x})\varphi(x)\right\rangle = \lim_{k\to\infty} \int |x|^{1/2}\,\widehat{v}_k(x) \frac{1-e^{2\pi i h x}}{|x|^{1/2}}\varphi(x)\,dx = \int \mathcal{F}((-\Delta)^{1/4}v) \frac{1-e^{2\pi i h x}}{(2\pi\,|x|)^{1/2}}\varphi(x)\,dx.$$

We conclude that

$$\widehat{w}_\epsilon(x) = \mathcal{F}((-\Delta)^{1/4}v)(x) \int_\epsilon^\infty \frac{2-2\cos(2\pi h x)}{h^{3/2}(2\pi\,|x|)^{1/2}}\,dh$$

and, for $x \neq 0$, the last integral equals $\int_{\epsilon|x|}^\infty \frac{2-2\cos(2\pi t)}{(2\pi)^{1/2}t^{3/2}}\,dt$, which converges to some positive constant from below, as $\epsilon \to 0$. ☐

As for the formal inverse, the Riesz potential operator $(-\Delta)^{-1/4}$, notice that $\mathcal{F}^{-1}(|\xi|^{-1/2}) = c\,|x|^{-1/2}$ for some $c \in \mathbb{R}$ (indeed, $|x|^{-1/2}$ is the only $-\frac{1}{2}$-homogeneous tempered distribution up to multiples, see e.g. [**Gra14C**]).

Since $|x|^{-1/2} \in L^{2,\infty}(\mathbb{R})$, we get $(-\Delta)^{-1/4}(L^1(\mathbb{R})) \subseteq L^{2,\infty}(\mathbb{R})$ and $(-\Delta)^{-1/4}(L^{2,1}(\mathbb{R})) \subseteq L^\infty(\mathbb{R})$.[5] Also, $(-\Delta)^{-1/4}$ maps $\mathcal{H}^1(\mathbb{R})$ into $L^{2,1}(\mathbb{R})$: this is a straightforward consequence of the atomic decomposition property of $\mathcal{H}^1(\mathbb{R})$ (see [**Ste93**]).

Finally, we define the Hilbert–Riesz transform of $v \in \mathcal{S}'(\mathbb{R})$ as

$$\mathcal{R}v := \lim_{\epsilon\to 0} \mathcal{F}^{-1}\left[-i\frac{\xi}{(\epsilon^2+|\xi|^2)^{1/2}}\widehat{v}\right],$$

whenever the limit exists. A well-known consequence of Hörmander–Mikhlin estimates is the fact that this limit exists on $L^p(\mathbb{R})$ and $\mathcal{R}$ maps $L^p(\mathbb{R})$ continuously into itself, for $1 < p < \infty$.

The same holds for $H^{s,p}(\mathbb{R})$ and $\dot{H}^{-1/2}(\mathbb{R})$, being the former isomorphic to $L^p(\mathbb{R})$ via $v \mapsto \mathcal{F}^{-1}[(1+4\pi^2\,|\xi|^2)^{s/2}\mathcal{F}v]$ and the latter to $L^2(\mathbb{R})$ via $v \mapsto \mathcal{F}^{-1}((2\pi\,|\xi|)^{-1/2}v)$.

Moreover, $\mathcal{R}$ also maps $\mathcal{H}^1(\mathbb{R})$ continuously into itself: this follows from [**Gra14M**] and $\mathcal{R}(\mathcal{R}v) = -v$ for $v \in L^1(\mathbb{R})$.

[5] For $v \in L^{2,1}(\mathbb{R})$ the fractional Laplacian $(-\Delta)^{-1/4}v$ exists and equals $c\,|x|^{-1/2} * v$: indeed, from [**Gra14M**] one easily deduces the weak* convergence of $\mathcal{F}^{-1}[(\epsilon^2+4\pi^2\,|\xi|^2)^{-1/4}]$ to $\mathcal{F}^{-1}((2\pi\,|\xi|)^{-1/2})$ in $L^{2,\infty}(\mathbb{R})$.

B.2. **Littlewood–Paley decomposition.** We briefly recall a well-known tool in harmonic analysis, the Littlewood–Paley dyadic decomposition. This decomposition can be obtained as follows. Let $\chi \in C_c^\infty(B(0,2))$ be an even function, with $\chi = 1$ on $B(0,1)$. Let $\varrho := \chi - \chi(2\cdot)$ and observe that the support of ϱ is included in the annulus $B(0,2) \setminus B(0,1/2)$.

Let $\varrho_0 := \chi$ and $\varrho_j := \varrho(2^{-j}\cdot)$ for $j > 0$, so that the support of ϱ_j, for $j > 0$, is contained in $B(0,2^{j+1}) \setminus B(0,2^{j-1})$. The functions $(\rho_j)_{j\in\mathbb{N}}$ realize a so-called *inhomogeneous* dyadic partition of unity, i.e. $\sum_{j=0}^\infty \rho_j = 1$ pointwise. We further denote $\chi_j(\xi) := \sum_{k=0}^j \varrho_k = \chi(2^{-j}\cdot)$.

For every $v \in \mathcal{S}'(\mathbb{R})$ we define the inhomogeneous Littlewood–Paley projection operators:

$$v_j = \mathcal{F}^{-1}[\varrho_j \widehat{v}], \quad v^j = \mathcal{F}^{-1}[\chi_j \widehat{v}]. \tag{17}$$

Roughly, v_j and v^j mimic a frequency projection to the annulus $B(0,2^j) \setminus B(0,2^{j-1})$ and to the ball $B(0,2^j)$, respectively.

We observe that $v^j = \sum_{k=0}^j v_k$ and $v = \sum_{k=0}^\infty v_k$ in the distributional sense. Given $v, w \in \mathcal{S}'(\mathbb{R})$, we can formally split their product in the following way:

$$vw = \Pi_1(v,w) + \Pi_2(v,w) + \Pi_3(v,w), \tag{18}$$

where

$$\Pi_1(v,w) := \sum_{j=3}^{+\infty} v_j w^{j-3}, \quad \Pi_2(v,w) := \sum_{j=3}^{+\infty} v^{j-3} w_j, \quad \Pi_3(v,w) := \sum_{j=0}^{\infty} v_j \sum_{|k-j|<3} w_k.$$

We observe that the support of $\mathcal{F}[v_j w^{j-3}]$ is contained in the sum of the supports of $\mathcal{F}v_j$ and $\mathcal{F}w^{j-3}$, i.e. in the annulus $B(0,2^{j+2}) \setminus B(0,2^{j-2})$ (for $j \geq 3$). A similar remark applies to $\mathcal{F}[v^{j-3}w_j]$.

Next we recall the definition of the inhomogeneous Besov spaces $B^s_{p,q}(\mathbb{R})$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R})$ in terms of the above dyadic decomposition.

Definition B.6. Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we set

$$\begin{aligned} \|v\|_{B^s_{p,q}} &:= \left(\sum_{j=0}^\infty 2^{jsq} \|v_j\|^q_{L^p} \right)^{1/q} && \text{if } q < \infty, \\ \|v\|_{B^s_{p,q}} &:= \sup_{j\in\mathbb{N}} 2^{js} \|v_j\|_{L^p} && \text{if } q = \infty. \end{aligned}$$

When $1 \leq p, q < \infty$ we also set

$$\|v\|_{F^s_{p,q}} = \left\| \left(\sum_{j=0}^\infty 2^{jsq} |v_j|^q \right)^{1/q} \right\|_{L^p}.$$

The space of all $v \in \mathcal{S}'(\mathbb{R})$ for which $\|v\|_{B^s_{p,q}} < \infty$ is the inhomogeneous Besov space with indices s, p, q and is denoted by $B^s_{p,q}(\mathbb{R})$. The space of all $v \in \mathcal{S}'(\mathbb{R})$ for which $\|v\|_{F^s_{p,q}} < \infty$ is the inhomogeneous Triebel–Lizorkin space with indices s, p, q and is denoted by $F^s_{p,q}(\mathbb{R})$. These spaces do not depend on the choice of χ: see [**Tri83**].

A well-known fact is that $H^{s,p}(\mathbb{R}) = F^s_{p,2}(\mathbb{R})$, with equivalent norms: see e.g. [**Tri83**].

Corollary B.7. If $s > \frac{1}{p}$, then $H^{s,p}(\mathbb{R}) \subseteq L^\infty(\mathbb{R}) \cap C^{k,\alpha}(\mathbb{R})$, for all $k \in \mathbb{N}$ and $0 < \alpha < 1$ with $k + \alpha \leq s - \frac{1}{p}$.

Proof. By [**Tri83**] we can assume $k=0$, as well as $s=\alpha+\frac{1}{p}$. Setting $\widetilde{\varrho}_j := \varrho_{j-1}+\varrho_j+\varrho_{j+1}$ (with $\varrho_{-1}:=0$), we have $v_j=\mathcal{F}^{-1}(\widetilde{\varrho}_j\mathcal{F}v_j)$ and $\left\|\mathcal{F}^{-1}\widetilde{\varrho}_j\right\|_{L^{p'}}\lesssim 2^{j/p}$, $\left\|\nabla(\mathcal{F}^{-1}\widetilde{\varrho}_j)\right\|_{L^{p'}}\lesssim 2^{j+j/p}$ (as $\mathcal{F}^{-1}\widetilde{\varrho}_j=2^{j-2}(\mathcal{F}^{-1}\widetilde{\varrho}_2)(2^{j-2}\cdot)$ for $j\ge 2$). Hence, given $0<h<1$,

$$\sum_{j=0}^{\infty}\|v_j\|_{L^\infty}=\sum_{j=0}^{\infty}\left\|(\mathcal{F}^{-1}\widetilde{\varrho}_j)*v_j\right\|_{L^\infty}\lesssim\sum_{j=0}^{\infty}2^{j/p}\|v_j\|_{L^p}\le\sum_{j=0}^{\infty}2^{-j(s-1/p)}\|v\|_{F^s_{p,2}}\lesssim\|v\|_{H^{s,p}},$$

$$\sum_{j=0}^{\infty}\|v_j(\cdot+h)-v_j\|_{L^\infty}\lesssim\sum_{2^jh\le1}h\|\nabla v_j\|_{L^\infty}+\sum_{2^jh>1}\|v_j\|_{L^\infty}\le\sum_{2^jh\le1}h\left\|\nabla(\mathcal{F}^{-1}\widetilde{\varrho}_j)*v_j\right\|_{L^\infty}$$

$$+\sum_{2^jh>1}\left\|(\mathcal{F}^{-1}\widetilde{\varrho}_j)*v_j\right\|_{L^\infty}\lesssim\left(\sum_{2^jh\le1}h2^{j(1+1/p-s)}+\sum_{2^jh>1}2^{-j(s-1/p)}\right)\|v\|_{F^s_{p,2}}\lesssim h^\alpha\|v\|_{H^{s,p}}. \qquad \square$$

Similarly, one can form the *homogeneous* Littlewood–Paley decomposition using instead $\varrho_j:=\varrho(2^{-j}\cdot)$ and $\chi_j:=\chi(2^{-j}\cdot)$, for all $j\in\mathbb{Z}$, and defining v_j and v^j as in (17). One then has the formal identities

$$v=\sum_{j\in\mathbb{Z}}v_j,\quad v^j=\sum_{k\le j}v_k,\quad vw=\Pi_1(v,w)+\Pi_2(v,w)+\Pi_3(v,w),$$

but notice that not even the first two are always true distributionally: for instance they fail when $v=1$ (in which case $v_k=0$ for all $k\in\mathbb{Z}$). This reflects the fact that $\sum_{j\in\mathbb{Z}}\varrho_j=\mathbf{1}_{\mathbb{R}\setminus\{0\}}$ and $\sum_{k\le j}\varrho_k=\mathbf{1}_{\mathbb{R}\setminus\{0\}}\chi_j$. Using this homogeneous decomposition, with the same formulas as above one can define the homogeneous Besov and Triebel–Lizorkin spaces $\dot{B}^s_{p,q}(\mathbb{R})$ and $\dot{F}^s_{p,q}(\mathbb{R})$ (the above norms now become merely seminorms).

If $v\in L^p(\mathbb{R})$ and $1<p<\infty$, then $\|v\|_{L^p}\lesssim\|v\|_{\dot{F}^0_{p,2}}$ and $\|v\|_{\dot{F}^0_{p,2}}\lesssim\|v\|_{L^p}$: see [**Gra14C**].

B.3. **Spaces on the unit circle $\mathcal{S}^1$.** We let $\mathcal{D}(\mathcal{S}^1):=C^\infty(\mathcal{S}^1)$ be the Fréchet space of smooth functions on $\mathcal{S}^1=\mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{D}'(\mathcal{S}^1)$ its topological dual. The product of two elements in $\mathcal{D}'(\mathcal{S}^1)$ is defined as before for $\mathbb{R}$. For $v\in\mathcal{D}'(\mathcal{S}^1)$ and $k\in\mathbb{Z}$ we let $\widehat{v}(k):=\frac{1}{2\pi}\left\langle v,e^{-ikx}\right\rangle$.

Notice that, for all $v\in\mathcal{D}'(\mathcal{S}^1)$, there exists some $N>0$ such that $|\widehat{v}(k)|\lesssim(1+|k|)^N$. Also, we recall that $v\in C^\infty(\mathcal{S}^1)$ if and only if the Fourier coefficients $\widehat{v}(k)$ have rapid decay, i.e. $\sup_k(1+|k|)^N|\widehat{v}(k)|<\infty$ for all $N>0$.

Given $v\in\mathcal{D}'(\mathcal{S}^1)$, we define $(-\Delta)^sv$ to be the distributional limit of $\sum_{k=-N}^{N}|k|^{2s}\widehat{v}(k)e^{ikx}$, as $N\to\infty$. Observe that $(-\Delta)^sv$ can be characterized as the unique $w\in\mathcal{D}'(\mathcal{S}^1)$ such that $\widehat{w}(k)=|k|^{2s}\widehat{v}(k)$, for all $k\in\mathbb{Z}$.

Given $s\in\mathbb{R}$, we define the Sobolev space

$$H^s(\mathcal{S}^1):=\Big\{v\in\mathcal{D}'(\mathcal{S}^1):\|v\|^2_{H^s}:=\sum_{k\in\mathbb{Z}}(1+|k|^2)^s|\widehat{v}(k)|^2<\infty\Big\}.$$

We observe that $\mathcal{D}'(\mathcal{S}^1)=\bigcup_{s\in\mathbb{R}}H^s(\mathcal{S}^1)$. Also, the Fréchet space structure of $\mathcal{D}(\mathcal{S}^1)$ is equivalent to the one given by all H^s-norms with $s\in\mathbb{N}$, by the embeddings $C^s(\mathcal{S}^1)\subseteq H^s(\mathcal{S}^1)\subseteq C^{s-1}(\mathcal{S}^1)$. Hence, by the uniform boundedness principle, any sequence v_j converging in $\mathcal{D}'(\mathcal{S}^1)$ will form a bounded set in some $H^{-s}(\mathcal{S}^1)$, with $s\in\mathbb{N}$ (by the canonical duality with $H^s(\mathcal{S}^1)$).

Lemma B.8. The space $H^{1/2}(\mathcal{S}^1)$ is the set of traces of $H^1(\mathbb{D})$. Moreover, for $v \in L^2(\mathcal{S}^1)$

$$\iint_{(\mathcal{S}^1)^2} \frac{\left|v(e^{i\theta}) - v(e^{i\tau})\right|^2}{\left|e^{i\theta} - e^{i\tau}\right|^2}\, d\theta\, d\tau = 4\pi^2 \sum_{k\in\mathbb{Z}} |k|\, |\widehat{v}(k)|^2 . \tag{19}$$

Proof. Given $u \in C^\infty(\overline{\mathbb{D}})$, let $v := u\Big|_{\mathcal{S}^1}$ be its trace and

$$\widetilde{v}(re^{i\theta}) := \sum_{k\in\mathbb{Z}} \widehat{v}(k) r^{|k|} e^{ik\theta} = \sum_{k<0} \widehat{v}(k) \left(\overline{re^{i\theta}}\right)^{|k|} + \sum_{k\geq 0} \widehat{v}(k)(re^{i\theta})^k,$$

which lies in $C^\infty(\overline{\mathbb{D}})$, is harmonic and has trace v. We have $\int_{\mathbb{D}} \langle \nabla\widetilde{v}, \nabla(u-\widetilde{v})\rangle = 0$ by the divergence theorem, so

$$\int_{\mathbb{D}} |\nabla u|^2 = \int_{\mathbb{D}} |\nabla\widetilde{v}|^2 + 2\int_{\mathbb{D}} \langle \nabla\widetilde{v}, \nabla(u-\widetilde{v})\rangle + \int_{\mathbb{D}} |\nabla(u-\widetilde{v})|^2 \geq \int_{\mathbb{D}} |\nabla\widetilde{v}|^2 .$$

A straightforward computation shows that the last integral equals $2\pi\sum_{k\in\mathbb{Z}} |k|\,|\widehat{v}(k)|^2$, so by density of smooth functions we deduce that the trace of a function in $H^1(\mathcal{D})$ lies in $H^{1/2}(\mathcal{S}^1)$. Conversely, given $v \in H^{1/2}(\mathcal{S}^1)$ one checks that $\widetilde{v}$, with the above definition, is in $H^1(\mathbb{D})$. It has trace v since $\widetilde{v} \in C^\infty(\mathbb{D})$ and, as $\tau \uparrow 1$, $\widetilde{v}(\tau\cdot) \to \widetilde{v}$ in $H^1(\mathbb{D})$, as well as $v(\tau\cdot)\Big|_{\mathcal{S}^1} \to v$ in $L^2(\mathcal{S}^1)$. Finally, the left-hand side of (19) equals

$$\int_{\mathcal{S}^1} \frac{\left\|v - v(e^{i\sigma}\cdot)\right\|_{L^2}^2}{|1-e^{i\sigma}|^2}\, d\sigma = 2\pi \int_{\mathcal{S}^1} \frac{\sum_k |\widehat{v}(k)|^2 \left|1-e^{ik\sigma}\right|^2}{|1-e^{i\sigma}|^2} = 2\pi \sum_k |\widehat{v}(k)|^2 \int_{\mathcal{S}^1} \left|\sum_{\ell=0}^{|k|-1} e^{i\ell\sigma}\right|^2 d\sigma. \qquad \square$$

B.4. **Spaces on a boundary ∂S.** Given a smooth compact Riemannian surface (S, g) with boundary, we define the spaces $H^s(\partial S)$ by isometrically identifying each boundary component with (a dilation of) $\mathcal{S}^1$. Lemma B.8, together with a partition of unity argument, immediately implies the following result.

Lemma B.9. We have

$$H^{1/2}(\partial S) = \left\{ v \in L^2(\partial S) : \iint_{(\partial S)^2} \frac{|v(x)-v(y)|^2}{d(x,y)^2}\, \mathrm{dvol}_g(x)\, \mathrm{dvol}_g(y) < \infty \right\},$$

$d(x, y)$ denoting the Riemannian distance. Moreover, the traces of functions in $H^1(S)$ are precisely the functions in $H^{1/2}(\partial S)$. In particular, each $v \in H^{1/2}(\partial S)$ has a unique harmonic extension $\widetilde{v} \in H^1(S)$.

Appendix C. Commutator estimates

We introduce the following commutators for functions defined on the real line:

$$\begin{aligned}
T(Q, v) &:= (-\Delta)^{1/4}(Qv) + ((-\Delta)^{1/4}Q)v - Q((-\Delta)^{1/4}v),\\
U(Q, v) &:= -\mathcal{R}(-\Delta)^{1/4}(Qv) + (\mathcal{R}(-\Delta)^{1/4}Q)v + Q(\mathcal{R}(-\Delta)^{1/4}v),\\
T^*(P, Q) &:= ((-\Delta)^{1/4}P)Q + P((-\Delta)^{1/4}Q) - (-\Delta)^{1/4}(PQ),\\
U^*(P, Q) &:= (\mathcal{R}(-\Delta)^{1/4}P)Q + P(\mathcal{R}(-\Delta)^{1/4}Q) - \mathcal{R}(-\Delta)^{1/4}(PQ),\\
\Lambda(Q, v) &:= Qv + \mathcal{R}(Q\mathcal{R}v),\\
F(f, v) &:= \mathcal{R}f\mathcal{R}v - fv.
\end{aligned}$$

The notation T^* and U^* is motivated by the formal identities

$$\int PT(Q,v) = \int T^*(P,Q)v, \quad \int PU(Q,v) = \int U^*(P,Q)v.$$

Using the technology of Littlewood–Paley decomposition and paraproducts, one can establish the following estimates of integrability by compensation.

Theorem C.1. If $P, Q \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$, we have $T^*(P,Q), U^*(P,Q) \in L^{2,1}(\mathbb{R})$ and

$$\|T^*(P,Q)\|_{L^{2,1}}, \|U^*(P,Q)\|_{L^{2,1}} \lesssim \|P\|_{\dot{H}^{1/2}} \|Q\|_{\dot{H}^{1/2}}.$$

Proof. By [**DLR09**] we have $(-\Delta)^{1/4}T^*(P,Q) \in \mathcal{H}^1(\mathbb{R})$, with

$$\left\|(-\Delta)^{1/4}T^*(P,Q)\right\|_{\mathcal{H}^1} \lesssim \|P\|_{\dot{H}^{1/2}} \|Q\|_{\dot{H}^{1/2}}.$$

The estimate for T^* follows from the fact that $(-\Delta)^{-1/4}(\mathcal{H}^1(\mathbb{R})) \subseteq L^{2,1}(\mathbb{R})$. The estimate for U^* can be obtained in a completely analogous way. It can also be deduced from Theorem C.5 below, since

$$U^*(P,Q) = \mathcal{R}T^*(P,Q) + \Lambda(P, \mathcal{R}(-\Delta)^{1/4}Q) + \Lambda(Q, \mathcal{R}(-\Delta)^{1/4}P)$$

and $\mathcal{R}$ maps the spaces $L^2(\mathbb{R})$ and $L^{2,1}(\mathbb{R})$ into themselves continuously. $\square$

Theorem C.2. If $Q \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$ and $v \in L^2(\mathbb{R})$, we have $T(Q,v), U(Q,v) \in \mathcal{H}^1(\mathbb{R})$ and

$$\|T(Q,v)\|_{\mathcal{H}^1}, \|U(Q,v)\|_{\mathcal{H}^1} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2}.$$

Proof. For the estimate of $T(Q,v)$, we refer the reader to the proof of [**DaL15**] (where one just replaces $(-\Delta)^{1/4}u$ with v). The estimate of $U(Q,v)$ can be achieved with a completely analogous proof. It also follows from the identity

$$U(Q,v) = -T(Q, \mathcal{R}v) - F((-\Delta)^{1/4}Q, \mathcal{R}v) + (-\Delta)^{1/4}\Lambda(Q, \mathcal{R}v)$$

and Theorem C.6, together with the estimate $\left\|(-\Delta)^{1/4}\Lambda(Q,\mathcal{R}v)\right\|_{\mathcal{H}^1} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2}$ (see the proof of Theorem C.5). $\square$

The two following results now follow from Theorems C.1 and C.2 by a duality argument.

Corollary C.3. If $P, Q \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$, we have

$$\|T^*(P,Q)\|_{L^2}, \|U^*(P,Q)\|_{L^2} \lesssim \|P\|_{\dot{H}^{1/2}} \left\|(-\Delta)^{1/4}Q\right\|_{L^{2,\infty}}.$$

Proof. Since $T(P,Q)$ vanishes if P or Q is constant, we can assume that $P, Q \in \mathcal{S}(\mathbb{R})$ with $\widehat{Q} \in C_c^\infty(\mathbb{R} \setminus \{0\})$ (see Lemma B.2 and Remark B.3). For any $v \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}\int T^*(P,Q)v &= \int T^*(Q,P)v = \int QT(P,v) = \int (-\Delta)^{1/4}Q(-\Delta)^{-1/4}T(P,v) \\ &\lesssim \left\|(-\Delta)^{1/4}Q\right\|_{L^{2,\infty}} \left\|(-\Delta)^{-1/4}T(P,Q)\right\|_{L^{2,1}} \lesssim \left\|(-\Delta)^{1/4}Q\right\|_{L^{2,\infty}} \|T(P,Q)\|_{\mathcal{H}^1} \\ &\lesssim \left\|(-\Delta)^{1/4}Q\right\|_{L^{2,\infty}} \|P\|_{\dot{H}^{1/2}} \|v\|_{L^2},\end{aligned}$$

where we used Theorem C.2 and the fact that $(-\Delta)^{-1/4}(\mathcal{H}^1(\mathbb{R})) \subseteq L^{2,1}(\mathbb{R})$. A similar argument applies for U^*. $\square$

Corollary C.4. If $Q \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$ and $v \in L^2(\mathbb{R})$, we have $T(Q,v), U(Q,v) \in \dot{H}^{-1/2}(\mathbb{R})$ and

$$\|T(Q,v)\|_{\dot{H}^{-1/2}}, \|U(Q,v)\|_{\dot{H}^{-1/2}} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^{2,\infty}}.$$

Proof. Since $T(Q,v)$ vanishes when Q is constant, we can assume that $Q, v \in \mathcal{S}(\mathbb{R})$. For any $P \in \mathcal{S}(\mathbb{R})$ we get

$$\int PT(Q,v) = \int T^*(P,Q)v \lesssim \|T^*(P,Q)\|_{L^{2,1}} \|v\|_{L^{2,\infty}} \lesssim \|P\|_{\dot{H}^{1/2}} \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^{2,\infty}} ,$$

thanks to Theorem C.1. A similar argument applies for U. ☐

Theorem C.5. If $Q \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$ and $v \in L^2(\mathbb{R})$, we have $\Lambda(Q,v) \in L^{2,1}(\mathbb{R})$ and

$$\|\Lambda(Q,v)\|_{L^{2,1}} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2} .$$

Proof. By [**DLR11**] (which contains a wrong sign in the statement) we know that $(-\Delta)^{1/4}\Lambda(Q,v) \in \mathcal{H}^1(\mathbb{R})$, with

$$\left\|(-\Delta)^{1/4}\Lambda(Q,v)\right\|_{\mathcal{H}^1} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2} ,$$

and thus $\|\Lambda(Q,v)\|_{L^{2,1}} = \left\|(-\Delta)^{-1/4}(-\Delta)^{1/4}\Lambda(Q,v)\right\|_{L^{2,1}} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2}$. ☐

The following inequality is due to Coifman–Rochberg–Weiss.

Theorem C.6. If $f, v \in L^2(\mathbb{R})$, we have $F(f,v) \in \mathcal{H}^1(\mathbb{R})$ and

$$\|F(f,v)\|_{\mathcal{H}^1} \lesssim \|f\|_{L^2} \|v\|_{L^2} .$$

Proof. The Hilbert–Riesz transform $\mathcal{R}$ satisfies the identity $\mathcal{R}(fv - \mathcal{R}f\mathcal{R}v) = f\mathcal{R}v + v\mathcal{R}f$: indeed, taking the Fourier transform at $\xi \in \mathbb{R}$, this amounts to say that for a.e. ξ

$$-i\,\mathrm{sgn}(\xi)\int (1+\mathrm{sgn}(\xi-\zeta)\,\mathrm{sgn}(\zeta))\widehat{f}(\xi-\zeta)\widehat{v}(\zeta)\,d\zeta = -i\int(\mathrm{sgn}(\xi-\zeta)+\mathrm{sgn}(\zeta))\widehat{f}(\xi-\zeta)\widehat{v}(\zeta).$$

If $\xi > 0$, $1+\mathrm{sgn}(\xi-\zeta)\,\mathrm{sgn}(\zeta)-\mathrm{sgn}(\xi-\zeta)-\mathrm{sgn}(\zeta) = (1-\mathrm{sgn}(\xi-\zeta))(1-\mathrm{sgn}(\zeta))$ vanishes identically (since either $\zeta > 0$ or $\xi-\zeta > 0$). On the other hand, if $\xi < 0$, $1+\mathrm{sgn}(\xi-\zeta)\,\mathrm{sgn}(\zeta)+\mathrm{sgn}(\xi-\zeta)+\mathrm{sgn}(\zeta) = (1+\mathrm{sgn}(\xi-\zeta))(1+\mathrm{sgn}(\zeta))$ vanishes also identically (since either $\zeta < 0$ or $\xi-\zeta < 0$). In both cases we get

$$\mathrm{sgn}(\xi)(1+\mathrm{sgn}(\xi-\zeta)\,\mathrm{sgn}(\zeta)) = \mathrm{sgn}(\xi-\zeta)+\mathrm{sgn}(\zeta)$$

and the identity follows. Thus we have $\|F(f,v)\|_{L^1} \lesssim \|f\|_{L^2}\|v\|_{L^2}$ and

$$\mathcal{R}F(f,v) = -f\mathcal{R}v - v\mathcal{R}f \in L^1(\mathbb{R}), \quad \|\mathcal{R}F(f,v)\|_{L^1} \lesssim \|f\|_{L^2}\|v\|_{L^2} .$$

The claim follows from [**Gra14M**]. ☐

Appendix D. Hölder continuity of $\frac{1}{2}$-harmonic maps

In this section we obtain the Hölder continuity of $\frac{1}{2}$-harmonic maps on ∂S with values into (at least) C^2-smooth closed manifolds.

Theorem D.1. Let $\mathcal{N} \subset \mathbb{R}^m$ be a C^k-smooth closed embedded manifold, with $k \geq 2$, and let $u \in H^{1/2}(\partial S, \mathcal{N})$ be $\frac{1}{2}$-harmonic. Then u is Hölder continuous.

The strategy for the proof of Theorem D.1 is similar to the one used to get the Hölder continuity of $\frac{1}{2}$-harmonic maps defined on $\mathbb{R}$ (see [**DLR09**, **DLR11**, **Sch12**]). We provide here the details for the reader's convenience. The proof can be described (roughly speaking) by the following steps.

1. By means of a stereographic projection we can reduce to a problem on $\mathbb{R}$, as it was already observed in [**DaL15**, **DaL17**].

2. We rewrite the Euler equation on $\mathbb{R}$ as a Schrödinger-type linear system with antisymmetric potential satisfied by $(-\Delta)^{1/4}w$ (where $w := u \circ \psi_\ell^{-1} \circ \Pi^{-1}$, Π^{-1} being the inverse of the stereographic projection.

3. We show that $(-\Delta)^{1/4}w \in L^p_{loc}(\mathbb{R})$ for some $p > 2$, giving $u \in C^{0,\delta}_{loc}(\mathbb{R})$ for some $0 < \delta < 1$.

Lemma D.2. Let $u \in H^{1/2}(\partial S, \mathcal{N})$ be a $\frac{1}{2}$-harmonic map and let $\Pi : \mathcal{S}^1 \setminus \{i\} \to \mathbb{R}$ be the stereographic projection. Then $w := u \circ \psi_\ell^{-1} \circ \Pi^{-1} \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ and w satisfies

$$P^T(w)(-\Delta)^{1/2}w + \frac{2}{1+x^2}P^T(w)\left(R_\ell((f_j)_{j=1}^k) \circ \Pi^{-1}\right) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}), \tag{20}$$

$$P^N(w)\nabla w = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{21}$$

Proof. Step 1. We first prove (20).

Claim: $w \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ and $(-\Delta)^{1/2}w = \frac{2}{1+x^2}((-\Delta)^{1/2}(w \circ \Pi)) \circ \Pi^{-1}$ in distributional sense.

Proof of the claim: let $\mathbb{D} := \{|z| < 1\}$ and $\mathbb{H} := \{\Im z > 0\}$ be the standard unit disk and upper half-plane in $\mathbb{C}$ and notice that the map

$$\widetilde{\Pi} : \mathbb{D} \to \mathbb{H}, \quad \widetilde{\Pi}(z) := \left(\frac{2}{z-i} - i\right)$$

is conformal, with trace Π on $\mathcal{S}^1 \setminus \{i\}$. Hence, by conformal invariance of the Dirichlet energy, this map gives a bijection between $H^1(\mathbb{D})$ and $\dot{H}^1(\mathbb{H}) := \{w \in W^{1,2}_{loc}(\mathbb{H}) : \int_{\mathbb{H}} |\nabla w|^2 \, dx < \infty\}$. Moreover, Π gives a bijection between $H^{1/2}(\mathcal{S}^1)$ and $\dot{H}^{1/2}(\mathbb{R})$: indeed, for a real measurable function f on $\mathbb{R}$,

$$\iint_{\mathbb{R}^2} \frac{|f(x)-f(y)|^2}{|x-y|^2} \, dx \, dy = \iint_{(\mathcal{S}^1)^2} \frac{\left|f \circ \Pi(e^{i\theta}) - f \circ \Pi(e^{i\tau})\right|^2}{|e^{i\theta} - e^{i\tau}|^2} \, d\theta \, d\tau, \tag{22}$$

since $\left|\Pi'(e^{i\theta})\right| = \frac{2}{|e^{i\theta}-i|^2}$ and $\left|\Pi(e^{i\theta}) - \Pi(e^{i\tau})\right|^{-2} = \frac{|e^{i\theta}-i|^2|e^{i\tau}-i|^2}{4|e^{i\theta}-e^{i\tau}|^2}$. In particular we get that $w \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$. We infer that $\dot{H}^{1/2}(\mathbb{R})$ is precisely the image of the trace of $\dot{H}^1(\mathbb{H})$ and that any $f \in \dot{H}^{1/2}(\mathbb{R})$ is the trace of a unique harmonic map in $\dot{H}^1(\mathbb{H})$ (since the corresponding statements for the unit disk hold).

Given any $f \in C^\infty(\mathcal{S}^1)$, the normal derivative of its harmonic extension $\widetilde{f} \in H^1(\mathbb{D}, \mathbb{R}^m)$ at the boundary is given by $\frac{\partial \widetilde{f}}{\partial \nu} = (-\Delta)^{1/2}f$, as is readily checked using the formula $\widetilde{f}(re^{i\theta}) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) r^{|n|} e^{in\theta}$. The same formula also shows that $\|\widetilde{f}\|_{H^1(\mathbb{D})} = \|(-\Delta)^{1/4}f\|_{L^2}$.

By Lemma B.2, w can be approximated in $\mathcal{S}'(\mathbb{R}, \mathbb{R}^m)$ by a sequence $w_n = h_n + c_n \in \mathcal{S}(\mathbb{R}, \mathbb{R}^m) + \mathbb{R}^m$ such that $w_n \to w$ in $\dot{H}^{1/2}(\mathbb{R}, R^m)$ and in $\mathcal{S}'(\mathbb{R}, \mathbb{R}^m)$. The functions $f_n := w_n \circ \Pi$ extend smoothly to all the circle. By conformality of $\widetilde{\Pi}$, $\widetilde{w}_n := \widetilde{f}_n \circ \widetilde{\Pi}^{-1}$ is the unique harmonic extension of w_n in $\dot{H}(\mathbb{H})$ and its normal derivative is

$$\frac{\partial \widetilde{w}_n}{\partial \nu} = \left|\Pi' \circ \Pi^{-1}\right|^{-1} \frac{\partial \widetilde{f}_n}{\partial \nu} \circ \Pi^{-1} = \frac{2}{x^2+1} \frac{\partial \widetilde{f}_n}{\partial \nu} \circ \Pi^{-1}.$$

By uniqueness, $\widetilde{w}_n(x+iy) = \int_{\mathbb{R}} e^{-2\pi y|\xi|} e^{2\pi i x\xi} \, \widehat{h}_n(\xi) \, d\xi + c_n$ and thus $\frac{\partial \widetilde{w}_n}{\partial \nu}(x) = (-\Delta)^{1/2}w_n$.

From (22) and (19), $(-\Delta)^{1/4}f_n \to (-\Delta)^{1/4}(w \circ \Pi)$ in $L^2(\mathcal{S}^1, \mathbb{R}^m)$. Hence,

$$(-\Delta)^{1/2}w = \lim_{n\to\infty} (-\Delta)^{1/2}w_n = \lim_{n\to\infty} \frac{\partial \widetilde{w}_n}{\partial \nu} = \lim_{n\to\infty} \frac{2}{x^2+1} \frac{\partial \widetilde{f}_n}{\partial \nu} \circ \Pi^{-1}$$

$$= \lim_{n\to\infty} \frac{2}{x^2+1}((-\Delta)^{1/2} f_n)\circ \Pi^{-1} = \frac{2}{x^2+1}((-\Delta)^{1/2}(w\circ\Pi))\circ\Pi^{-1}$$

in the distributional sense. Using Lemma 2.3 we can conclude that (20) holds.

Step 2. Next we show (21). To this aim let us fix a nonnegative bump function $\rho \in C_c^\infty(B(0,1))$ with $\int \rho = 1$ and let $w_\epsilon := \rho_\epsilon * w$, where $\rho_\epsilon := \epsilon^{-1}\rho(\epsilon^{-1}\cdot)$. From (16) it immediately follows that $w_\epsilon \to w$ in $\dot H^{1/2}(\mathbb{R},\mathbb{R}^m)$, i.e.

$$\frac{w_\epsilon(x)-w_\epsilon(y)}{|x-y|} \to \frac{w(x)-w(y)}{|x-y|} \qquad \text{in } L^2(\mathbb{R}^2,\mathbb{R}^m). \tag{23}$$

In particular, for some sequence $\epsilon_j \downarrow 0$ there exists $h \in L^2(\mathbb{R}^2)$ such that $\frac{|w_{\epsilon_j}(x)-w_{\epsilon_j}(y)|}{|x-y|} \le h(x,y)$ and $w_{\epsilon_j} \to w$ a.e. Moreover, since $\mathcal{N}$ is a C^2 submanifold, there exists a neighborhood $U \supseteq \mathcal{N}$ such that the map $p \in C^1(U,\mathcal{N})$, associating to $x \in U$ the unique nearest point $p(x)$ on $\mathcal{N}$, is defined. Notice that $\operatorname{dist}(w_\epsilon,\mathcal{N}) \to 0$ in $L^\infty(\mathbb{R})$, as

$$\begin{aligned}\operatorname{dist}(w_\epsilon(x),\mathcal{N})^2 &\le \int |w_\epsilon(x)-w(x-z)|^2\rho_\epsilon(z)\,dz \le \iint |w(x-y)-w(x-z)|^2 \rho_\epsilon(y)\rho_\epsilon(z)\,dy\,dz \\ &\lesssim \epsilon^{-2}\iint_{B(0,\epsilon)^2} |w(x-y)-w(x-z)|^2\,dy\,dz \lesssim \iint_{B(x,\epsilon)^2} \frac{|w(y)-w(z)|^2}{|y-z|^2}\,dy\,dz,\end{aligned}$$

which converges to 0 uniformly in x. Thus, eventually $p(w_{\epsilon_j}) \in \dot H^{1/2}(\mathbb{R},\mathcal{N})$ is defined. Since $P^N \circ p(w_{\epsilon_j})\nabla(p(w_{\epsilon_j})) = 0$, it suffices to show that

$$P^N \circ p(w_{\epsilon_j}) \to P^N \circ p(w) = P^N(w), \quad p(w_{\epsilon_j}) \to p(w) = w$$

in $\dot H^{1/2}(\mathbb{R},\mathbb{R}^m)$. This immediately follows by dominated convergence, since the maps $P^N \circ p$ and p are Lipschitz (up to shrinking U).

We finally remark that $h := -\frac{2}{1+x^2}P^T(w)\left(R_\ell((f_j)_{j=1}^k)\circ\Pi^{-1}\right)$ lies in $L^1 \cap L^\infty(\mathbb{R},\mathbb{R}^m)$. ☐

Being $w \in \dot H^{1/2}(\mathbb{R},\mathcal{N})$, the quantity $P^N(-\Delta)^{1/4}w$ enjoys special regularity properties. This has already been observed in [**DLLR16**, **MS17**].

Lemma D.3. For any $w \in \dot H^{1/2}(\mathbb{R},\mathcal{N})$ it holds

$$\left|P^N(w)(-\Delta)^{1/4}w\right| \lesssim |T^*(w;w)| \quad \text{a.e.}$$

Proof. Since w takes values in the C^2 submanifold $\mathcal{N}$, it holds

$$\left|P^N(w(x))(w(x)-w(x+y))\right| \lesssim |w(x)-w(x+y)|^2$$

and, in view of Lemma B.5, we deduce that for some sequence $\epsilon_j \downarrow 0$

$$\left|P^N(w)(-\Delta)^{1/4}w\right|(x) \lesssim \liminf_{j\to\infty}\int_{\mathbb{R}\setminus B(0,\epsilon_j)} \frac{|w(x)-w(x+y)|^2}{|y|^{3/2}}\,dy,$$

$$\begin{aligned}T^*(w;w)(x) &= (-\Delta)^{1/4}w\cdot w + w\cdot(-\Delta)^{1/4}w - (-\Delta)^{1/4}(w\cdot w) \\ &= c\lim_{j\to\infty}\int_{\mathbb{R}\setminus B(0,\epsilon_j)} \frac{|w(x)-w(x+y)|^2}{|y|^{3/2}}\,dy,\end{aligned}$$

thanks to the identity (with $z := x+y$)

$$(w(x)-w(z))\cdot w(x) + w(x)\cdot(w(x)-w(z)) - (w(x)\cdot w(x) - w(z)\cdot w(z)) = |w(x)-w(z)|^2. \quad ☐$$

In what follows, given $x_0 \in \mathbb{R}$ and $r > 0$, we set $B := B(x_0, r)$, $A_0 := B(x_0, 2r)$ and, for $j \geq 1$, $A_j := B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)$. We now give some preliminary estimates.

Lemma D.4. For any $w \in \dot{H}^{1/2}(\mathbb{R})$ and any $1 \leq p < \infty$ we have

$$r^{-1/p} \|w - (w)_B\|_{L^p(B)} \lesssim \|w\|_{\dot{H}^{1/2}}, \tag{24}$$

$$\begin{aligned} r^{-1/2} \|w - (w)_B\|_{L^2(B)} &\lesssim r^{-3/4} \left(\iint_{B^2} \frac{|w(x) - w(y)|^2}{|x - y|^{1/2}} \, dx \, dy \right)^{1/2} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j/2} \big\| (-\Delta)^{1/4} w \big\|_{L^{2,\infty}(A_j)}. \end{aligned} \tag{25}$$

Proof. By translating and rescaling, we can assume $x_0 = 0$ and $r = 1$. Moreover, we can suppose $w = (-\Delta)^{-1/4} v = c\,|x|^{-1/2} * v$ for some $v \in \mathcal{S}(\mathbb{R})$, with $\widehat{v} \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Proof of (24): letting $w_1 := (-\Delta)^{-1/4}(v \mathbf{1}_{A_0})$, $w_2 := (-\Delta)^{-1/4}(v \mathbf{1}_{\mathbb{R} \setminus A_0})$ and using Young's inequality, the mean value theorem and Hölder's inequality,

$$\begin{aligned} &\|w - (w)_B\|_{L^p(B)} \lesssim \|w_1\|_{L^p(B)} + \sup_{x, x' \in B} \left| w_2(x) - w_2(x') \right| \\ &\lesssim \left\| (|x|^{-1/2} \mathbf{1}_{B(0,3)}) * (v \mathbf{1}_{A_0}) \right\|_{L^p(B)} + \sup_{x, x' \in B} \int_{\mathbb{R} \setminus A_0} \left| |x - y|^{-1/2} - |x' - y|^{-1/2} \right| |v(y)| \, dy \\ &\lesssim \left\| |x|^{-1/2} \right\|_{L^{2p/(p+2)}(B(0,3))} \|v\|_{L^2(A_0)} + \int_{\mathbb{R} \setminus A_0} |y|^{-3/2} |v(y)| \, dy \lesssim \|v\|_{L^2} \end{aligned}$$

(assuming without loss of generality $p \geq 2$), which proves the first part.

Proof of (25): Jensen's inequality gives

$$\begin{aligned} \|w - (w)_B\|_{L^2(B)}^2 &\lesssim \iint_{B^2} |w(x) - w(y)|^2 \, dx \, dy \lesssim \iint_{B^2} \frac{|w(x) - w(y)|^2}{|x - y|^{1/2}} \, dx \, dy \\ &\lesssim \int_0^2 \int_B \frac{|w(x + h) - w(x)|^2}{h^{1/2}} \, dx \, dh. \end{aligned} \tag{26}$$

Setting $f_h(z) := (|z + h|^{-1/2} + |z|^{-1/2}) \mathbf{1}_{B(0,2h)}(z)$,

$$\begin{aligned} |w(x + h) - w(x)| &\lesssim \int \left| |x + h - y|^{-1/2} - |x - y|^{-1/2} \right| |v(y)| \, dy \\ &\lesssim f_h * |v| \, (x) + \int_{\mathbb{R} \setminus B(x, 2h)} \left| |x + h - y|^{-1/2} - |x - y|^{-1/2} \right| |v(y)| \, dy \\ &\lesssim f_h * |v| \, (x) + h \int_{\mathbb{R} \setminus B(x, 2h)} |x - y|^{-3/2} |v(y)| \, dy, \end{aligned} \tag{27}$$

where we used again the mean value theorem. Notice that, by Young's inequality,

$$\begin{aligned} &\int_0^2 h^{-1/2} \int_B |f_h * |v| \, (x)|^2 \, dx \, dh = \int_0^2 h^{-1/2} \left\| f_h * (|v| \, \mathbf{1}_{B(0,5)}) \right\|_{L^2(B)}^2 \, dh \\ &\leq \int_0^2 h^{-1/2} \|f_h\|_{L^{4/3}}^2 \|v\|_{L^{4/3}(B(0,5))}^2 \, dh \lesssim \|v\|_{L^{2,\infty}(B(0,5))}^2, \end{aligned} \tag{28}$$

since $\|f_h\|_{L^{4/3}} \lesssim h^{1/4}$. On the other hand, by Hölder's inequality,

$$\int_{A_0\setminus B(x,2h)} |x-y|^{-3/2} |v(y)| \, dy \lesssim \Big(\int_{2h}^{\infty} t^{-9/2} \, dt \Big)^{1/3} \|v\|_{L^{3/2}(A_0)} \lesssim h^{-7/6} \|v\|_{L^{2,\infty}(A_0)} , \tag{29}$$

while, since $|x-y|^{-3/2} \lesssim 2^{-3j/2}$ for $x \in B$ and $y \in A_j$ (when $j \ge 1$),

$$\int_{\mathbb{R}\setminus A_0} |x-y|^{-3/2} |v(y)| \, dy = \sum_{j=1}^{\infty} \int_{A_j} |x-y|^{-3/2} |v(y)| \, dy \lesssim \sum_{j=1}^{\infty} 2^{-j} \|v\|_{L^{2,\infty}(A_j)} . \tag{30}$$

By combining (26)–(30) and by applying Cauchy–Schwarz inequality we conclude that

$$\begin{aligned} \int_0^2 \int_{B_1} \frac{|w(x+h)-w(x)|^2}{h^{1/2}} \, dx \, dh &\lesssim \|v\|^2_{L^{2,\infty}(B(0,5))} + \int_0^2 h^{-5/6} \|v\|^2_{L^{2,\infty}(A_0)} \, dh \\ + \int_0^2 h^{3/2} \bigg(\sum_{j=1}^{\infty} 2^{-j} \|v\|_{L^{2,\infty}(A_j)} \bigg)^2 dh &\lesssim \|v\|^2_{L^{2,\infty}(A_0)} + \sum_{j=1}^{\infty} 2^{-j} \|v\|^2_{L^{2,\infty}(A_j)} . \end{aligned}$$

The claim follows. □

Lemma D.5. Given $w \in \dot H^{1/2} \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$, we can estimate

$$\begin{aligned} \|T^*(w;w)\|_{L^2(B)} \lesssim{}& \Big(\|w\|_{\dot H^{1/2}(B(x_0,4r))} + \big\|(-\Delta)^{1/4} w\big\|_{L^{2,\infty}(B(x_0,4r))} \Big) \big\|(-\Delta)^{1/4} w\big\|_{L^{2,\infty}(A_0)} \\ &+ \sum_{j=1}^{\infty} 2^{-j/4} \Big(\|w\|_{\dot H^{1/2}(B(x_0,4r))} + \big\|(-\Delta)^{1/4} w\big\|_{L^{2,\infty}(A_j)} \Big) \big\|(-\Delta)^{1/4} w\big\|_{L^{2,\infty}(A_j)}, \end{aligned}$$

where $\|w\|^2_{\dot H^{1/2}(B(x_0,4r))} := \iint_{B(x_0,4r)^2} \frac{|w(x)-w(y)|^2}{|x-y|^2} \, dx \, dy$.

Proof. Again we can assume $x_0 = 0$, $r = 1$. Given $\rho \in C_c^\infty(B(0,3))$ with $\rho = 1$ on $B(0,2)$, we define $w_0 := w - (w)_{B(0,4)}$ and observe that $T^*(w;w) = T^*(w_0;w_0)$, since T^* vanishes when one of the arguments is constant, while $\|\rho w_0\|^2_{\dot H^{1/2}}$ equals

$$\begin{aligned} \|\rho w_0\|^2_{\dot H^{1/2}} &\lesssim \iint_{B(0,4)} \frac{|\rho(x) w_0(x) - \rho(y) w_0(y)|^2}{|x-y|^2} + \iint_{B(0,4)\times(\mathbb{R}\setminus B(0,4))} \frac{|\rho(x) w_0(x)|^2}{|x-y|^2} \\ &\lesssim \iint_{B(0,4)} \frac{|w_0(x) - w_0(y)|^2}{|x-y|^2} + \int_{B(0,4)} \big|w_0 - (w_0)_{B(0,4)}\big|^2 \\ &\lesssim \iint_{B(0,4)^2} |w_0(x) - w_0(y)|^2 \left(|x-y|^{-2} + 1\right) \lesssim \|w\|^2_{\dot H^{1/2}(B(0,4))} , \end{aligned} \tag{31}$$

where we split $\rho(x)w_0(x) - \rho(y)w_0(y) = \rho(x)(w_0(x) - w_0(y)) + (\rho(x) - \rho(y))w_0(y)$ and used the fact that $(w_0)_{B(0,4)} = 0$. Next we write

$$T^*(w;w) = T^*(\rho w_0; \rho w_0) + T^*((1-\rho)w_0; \rho w_0) + T^*(w_0; (1-\rho)w_0),$$

so that Corollary C.3 gives

$$\begin{aligned} \|T^*(w;w)\|_{L^2(B)} \lesssim{}& \|\rho w_0\|_{\dot H^{1/2}} \big\|(-\Delta)^{1/4}(\rho w_0)\big\|_{L^{2,\infty}} + \|T^*((1-\rho)w_0; \rho w_0)\|_{L^2(B)} \\ &+ \|T^*(w_0; (1-\rho)w_0)\|_{L^2(B)} . \end{aligned}$$

Estimate of $\|\rho w_0\|_{\dot H^{1/2}} \left\|(-\Delta)^{1/4}(\rho w_0)\right\|_{L^{2,\infty}}$: from (31) we get $\|\rho w_0\|_{\dot H^{1/2}} \lesssim \|w\|_{\dot H^{1/2}(B(0,4))}$. Also,

$$\left\|(-\Delta)^{1/4}(\rho w_0)\right\|_{L^{2,\infty}(\mathbb{R})}^2 \lesssim \left\|\rho(-\Delta)^{1/4}w\right\|_{L^{2,\infty}(\mathbb{R})}^2 + \int \left|\int \frac{(\rho(x)-\rho(y))w_0(y)}{|x-y|^{3/2}}\,dy\right|^2 dx \tag{32}$$

(see Lemma B.5). It suffices to bound the last term of (32). Splitting $(\rho(x)-\rho(y))w_0(y) = -(\rho(x)-\rho(y))(w_0(x)-w_0(y)) + (\rho(x)-\rho(y))w_0(x)$ and using Cauchy–Schwarz, as well as $|\rho(x)-\rho(y)| \lesssim |x-y|$,

$$\begin{aligned}
&\int_{B(0,4)} \left|\int_{B(0,4)} \frac{(\rho(x)-\rho(y))w_0(y)}{|x-y|^{3/2}}\,dy\right|^2 dx \lesssim \|w_0\|_{L^2(B(0,4))}^2 \\
&\quad + \int_{B(0,4)} \left(\int_{B(0,4)} \frac{|\rho(x)-\rho(y)|^2}{|x-y|^{5/2}}\,dy\right)\left(\int_{B(0,4)} \frac{|w(x)-w(y)|^2}{|x-y|^{1/2}}\,dy\right) dx \\
&\lesssim \|w_0\|_{L^2(B(0,4))}^2 + \iint_{B(0,4)} \frac{|w(x)-w(y)|^2}{|x-y|^{1/2}}\,dx\,dy.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}\setminus B(0,4)} \left|\int_{B(0,4)} -\frac{\rho(y)w_0(y)}{|x-y|^{3/2}}\,dy\right|^2 dx &\lesssim \int_{B(0,3)} \left(\int_{\mathbb{R}\setminus B(0,4)} \frac{|w_0(y)|^2}{|x-y|^3}\,dx\right) dy \\
&\lesssim \|w_0\|_{L^2(B(0,4))}^2 .
\end{aligned}$$

Now we use the elementary inequality

$$\begin{aligned}
\|w_0\|_{L^1(B(0,2^j))} &\lesssim \left\|w_0 - (w_0)_{B(0,2^j)}\right\|_{L^1(B(0,2^j))} + \sum_{\ell=3}^{j} 2^j \left|(w_0)_{B(0,2^\ell)} - (w_0)_{B(0,2^{\ell-1})}\right| \\
&\lesssim \sum_{\ell=2}^{j} 2^{j-\ell/2}\left\|w-(w)_{B(0,2^\ell)}\right\|_{L^2(B(0,2^\ell))}
\end{aligned}$$

(for $j \ge 2$) and we get

$$\begin{aligned}
&\left(\int_{B(0,4)} \left|\int_{\mathbb{R}\setminus B(0,4)} \frac{\rho(x)w_0(y)}{|x-y|^{3/2}}\,dy\right|^2 dx\right)^{1/2} \lesssim \sum_{j=2}^{\infty} 2^{-3j/2}\,\|w_0\|_{L^1(A_j)} \\
&\lesssim \sum_{j=2}^{\infty}\sum_{\ell=2}^{j+1} 2^{-j/2-\ell/2}\left\|w-(w)_{B(0,2^\ell)}\right\|_{L^2(B(0,2^\ell))} \lesssim \sum_{\ell=2}^{\infty} 2^{-\ell}\left\|w-(w)_{B(0,4)}\right\|_{L^2(B(0,2^\ell))}.
\end{aligned}$$

Thus, applying Lemma D.4 to $B(0,4)$ and $B(0,2^\ell)$, we get

$$\begin{aligned}
&\left\|(-\Delta)^{1/4}(\rho w_0)\right\|_{L^{2,\infty}} \lesssim \left\|(-\Delta)^{1/4}w\right\|_{L^{2,\infty}(B(0,3))} \\
&+ \sum_{\ell=2}^{\infty} 2^{-\ell/2}\left(\sum_{p=0}^{\ell}\left\|(-\Delta)^{1/4}w\right\|_{L^{2,\infty}(A_p)} + \sum_{p=\ell+1}^{\infty} 2^{(\ell-p)/2}\left\|(-\Delta)^{1/4}w\right\|_{L^{2,\infty}(A_p)}\right) \\
&\lesssim \sum_{p=0}^{\infty}(p+1)2^{-p/2}\left\|(-\Delta)^{1/4}w\right\|_{L^{2,\infty}(A_p)} \lesssim \sum_{p=0}^{\infty} 2^{-p/4}\left\|(-\Delta)^{1/4}w\right\|_{L^{2,\infty}(A_p)}.
\end{aligned}$$

Estimate of $\|T^*(w_0;(1-\rho)w_0)\|_{L^2(B)}$: by Lemma B.5 we have

$$
\begin{aligned}
&\left\|w_0\cdot(-\Delta)^{1/4}((1-\rho)w_0)\right\|_{L^2(B)} \lesssim \left\|w_0-(w_0)_{B(0,4)}\right\|_{L^2(B(0,4))}\left\|(-\Delta)^{1/4}((1-\rho)w_0)\right\|_{L^\infty(B)}\\
&\lesssim \|w\|_{\dot H^{1/2}(B(0,4))}\sum_{j=1}^{\infty}2^{-3j/2}\|w_0\|_{L^1(A_j)}\\
&\lesssim \|w\|_{\dot H^{1/2}(B(0,4))}\sum_{p=0}^{\infty}2^{-p/4}\left\|(-\Delta)^{1/4}w\right\|_{L^{2,\infty}(A_p)},
\end{aligned}
$$

where the last inequality is obtained as before. Hence, $\left\|w_0\cdot(-\Delta)^{1/4}((1-\rho)w_0)\right\|_{L^2(B)}$ has the desired upper bound. Similarly, using Cauchy–Schwarz inequality twice,

$$
\begin{aligned}
&\left\|(-\Delta)^{1/4}((1-\rho)\,|w_0|^2)\right\|_{L^2(B)} \lesssim \left\|(-\Delta)^{1/4}((1-\rho)\,|w_0|^2)\right\|_{L^\infty(B)}\\
&\lesssim \sum_{j=1}^{\infty}2^{-3j/2}\|w_0\|^2_{L^2(A_j)}\\
&\lesssim \sum_{j=1}^{\infty}2^{-3j/2}\left(\sum_{\ell=2}^{j+1}2^{j/2-\ell/2}\left\|w-(w)_{B(0,2^\ell)}\right\|_{L^2(B(0,2^\ell))}\right)^2\\
&\lesssim \sum_{j=1}^{\infty}\sum_{\ell=2}^{j+1}2^{-j/2-\ell}\ell^2\left\|w-(w)_{B(0,2^\ell)}\right\|^2_{L^2(B(0,2^\ell))}\\
&\lesssim \sum_{\ell=2}^{\infty}2^{-3\ell/2}\ell^2\left\|w-(w)_{B(0,2^\ell)}\right\|^2_{L^2(B(0,2^\ell))}\\
&\lesssim \sum_{\ell=2}^{\infty}2^{-\ell/2}\ell^2\left(\sum_{p=0}^{\ell}(p+1)^2\left\|(-\Delta)^{1/4}w\right\|^2_{L^{2,\infty}(A_p)}+\sum_{p=\ell+1}^{\infty}(p+1)^2 2^{\ell-p}\left\|(-\Delta)^{1/4}w\right\|^2_{L^{2,\infty}(A_p)}\right)\\
&\lesssim \sum_{p=0}^{\infty}2^{-p/4}\left\|(-\Delta)^{1/4}w\right\|^2_{L^{2,\infty}(A_p)}.
\end{aligned}
$$

Estimate of $T^*((1-\rho)w_0;\rho w_0)=T^*(\rho w_0;(1-\rho)w_0)$: analogous. □

Lemma D.6. Let $P\in \dot H^{1/2}\cap L^\infty(\mathbb{R})$ and $v\in L^2(\mathbb{R})$. Then, uniformly in $s\geq 1$,

$$
\begin{aligned}
\left\|(-\Delta)^{-1/4}T(P,v)\right\|_{L^2(B)} \lesssim{}& \|v\|_{L^{2,\infty}(B(0,2^s))}\sum_{j=s}^{\infty}2^{s/2-j/4}\|P\|_{\dot H^{1/2}(B(0,2^j r))}\\
&+\|P\|_{\dot H^{1/2}}\sum_{j=s}^{\infty}2^{-j/4}\|v\|_{L^{2,\infty}(A_j)}.
\end{aligned}
$$

Proof. The proof is similar to the one of Lemma D.5, but is substantially simpler: as before we assume $x_0=0$, $r=1$. Setting $P_0:=P-(P)_{B(0,2^{s+3})}$, notice that $T(P,v)=T(P_0,v)$.

Let $\rho\in C_c^\infty(B(0,2^{s+2}))$ with $\rho=1$ on $B(0,2^{s+1})$ and $|\rho'|\lesssim 2^{-s}$. By Corollary C.4 we can write

$$
(-\Delta)^{-1/4}T(P,v) =(-\Delta)^{-1/4}T(\rho P_0, v\mathbf{1}_{B(0,2^s)})+(-\Delta)^{-1/4}T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)})
$$

$$+\sum_{j=s}^{\infty}(-\Delta)^{-1/4}T(P_0, v\mathbf{1}_{A_j})$$

in $L^2(\mathbb{R})$ and as before

$$\left\|(-\Delta)^{-1/4}T(\rho P_0, v\mathbf{1}_{B(0,2^s)})\right\|_{L^2} \lesssim \|\rho P_0\|_{\dot{H}^{1/2}} \|v\|_{L^{2,\infty}(B(0,2^s))} \lesssim \|P\|_{\dot{H}^{1/2}(B(0,2^{s+3}))} \|v\|_{L^{2,\infty}(B(0,2^s))} .$$

To estimate the two other pieces, fix any $j \geq 1$ and let $\chi_j \in C_c^\infty(B(0,2^{j+2}))$ with

$$\chi_j = 1 \quad \text{on } B(0,2^{j+1}) \setminus B(0,(5/6)2^j), \qquad \chi_j = 0 \quad \text{on } B(0,(4/6)2^j) \cup (\mathbb{R} \setminus B(0,2^{j+2}))$$

and $\left\|\chi_j'\right\|_{L^\infty} \lesssim 2^{-j}$. In particular, χ_j vanishes in a neighborhood of $\overline{B}$. Next we are going to use [**Gra14C**] (which implies that, on $\mathbb{R} \setminus \overline{A_j}$, $(-\Delta)^{1/4}(P_0 v\mathbf{1}_{A_j})$ is smooth and bounded by $|x|^{-3/2} * \left|P_0 v\mathbf{1}_{A_j}\right|$) and the fact that,[6] by Lemma D.4,

$$\|P_0\|_{L^4(A_\ell)} \lesssim \sum_{p=0}^{\max(\ell+1,s+3)} 2^{\ell/4-p/4}\left\|P_0 - (P_0)_{B(0,2^p)}\right\|_{L^4(B(0,2^p))} \lesssim \max(\ell+1,s)2^{\ell/4} \|P_0\|_{\dot{H}^{1/2}} .$$

We split $T(P_0, v\mathbf{1}_{A_j}) = (1-\chi_j)T(P_0, v\mathbf{1}_{A_j}) + \chi_j T(P_0, v\mathbf{1}_{A_j})$. For all $\ell \geq 0$ we have

$$\begin{aligned}
&\left\|(1-\chi_j)T(P_0, v\mathbf{1}_{A_j})\right\|_{L^2(A_\ell)} \\
&\lesssim 2^{\ell/2}\left\|(1-\chi_j)(-\Delta)^{1/4}(P_0 v\mathbf{1}_{A_j})\right\|_{L^\infty(A_\ell)} + 2^{\ell/4}\|P_0\|_{L^4(A_\ell)} \left\|(1-\chi_j)(-\Delta)^{1/4}(v\mathbf{1}_{A_j})\right\|_{L^\infty(A_\ell)} \\
&\lesssim 2^{-3\max(j,\ell)/2}\left(2^{\ell/2}\|P_0\|_{L^4(A_j)} + 2^{\ell/4}2^{j/4}\|P_0\|_{L^4(A_\ell)}\right)\|v\|_{L^{4/3}(A_j)} \\
&\lesssim \max(j+1,\ell+1,s)2^{j/2+\ell/2}2^{-3\max(j,\ell)/2}\|P\|_{\dot{H}^{1/2}}\|v\|_{L^{2,\infty}(A_j)} .
\end{aligned}$$

Since $T(P_0, v\mathbf{1}_{A_j}) \in L^1(\mathbb{R})$ by Theorem C.2, it follows that for all $j \geq s$

$$\begin{aligned}
&\left\|(-\Delta)^{-1/4}((1-\chi_j)T(P_0, v\mathbf{1}_{A_j}))\right\|_{L^2(B)} \\
&\lesssim \left(\left\|\,|x|^{-1/2} * ((1-\chi_j)T(P_0, v\mathbf{1}_{A_j})\mathbf{1}_{A_0})\right\|_{L^2(B)} + \sum_{\ell=1}^{\infty} 2^{-\ell/2}\left\|(1-\chi_j)T(P_0, v\mathbf{1}_{A_j})\right\|_{L^1(A_\ell)}\right) \\
&\lesssim \sum_{\ell=0}^{\infty}\left\|(1-\chi_j)T(P_0, v\mathbf{1}_{A_j})\right\|_{L^2(A_\ell)} \\
&\lesssim \|P\|_{\dot{H}^{1/2}}\|v\|_{L^{2,\infty}(A_j)}\left(\sum_{\ell=0}^{j}(j+1)2^{-j+\ell/2} + \sum_{\ell=j+1}^{\infty}(\ell+1)2^{j/2-\ell}\right) \\
&\lesssim 2^{-j/4}\|P\|_{\dot{H}^{1/2}}\|v\|_{L^{2,\infty}(A_j)} .
\end{aligned}$$

Similarly, as $T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)}) = (-\Delta)^{1/4}((1-\rho)P_0)v\mathbf{1}_{B(0,2^s)} - (1-\rho)P_0(-\Delta)^{1/4}(v\mathbf{1}_{B(0,2^s)})$, using Lemma B.5 we get

$$\begin{aligned}
\left\|T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)})\right\|_{L^{2,\infty}(B(0,2^s))} &\lesssim \sum_{p=s}^{\infty} 2^{-p}\|P_0\|_{L^2(A_p)}\|v\|_{L^{2,\infty}(B(0,2^s))} \\
&\lesssim \sum_{p=s}^{\infty} p2^{-p/2}\|P\|_{\dot{H}^{1/2}(B(0,2^{p+1}))}\|v\|_{L^{2,\infty}(B(0,2^s))} ,
\end{aligned}$$

[6] Since $(P_0)_{B(0,2^{s+3})} = 0$, if $\ell \geq s+2$ we can write $P_0 = P_0 - (P_0)_{B(0,2^{\ell+1})} + \sum_{p=s+4}^{\ell+1}((P_0)_{B(0,2^p)} - (P_0)_{B(0,2^{p-1})})$ and if $\ell \leq s+2$ we write $P_0 = P_0 - (P_0)_{B(0,2^{\ell+1})} + \sum_{p=\ell+2}^{s+3}((P_0)_{B(0,2^{p-1})} - (P_0)_{B(0,2^p)})$.

$$\begin{aligned}\left\|T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)})\right\|_{L^2(A_\ell)} &\lesssim 2^{-3\ell/2}\left\|P_0\right\|_{L^2(A_\ell)}\left\|v\right\|_{L^1(B(0,2^s))}\\ &\lesssim \ell 2^{s/2-\ell}\left\|P\right\|_{\dot H^{1/2}(B(0,2^{\ell+3}))}\left\|v\right\|_{L^{2,\infty}(B(0,2^s))}\end{aligned}$$

for $\ell \geq s$ (notice that $1-\rho$ vanishes near $\overline{B}(0,2^s)$). Since

$$\begin{aligned}&\left\|(-\Delta)^{-1/4}(T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)})\mathbf{1}_{B(0,2^s)})\right\|_{L^2(B)}\\ &\lesssim \left\||x|^{-1/2}\right\|_{L^{4/3}(B(0,2^{s+1}))}\left\|T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)})\right\|_{L^{4/3}(B(0,2^s))}\\ &\lesssim 2^{s/2}\left\|T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)})\right\|_{L^{2,\infty}(B(0,2^s))},\end{aligned}$$

we get

$$\left\|(-\Delta)^{-1/4}T((1-\rho)P_0, v\mathbf{1}_{B(0,2^s)})\right\|_{L^2(B)} \lesssim \sum_{j=s}^{\infty} 2^{s/2-j/4}\left\|P\right\|_{\dot H^{1/2}(B(0,2^j))}\left\|v\right\|_{L^{2,\infty}(B(0,2^\ell))}.$$

Finally, we estimate $\left\|(-\Delta)^{-1/4}(\chi_j T(P_0, v\mathbf{1}_{A_j}))\right\|_{L^2(B)}$ by duality: given $\psi \in C_c^\infty(B)$ with $\|\psi\|_{L^2} \leq 1$,

$$\begin{aligned}\left\langle(-\Delta)^{-1/4}(\chi_j T(P_0, v\mathbf{1}_{A_j})), \psi\right\rangle &= \int \chi_j T(P_0, v\mathbf{1}_{A_j})(-\Delta)^{-1/4}\psi\\ &= \int (-\Delta)^{-1/4}T(P_0, v\mathbf{1}_{A_j})(-\Delta)^{1/4}(\chi_j(-\Delta)^{-1/4}\psi).\end{aligned}$$

The first identity holds since $T(P_0, v\mathbf{1}_{A_j}) \in L^1(\mathbb{R})$ (by Theorem C.2), while the second is justified by $\chi_j(-\Delta)^{-1/4}\psi \in C_c^\infty(\mathbb{R})$, $(-\Delta)^{-1/4}T(P_0, v\mathbf{1}_{A_j}) \in L^2(\mathbb{R})$ (by Corollary C.4) and Plancherel's theorem.

We observe that, on the support of χ_j, $(-\Delta)^{-1/4}\psi$ is bounded by $2^{-j/2}$ and its derivative by $2^{-3j/2}$ (as $(-\Delta)^{-1/4}\psi = c\,|x|^{-1/2} * \psi$), so $f := \chi_j(-\Delta)^{-1/4}\psi$ has $\|f\|_{L^\infty} \lesssim 2^{-j/2}$, $\|f'\|_{L^\infty} \lesssim 2^{-3j/2}$ and

$$\left\|(-\Delta)^{1/4}f\right\|_{L^2}^2 \lesssim \int |\xi|\left|\widehat{f}(\xi)\right|^2 d\xi \lesssim \int (2^{-j}+2^j\xi^2)\left|\widehat{f}(\xi)\right|^2 d\xi \lesssim 2^{-j}\left\|f\right\|_{L^2}^2 + 2^j\left\|f'\right\|_{L^2}^2 \lesssim 2^{-j}.$$

Moreover, by Corollary C.4, $\left\|(-\Delta)^{-1/4}T(P_0, v\mathbf{1}_{A_j})\right\|_{L^2} \lesssim \|P\|_{\dot H^{1/2}}\|v\|_{L^{2,\infty}(A_j)}$. We deduce that

$$\left\|(-\Delta)^{-1/4}(\chi_j T(P_0, v\mathbf{1}_{A_j}))\right\|_{L^2} \lesssim 2^{-j/2}\left\|P\right\|_{\dot H^{1/2}}\left\|v\right\|_{L^{2,\infty}(A_j)}. \qquad \square$$

D.1. **Rewriting the Euler–Lagrange equation.** In Lemma D.2 we have seen that $w := u \circ \psi_\ell^{-1} \circ \Pi^{-1} \in \dot H^{1/2}(\mathbb{R}, \mathcal{N})$ satisfies

$$P^T(w)(-\Delta)^{1/2}w + \frac{2}{1+x^2}P^T(w)\left(R_\ell((f_j)_{j=1}^k) \circ \Pi^{-1}\right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Therefore we can write

$$(-\Delta)^{1/2}w = P^N(w)(-\Delta)^{1/2}w + h \tag{33}$$

where $h = -\frac{2}{1+x^2}P^T(w)\left(R_\ell((f_j)_{j=1}^k) \circ \Pi^{-1}\right) \in L^1 \cap L^\infty(\mathbb{R})$. We are going to reformulate (33) in the same spirit as it has been done in [**DLS17**, **MS17**].

This equivalent reformulation will be crucial in order to obtain the regularity of w. First of all, writing for simplicity P^T and P^N in place of $P^T(w)$ and $P^N(w)$,

$$\begin{aligned}P^N(-\Delta)^{1/2}w &= (-\Delta)^{1/4}(P^N(-\Delta)^{1/4}w) + (-\Delta)^{1/4}P^N(-\Delta)^{1/4}w - T(P^N, (-\Delta)^{1/4}w)\\ &= (-\Delta)^{1/4}(P^N(-\Delta)^{1/4}w) + (-\Delta)^{1/4}P^N P^N(-\Delta)^{1/4}w\\ &\quad + (-\Delta)^{1/4}P^N P^T(-\Delta)^{1/4}w - T(P^N, (-\Delta)^{1/4}w).\end{aligned}$$

Next, we observe that

$$\begin{aligned}(-\Delta)^{1/4}P^N P^T &= -P^N(-\Delta)^{1/4}P^T + T^*(P^N,P^T)\\ &= P^N(-\Delta)^{1/4}P^N + T^*(P^N,P^T) = \Omega_0 + \Omega_1 + (-\Delta)^{1/4}P^N P^N,\end{aligned}$$

where $\Omega_0 := P^N(-\Delta)^{1/4}P^N - (-\Delta)^{1/4}P^N P^N \in L^2(\mathbb{R},\mathfrak{so}(m))$, $\Omega_1 := T^*(P^N,P^T) \in L^{2,1}(\mathbb{R},\mathbb{R}^{m\times m})$. Hence, setting $v := (-\Delta)^{1/4}w \in L^2(\mathbb{R},\mathbb{R}^m)$, we arrive at

$$(-\Delta)^{1/4}v = \Omega_0 v + \Omega_1 v + (-\Delta)^{1/4}(P^N v) + 2(-\Delta)^{1/4}P^N(P^N v) - T(P^N,v) + h. \tag{34}$$

Theorem D.7. The map $v = (-\Delta)^{1/4}w$ has $(-\Delta)^{1/4}(P^T v), \mathcal{R}(-\Delta)^{1/4}(P^N v) \in L^1(\mathbb{R},\mathbb{R}^m)$ and there exists $\alpha > 0$ such that

$$\left\|(-\Delta)^{1/4}(P^T v)\right\|_{L^1(B(x_0,r))} + \left\|\mathcal{R}(-\Delta)^{1/4}(P^N v)\right\|_{L^1(B(x_0,r))} \lesssim r^\alpha,$$

for all $r > 0$, uniformly in $x_0 \in \mathbb{R}$.

Proof. Step 1. Fix any $x_0 \in \mathbb{R}$. We first proceed to locally remove the antisymmetric matrix Ω_0: if $R > 0$ is small enough, then we can write $\Omega_0 \mathbf{1}_{B(x_0,R)} = \frac{1}{2}(Q^{-1}(-\Delta)^{1/4}Q - (-\Delta)^{1/4}Q^{-1}Q)$ for some $Q \in \dot H^{1/2}(\mathbb{R},SO(m))$ with $\|Q\|_{\dot H^{1/2}} \lesssim \|\Omega_0\|_{L^2(B(x_0,R))}$ (see [**DLR11**] and [**DLS17**]). The map $\widetilde v := Qv$ then satisfies

$$\begin{aligned}(-\Delta)^{1/4}\widetilde v &= Q(-\Delta)^{1/4}v - (-\Delta)^{1/4}Qv + T(Q,v)\\ &= Q\Omega_0 v + Q\Omega_1 v + Q(-\Delta)^{1/4}(P^N v) + 2(Q(-\Delta)^{1/4}P^N)(P^N v)\\ &\quad - QT(P^N,v) + Qh - (-\Delta)^{1/4}Qv + T(Q,v).\end{aligned}$$

Using the identities

$$\begin{gathered}Q\Omega_0\mathbf{1}_{B(x_0,R)} - (-\Delta)^{1/4}Q = -\frac{Q}{2}T^*(Q^{-1},Q),\\ Q(-\Delta)^{1/4}(P^N v) = (-\Delta)^{1/4}(QP^N v) + (-\Delta)^{1/4}QP^N v - T(Q,P^N v),\\ QT(P^N,v) = T^*(Q,P^N)v + T(QP^N,v) - T(Q,P^N v),\end{gathered}$$

we get

$$\begin{aligned}(-\Delta)^{1/4}\widetilde v &= Q\Omega_0\mathbf{1}_{\mathbb{R}\setminus B(x_0,r)}v - \frac{Q}{2}T^*(Q^{-1},Q)v + Q\Omega_1 v + (-\Delta)^{1/4}(QP^N v)\\ &\quad + (-\Delta)^{1/4}QP^N v + 2(Q(-\Delta)^{1/4}P^N)(P^N v) - T^*(Q,P^N)v - T(QP^N,v)\\ &\quad + Qh + T(Q,v)\\ &= \widetilde\Omega_0\widetilde v + \widetilde\Omega_1\widetilde v + \widetilde\Omega_2 P^N v + (-\Delta)^{1/4}(QP^N v) + T(QP^T,v) + Qh,\end{aligned}$$

with $\widetilde\Omega_0 := Q\Omega_0\mathbf{1}_{\mathbb{R}\setminus B(x_0,r)}Q^{-1}$, $\widetilde\Omega_1 := Q\left(\Omega_1 - \frac{1}{2}T^*(Q^{-1},Q)\right)Q^{-1} - T^*(Q,P^N)Q^{-1}$ and $\widetilde\Omega_2 := (-\Delta)^{1/4}Q + 2Q(-\Delta)^{1/4}P^N$. Notice that $\widetilde\Omega_0, \widetilde\Omega_2 \in L^2(\mathbb{R},\mathbb{R}^{m\times m})$ and $\widetilde\Omega_1 \in L^{2,1}(\mathbb{R},\mathbb{R}^{m\times m})$. Recall that $\left|P^N v\right| \lesssim |T^*(w;w)|$ by Lemma D.3 (see also [**DLS17**] for related properties).

Step 2. Next, we use the last equation satisfied by $\widetilde v$ in order to estimate locally the $L^{2,\infty}$-norm of v. As $\widetilde v \in L^2(\mathbb{R},\mathbb{R}^m)$, we have

$$\begin{aligned}\widetilde v = (-\Delta)^{-1/4}(-\Delta)^{1/4}\widetilde v &= (-\Delta)^{-1/4}(\widetilde\Omega_0\widetilde v) + (-\Delta)^{-1/4}(\widetilde\Omega_1\widetilde v) + (-\Delta)^{-1/4}(\widetilde\Omega_2 P^N v)\\ &\quad + QP^N v + (-\Delta)^{-1/4}T(QP^T,v) + (-\Delta)^{-1/4}(Qh).\end{aligned}$$

Fix any radius $r \le \frac{R}{2}$ and an integer $s \ge 1$.

Notice that $(-\Delta)^{-1/4}(\widetilde{\Omega}_0\widetilde{v}) = c\,|x|^{-1/2} * (\widetilde{\Omega}_0\widetilde{v})$ restricts to an L^∞ function on $B = B(x_0, r)$ bounded by $c\left(\frac{R}{2}\right)^{-1/2}\left\|\widetilde{\Omega}_0\widetilde{v}\right\|_{L^1}$, as $\widetilde{\Omega}_0\widetilde{v}$ is supported far from B, while $(-\Delta)^{-1/4}(Qh) \in L^\infty(\mathbb{R})$ since $|x|^{-1/2} \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ and $h \in L^1 \cap L^\infty(\mathbb{R})$. Moreover, being $|x|^{-1/2} \in L^{2,\infty}(\mathbb{R})$,

$$
\begin{aligned}
&\left\|(-\Delta)^{-1/4}(\widetilde{\Omega}_1\widetilde{v})\right\|_{L^{2,\infty}(B)} \lesssim \left\|\widetilde{\Omega}_1\widetilde{v}\right\|_{L^1(B(x_0,2^s r))} + \sum_{j=s}^{\infty} r^{1/2}\left\||x|^{-1/2} * (\widetilde{\Omega}_1\widetilde{v}\mathbf{1}_{A_j})\right\|_{L^\infty(B)} \\
&\lesssim \left\|\widetilde{\Omega}_1\right\|_{L^{2,1}(B(x_0,2^s r))}\|v\|_{L^{2,\infty}(B(x_0,2^s r))} + \sum_{j=s}^{\infty} 2^{-j/2}\left\|\widetilde{\Omega}_1\widetilde{v}\right\|_{L^1(A_j)} \\
&\lesssim \Big(\|\Omega_1\|_{L^{2,1}(B(x_0,2^s r))} + \|\Omega_0\|_{L^2(B(x_0,R))}\Big)\|v\|_{L^{2,\infty}(B(x_0,2^s r))} + \sum_{j=s}^{\infty} 2^{-j/2}\|v\|_{L^{2,\infty}(A_j)},
\end{aligned}
$$

where we used Theorem C.1 and we neglected $\left\|\widetilde{\Omega}_1\right\|_{L^{2,1}}$ in the estimate of $\left\|\widetilde{\Omega}_1\widetilde{v}\right\|_{L^1(A_j)}$, as well as $\left\|P^N\right\|_{\dot{H}^{1/2}}$ (recall that $A_j = B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)$). Similarly, by Lemmas D.3 and D.5,

$$
\begin{aligned}
&\left\|(-\Delta)^{-1/4}(\widetilde{\Omega}_2 P^N v) + QP^N v\right\|_{L^{2,\infty}(B)} \lesssim \sum_{j=0}^{\infty} 2^{-j/2}\left\|P^N v\right\|_{L^2(B(x_0,2^j r))} \\
&\lesssim \sum_{j=0}^{\infty} 2^{-j/2}\Big(\|w\|_{\dot{H}^{1/2}(B(x_0,2^{j+2}r))} + \|v\|_{L^{2,\infty}(B(x_0,2^{j+2}r))}\Big)\|v\|_{L^{2,\infty}(B(x_0,2^{j+1}r))} \\
&\quad + \sum_{j=0}^{\infty}\sum_{\ell=j+1}^{\infty} 2^{-j/2-(\ell-j)/4}\Big(\|w\|_{\dot{H}^{1/2}(B(x_0,2^{j+2}r))} + \|v\|_{L^{2,\infty}(A_\ell)}\Big)\|v\|_{L^{2,\infty}(A_\ell)} \\
&\lesssim \Big(\|w\|_{\dot{H}^{1/2}(B(x_0,2^{s+1}r))} + \|v\|_{L^{2,\infty}(B(x_0,2^{s+1}r))}\Big)\|v\|_{L^{2,\infty}(B(x_0,2^s r))} \\
&\quad + \sum_{j=s+1}^{\infty} 2^{-j/4}\|v\|_{L^{2,\infty}(B(x_0,2^j r))},
\end{aligned}
$$

where we neglected $\left\|\widetilde{\Omega}_2\right\|_{L^2}$ and $\|v\|_{L^{2,\infty}}, \|w\|_{\dot{H}^{1/2}} \lesssim \|v\|_{L^2}$. Finally, using Lemma D.6 and neglecting $\left\|QP^T\right\|_{\dot{H}^{1/2}}$,

$$
\begin{aligned}
&\left\|(-\Delta)^{-1/4}T(QP^T, v)\right\|_{L^2(B)} \\
&\lesssim \|v\|_{L^{2,\infty}(B(0,2^s))}\sum_{j=s}^{\infty} 2^{s/2-j/4}\left\|QP^T\right\|_{\dot{H}^{1/2}(B(0,2^j r))} + \sum_{j=s}^{\infty} 2^{-j/4}\|v\|_{L^{2,\infty}(A_j)} \\
&\lesssim 2^{s/2}\left(\|\Omega_0\|_{L^2(B(x_0,R))} + \sum_{j=s}^{\infty} 2^{-j/4}\left\|P^T\right\|_{\dot{H}^{1/2}(B(x_0,2^j r))}\right)\|v\|_{L^{2,\infty}(B(x_0,2^s r))} \\
&\quad + \sum_{j=s}^{\infty} 2^{-j/4}\|v\|_{L^{2,\infty}(A_j)}.
\end{aligned}
$$

Combining the previous estimates, given ϵ we can fix R (depending on ϵ and s) so small that

$$
\begin{aligned}
\|v\|_{L^{2,\infty}(B(x_0,r))} &\leq \epsilon\|v\|_{L^{2,\infty}(B(x_0,2^s r))} + C\sum_{j=s+1}^{\infty} 2^{-j/4}\|v\|_{L^{2,\infty}(B(x_0,2^j r))} + Cr^{1/2} + CR^{-1/2}r^{1/2} \\
&\leq \epsilon\|v\|_{L^{2,\infty}(B(x_0,2^s r))} + C\sum_{j=s+1}^{\infty} 2^{-j/4}\|v\|_{L^{2,\infty}(B(x_0,2^j r))} + Cr^{1/4}
\end{aligned}
$$

for all sufficiently small r (depending on ϵ and s), with C independent of ϵ and s. Choosing s large enough, it follows that

$$\|v\|_{L^{2,\infty}(B(x_0,r))} \lesssim r^\beta. \tag{35}$$

for some $0 < \beta < \frac{1}{4}$ and all $r > 0$ small enough (see e.g. [**BRS16**], applied to the sequence $b_0 := \|v\|_{L^{2,\infty}}$, $b_k := \|v\|_{L^{2,\infty}(B(x_0,2^{-k}r_0))}$ for $k > 0$, with r_0 small enough). Hence, being $\|v\|_{L^2}$ finite, this holds for all $r > 0$. Notice that this inequality is uniform in x_0.

Step 3. We define $\zeta := (-\Delta)^{-1/4}(\widetilde{\Omega}_1\widetilde{v}) + (-\Delta)^{-1/4}(\widetilde{\Omega}_2 P^N v) \in L^{2,\infty}(\mathbb{R},\mathbb{R}^m)$ (where $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$, defined above, depend on x_0). From (35) and the preceding estimates we deduce

$$\left\|(-\Delta)^{1/4}\zeta\right\|_{L^1(B(x,r))} \lesssim r^\beta$$

for all $r > 0$ and all $x \in \mathbb{R}$. This Morrey-type estimate for the local L^1-norm of $(-\Delta)^{1/4}\zeta$ implies that $\zeta \in L^p_{loc}(\mathbb{R},\mathbb{R}^m)$ for some $2 < p < \infty$: indeed, arguing as in [**Ada75**], we have $\zeta = c\,|x|^{-1/2} * (-\Delta)^{1/4}\zeta$ and thus, for a.e. $x \in \mathbb{R}$ and all $r > 0$,

$$\begin{aligned}
|\zeta(x)| &\lesssim \sum_{j\in\mathbb{Z}} (2^j r)^{-1/2}\left\|(-\Delta)^{1/4}\zeta\right\|_{L^1(B(x,2^j r)\setminus B(x,2^{j-1}r))} \\
&\lesssim M((-\Delta)^{1/4}\zeta)(x)\sum_{j\le 0} 2^{j/2} r^{1/2} + \sum_{j>0}(2^j r)^{\beta-1/2} \\
&\lesssim r^{1/2} M((-\Delta)^{1/4}\zeta)(x) + r^{\beta-1/2}.
\end{aligned}$$

Optimizing this inequality in r, we infer that

$$|\zeta(x)| \lesssim M((-\Delta)^{1/4}\zeta)(x)^{(\frac{1}{2}-\beta)/(1-\beta)}.$$

for all $x \in \mathbb{R}$. The right-hand side lies in $L^{(1-\beta)/(\frac{1}{2}-\beta),\infty}(\mathbb{R})$ (as $(-\Delta)^{1/4}\zeta \in L^1(\mathbb{R},\mathbb{R}^m)$), so we get the claim for any $2 < p < \frac{1-\beta}{\frac{1}{2}-\beta}$. In particular, we get $\|\zeta\|_{L^2(B(x,r))} \lesssim r^{\beta'}$ for some $\beta' > 0$ and all $0 < r < \frac{R}{2}$. On the other hand,

$$\widetilde{v} - \zeta = (-\Delta)^{-1/4}(\widetilde{\Omega}_0\widetilde{v}) + QP^N v + (-\Delta)^{-1/4} T(QP^T, v) + (-\Delta)^{-1/4}(Qh)$$

and so the estimates derived in Step 2 give $\|\widetilde{v} - \zeta\|_{L^2(B(x_0,r))} \lesssim r^\beta$, for all $0 < r < \frac{R}{2}$. We deduce that, for all $0 < r < \frac{R}{2}$,

$$\|v\|_{L^2(B(x_0,r))} \le \|\widetilde{v} - \zeta\|_{L^2(B(x_0,r))} + \|\zeta\|_{L^2(B(x_0,r))} \lesssim r^\alpha,$$

with $\alpha := \min\{\beta,\beta'\}$. Hence $\|v\|_{L^2(B(x_0,r))} \lesssim r^\alpha$ for all $r > 0$, uniformly in x_0.

Step 4. Finally, Lemma D.2 gives the two identities

$$\begin{aligned}
(-\Delta)^{1/4}(P^T v) &= h - (-\Delta)^{1/4} P^T v + T(P^T, v), \\
\mathcal{R}(-\Delta)^{1/4}(P^N v) &= \mathcal{R}(-\Delta)^{1/4} P^N v - U(P^N, v),
\end{aligned}$$

as $\mathcal{R}(-\Delta)^{1/2} w = -\nabla w$. Arguing as in the proof of Lemma D.6, but using Theorem C.2 in place of Corollary C.4, we finally get

$$\left\|T(P^T,v)\right\|_{L^1(B)} \lesssim \|v\|_{L^2(A_0)} + \sum_{j=1}^\infty \left\|T(P^T, v\mathbf{1}_{A_j})\right\|_{L^1(B)} \lesssim \sum_{j=0}^\infty 2^{-j/2}\|v\|_{L^2(A^j)} \lesssim r^\alpha$$

and similarly $\left\|U(P^N,v)\right\|_{L^1(B)} \lesssim r^\alpha$. The statement follows. □

Corollary D.8. We have $v \in L^p_{loc}(\mathbb{R},\mathbb{R}^m)$ and $w \in C^{0,\gamma}_{loc}(\mathbb{R},\mathbb{R}^m)$, for some $p > 2$ and some $\gamma > 0$.

We include the standard proof for the reader's convenience.

Proof. Arguing as in Step 3 of the proof of Theorem D.7, we infer that

$$\int_{B(x_0,4)} \left|P^T v\right|^p + \int_{B(x_0,4)} \left|\mathcal{R}(P^N v)\right|^p \lesssim 1$$

for some $p > 2$, uniformly in x_0. If $\rho \in C_c^\infty(B(x_0,4))$ is a cut-off function with $\rho = 1$ on $B(x_0,2)$,

$$P^N v = -\mathcal{R}\mathcal{R}(P^N v) = -\mathcal{R}(\rho\mathcal{R}(P^N v)) - \mathcal{R}((1-\rho)\mathcal{R}(P^N v)).$$

Using [**Gra14C**] applied to $-i\,\mathrm{sgn}(\xi)$ (whose inverse Fourier transform is (-1)-homogeneous) and the fact that $(1-\rho)\mathcal{R}(P^N v) \in L^2(\mathbb{R},\mathbb{R}^m)$ is supported far from $B(x_0,1)$,

$$\left\|\mathcal{R}((1-\rho)\mathcal{R}(P^N v))\right\|_{L^\infty(B(x_0,1))} \lesssim 1$$

and, from the L^p-boundedness of the Hilbert–Riesz transform,

$$\left\|\mathcal{R}(\rho\mathcal{R}(P^N v))\right\|_{L^p} \lesssim \left\|\mathcal{R}(P^N v)\right\|_{L^p(B(x_0,4))}.$$

We deduce that $v = P^T v + P^N v$ also satisfies an estimate $\int_{B(x_0,1)} |v|^p \lesssim 1$, uniformly in x_0. In particular, $\left\|(-\Delta)^{1/4} w\right\|_{L^2(B(x_0,r))} \lesssim r^\gamma$ with $\gamma = \frac{1}{2} - \frac{1}{p} \in \left(0, \frac{1}{2}\right)$ (for $0 < r < 1$ and hence for all $r > 0$). Using Lemma D.4 we deduce that

$$\left(\fint_{B(x_0,r)} \left|w - (w)_{B(x_0,r)}\right|^2\right)^{1/2} \lesssim \sum_{j=0}^\infty 2^{-j/2}(2^j r)^\gamma \lesssim r^\gamma.$$

This is the integral characterization of Hölder continuity with exponent γ: see e.g. [**Gia83**]. □

Applying a rotation before taking the stereographic projection, we arrive at the following.

Corollary D.9. The map $u \circ \psi_\ell^{-1} : \mathcal{S}^1 \to \mathbb{R}^m$ is Hölder continuous and, being ℓ is arbitrary, u is Hölder continuous.

Appendix E. Higher regularity of $\frac{1}{2}$-harmonic maps

In this section we prove that $\frac{1}{2}$-harmonic maps $u \in H^{1/2}(\partial S, \mathcal{N})$ are $C_{loc}^{k-1,\delta}$, for any $0 < \delta < 1$, whenever $\mathcal{N}$ is a C^k-smooth closed manifold ($k \geq 2$). We mention that higher regularity of the so-called *half-wave* maps into S^2 has recently been obtained in [**LS17**].

Throughout the section, we will say that $a \in \mathcal{S}'(\mathbb{R})$ belongs to $H_{loc}^{s,p}(\mathbb{R})$ (with $s \geq 0$, $1 < p < \infty$) if $\psi a \in H^{s,p}(\mathbb{R})$ for any $\psi \in C_c^\infty(\mathbb{R})$.

Corollary D.8 shows that $(-\Delta)^{1/4} w \in L_{loc}^p(\mathbb{R},\mathbb{R}^m)$, where $w = u \circ \psi_\ell^{-1} \circ \Pi^{-1}$, for some $p > 2$. We now bootstrap this information to get higher regularity. We first prove two results concerning the regularity of the commutator $\mathcal{R}(ab) - a\mathcal{R}(b)$. The proofs will rely on the technique of splitting products into *paraproducts*, using the Littlewood–Paley decomposition (see Section B.2):

$$ab = \sum_j a_j b^{j-3} + \sum_j a^{j-3} b_j + \sum_{|j-k|\leq 2} a_j b_k, \qquad \widehat{a}_j = \varrho_j \widehat{a}, \quad \widehat{b}_j = \varrho_j \widehat{b}.$$

We will treat the first and third summands together, namely we will just decompose

$$ab = \sum_j a_j b^{j+2} + \sum_j a^{j-3} b_j.$$

Lemma E.1. Let $a \in \dot{H}^{1/2} \cap L^\infty(\mathbb{R})$ with $(-\Delta)^{1/4}a \in L^p(\mathbb{R})$, for some $2 < p < \infty$, and $b \in \dot{H}^{-1/2}(\mathbb{R})$. Then

$$\|\mathcal{R}(ab) - a\mathcal{R}(b)\|_{L^{2p/(p+2)}} \lesssim \left\|(-\Delta)^{1/4}a\right\|_{L^p} \|b\|_{\dot{H}^{-1/2}}.$$

Notice that $\mathcal{R}(ab)$ is defined, as for $\rho \in \mathcal{S}(\mathbb{R})$ the function $\mathcal{F}\left[-i\frac{\xi}{(\epsilon^2+|\xi|^2)^{1/2}}\check{\rho}\right]$ converges (as $\epsilon \to 0$) in $\dot{H}^{1/2} \cap L^\infty(\mathbb{R})$ and ab extends to a continuous functional on this space (see Appendix B).

Proof. Notice that the commutator vanishes if a is constant. Thus, as the proof of Lemma B.2 and Remark B.3 show, we can assume $\widehat{a}, \widehat{b} \in C_c^\infty(\mathbb{R} \setminus \{0\})$.[7] Using the homogeneous decomposition, we write

$$\mathcal{R}(ab) - a\mathcal{R}(b) = \sum_{j\in\mathbb{Z}}(\mathcal{R}(a^{j-3}b_j) - a^{j-3}\mathcal{R}(b_j)) + \sum_{j\in\mathbb{Z}}(\mathcal{R}(a_j b^{j+2}) - a_j\mathcal{R}(b^{j+2}))$$

and remark that the first sum vanishes since

$$\begin{aligned}\mathcal{F}(\mathcal{R}(a^{j-3}b_j) - a^{j-3}\mathcal{R}(b_j))(\xi) = &- i\,\mathrm{sgn}(\xi)\int \widehat{a^{j-3}}(\xi-\eta)\widehat{b_j}(\eta)\,d\eta \\ &+ i\int \widehat{a^{j-3}}(\xi-\eta)\,\mathrm{sgn}(\eta)\widehat{b_j}(\eta)\,d\eta = 0\end{aligned}$$

(as $\mathrm{sgn}(\eta) = \mathrm{sgn}(\xi)$ whenever $\widehat{a^{j-3}}(\xi-\eta)\widehat{b_j}(\eta) \neq 0$).

Since $\mathcal{R}$ is an isomorphism of $L^{2p/(p+2)}(\mathbb{R})$ and of $\dot{H}^{-1/2}(\mathbb{R})$, it suffices to bound $\sum_{j\in\mathbb{Z}} a_j b^{j+2}$ in $L^{2p/(p+2)}(\mathbb{R})$. We do this by duality: let $h \in \mathcal{S}(\mathbb{R})$ and observe that

$$\int \sum_{j\in\mathbb{Z}} a_j b^{j+2} h = \int \sum_{j\in\mathbb{Z}} a_j b^{j+2} h^{j+4} \lesssim \int \left(\sum_{j\in\mathbb{Z}} 2^j |a_j|^2\right)^{1/2} \left(\sum_{j\in\mathbb{Z}} 2^{-j}\left|b^{j+2}\right|^2\right)^{1/2} (Mh),$$

as $\mathcal{F}(a_j b^{j+2})$ is supported in $B(0, 2^{j+4})$ and as we have the elementary inequality $\left|h^{j+4}\right| \lesssim Mh$. Note that $\|Mh\|_{L^{2p/(p-2)}} \lesssim \|h\|_{L^{2p/(p-2)}}$, while the $\ell^2(\mathbb{Z})$-norm $\left(\sum_{j\in\mathbb{Z}}\left|2^{-j/2}b^{j+2}\right|^2\right)^{1/2}$ equals

$$\left(\sum_{j\in\mathbb{Z}}\left|\sum_{k=-\infty}^{2} 2^{-j/2} b_{j+k}\right|^2\right)^{1/2} \leq \sum_{k=-\infty}^{2}\left(\sum_{j\in\mathbb{Z}}\left|2^{-j/2}b_{j+k}\right|^2\right)^{1/2} \lesssim \left(\sum_{j\in\mathbb{Z}} 2^{-j}|b_j|^2\right)^{1/2}$$

(as $\sum_{j\in\mathbb{Z}}\left|2^{-j/2}b_{j+k}\right|^2 = 2^k\sum_{j\in\mathbb{Z}} 2^{-j}|b_j|^2$), so that, by Plancherel's identity,

$$\left\|\left(\sum_{j\in\mathbb{Z}} 2^{-j}\left|b^{j+2}\right|^2\right)^{1/2}\right\|_{L^2}^2 \lesssim \sum_{j\in\mathbb{Z}} 2^{-j}\int |b_j|^2 = \int \sum_{j\in\mathbb{Z}} 2^{-j}\rho_j^2\left|\widehat{b}\right|^2 \lesssim \int |\xi|^{-1}\left|\widehat{b}(\xi)\right|^2 = \|b\|_{\dot{H}^{-1/2}}^2.$$

To conclude, using [**Tri83**] with the multipliers $2^{j/2}|\xi|^{-1/2}(\varrho_{j-1} + \varrho_j + \varrho_{j+1})$ and [**Gra14C**], we infer

$$\left\|\left(\sum_{j\in\mathbb{Z}} 2^j|a_j|^2\right)^{1/2}\right\|_{L^p} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}}\left|(-\Delta)^{1/4}a_j\right|^2\right)^{1/2}\right\|_{L^p} \lesssim \left\|(-\Delta)^{1/4}a\right\|_{L^p}.$$

[7] We can assume $\widehat{a}$ has compact support in $\mathbb{R} \setminus \{0\}$, by replacing it with $\check{w}_k$ (defined as in Lemma B.2): the norm $\left\|(-\Delta)^{1/4}a\right\|_{L^p}$ stays controlled by Lemma B.4 and the same argument of Remark B.3; we can then choose $|\xi|^{1/2}\widehat{v_k}$ arbitrarily close to $|\xi|^{1/2}w_k$ in $L^{p'}(\mathbb{R})$, obtaining $(-\Delta)^{1/4}v_k$ close to $(-\Delta)^{1/4}\check{w}_k$ in $L^p(\mathbb{R})$.

To sum up, by Hölder's inequality we get the desired bound

$$\int \sum_{j\in\mathbb{Z}} a_j b^{j+2} h \lesssim \left\|(-\Delta)^{1/4} a\right\|_{L^p} \|b\|_{\dot{H}^{-1/2}} \|h\|_{L^{2p/(p-2)}} . \qquad \square$$

Lemma E.2. Let $a \in H^{s,p}(\mathbb{R})$ and $b \in L^q(\mathbb{R})$, with $s > \frac{1}{p}$, $1 < p, q < \infty$. Then, for any $\gamma > \frac{1}{p}$,

$$\|\mathcal{R}(ab) - a\mathcal{R}(b)\|_{H^{s-\gamma,q}} \lesssim \|a\|_{H^{s,p}} \|b\|_{L^q} .$$

Proof. We can assume $\widehat{a}, \widehat{b} \in C_c^\infty(\mathbb{R})$. We use the inhomogeneous Littlewood–Paley decomposition, so that $a = \sum_{j\geq 0} a_j$ and $b = \sum_{j\geq 0} b_j$, where $\widehat{a}_j = \varrho_j \widehat{a}$.

As in the previous proof, we need only estimate $\left\|\sum_{j\geq 0} a_j b^{j+2}\right\|_{H^{s-\gamma,q}}$, as $\mathcal{R}$ is an isomorphism of $H^{s-\gamma,q}(\mathbb{R})$ and of $L^q(\mathbb{R})$. We have $\|a_j\|_{L^\infty} \lesssim 2^{-j(s-1/p)} \|a\|_{H^{s,p}}$ (see the proof of Corollary B.7). Given $h \in \mathcal{S}(\mathbb{R})$, observe that $\mathcal{F}(a_j b^{j+2})$ vanishes outside $B(0, 2^{j+4})$, so

$$\left|\int \sum_{j\geq 0} a_j b^{j+2} h\right| = \left|\sum_{j\geq 0} \int a_j b^{j+2} h^{j+4}\right| \lesssim \|a\|_{H^{s,p}} \|Mb\|_{L^q} \|h\|_{F^{-(s-1/p)}_{q',1}} ,$$

thanks to the pointwise inequalities $\left|b^{j+2}\right| \lesssim Mb$ (Mb being the maximal function of b) and

$$\left|\sum_{j\geq 0} a_j b^{j+2} h^{j+4}\right| \lesssim \sum_{j\geq 0} 2^{-j(s-1/p)} \|a\|_{H^{s,p}} (Mb) \left|h^{j+4}\right| \lesssim \|a\|_{H^{s,p}} (Mb) \sum_{j\geq 0} 2^{-j(s-1/p)} |h_j| .$$

But $H^{-(s-\gamma),q'} = F^{-(s-\gamma)}_{q',2} \hookrightarrow F^{-(s-1/p)}_{q',1}$ (see [**Tri83**]), so

$$\left|\int \sum_{j\geq 0} a_j b^{j+2} h\right| \lesssim \|a\|_{H^{s,p}} \|b\|_{L^q} \|h\|_{H^{-(s-\gamma),q'}} . \qquad \square$$

We will implicitly use many times the following result.

Lemma E.3. If $u \in H^{s,p}_{loc}(\mathbb{R})$ for some $s \geq 1$ and $1 < p < \infty$, then $P^T(u) \in H^{\min(s,k-1),p}_{loc}(\mathbb{R})$.

Proof. We can assume that $1 \leq s \leq k$. The claim is trivial for $s \in \mathbb{N}$, while when $s > 1$ is not an integer it follows from [**BM01**], the map P^T being C^{k-1}-smooth. Notice that $u \in W^{1,sp}_{loc}(\mathbb{R})$ by [**BM01**] with $(p, q, s) := (sp, 2, 1)$, $(p_1, q_1, s_1) := (p, 2, s)$, $(p_2, q_2, s_2) := (\infty, \infty, 0)$ and the fact that $u \in H^{1,p}_{loc}(\mathbb{R}) \subseteq L^\infty_{loc}(\mathbb{R})$. $\square$

We also need the following lemmata, where we use the dyadic partition of unity $(\varrho_j)_{j=0}^\infty \subseteq C_c^\infty(\mathbb{R})$ introduced in Appendix B.

Lemma E.4. If $f \in \dot{H}^{1/2}(\mathbb{R})$ and $\rho \in C_c^\infty(B(0,1))$, we have

$$\left\langle (-\Delta)^{1/2} f, \rho\right\rangle = \sum_{j=0}^\infty \int ((-\Delta)^{1/2}\rho)(\varrho_j f). \tag{36}$$

Proof. Notice that $f \in L^1_{loc}(\mathbb{R})$ and $(-\Delta)^{1/2}\rho \in L^\infty(\mathbb{R})$, so each term in the right-hand side makes sense. By the remark after Lemma B.4, the left-hand side equals $2\pi \int |\xi|\, \widehat{f}(\xi)\overline{\widehat{\rho}(\xi)}\, d\xi$.

For any $j \geq 2$, by Lemma B.1 and the fact that $\mathcal{F}^{-1}(|\xi|) \in C^\infty(\mathbb{R} \setminus \{0\})$ is homogeneous of degree -2 (see [**Gra14C**]),

$$\left|\int ((-\Delta)^{1/2}\rho)(\varrho_j f)\right| \lesssim 2^{-2j} \|f\|_{L^1(B(0,2^{j+1})\setminus B(0,2^{j-1}))} \lesssim 2^{-j} \|f\|_{L^1(B(0,1))} + 2^{-j}(j+1) \|f\|_{\dot H^{1/2}} .$$

Therefore the sum in the right-hand side of (36) converges and is bounded by $\|f\|_{L^1(B(0,1))} + \|f\|_{\dot H^{1/2}}$. Hence, by Lemma B.2, it is enough to prove (36) on $\mathcal{S}(\mathbb{R}) + \mathbb{R}$.

If $f \in \mathcal{S}(\mathbb{R})$, the identity is trivially satisfied since in this case we have

$$\sum_{j=0}^{\infty} \int ((-\Delta)^{1/2}\rho)(\varrho_j f) = \int ((-\Delta)^{1/2}\rho) f = 2\pi \int |\xi|\, \widehat{f}(\xi)\overline{\widehat{\rho}(\xi)}\, d\xi.$$

If $f = c$ is constant then

$$\left\langle (-\Delta)^{1/2} c, \rho\right\rangle = 0 = 2\pi c \lim_{N\to\infty} \int |\xi|\, \widehat{\rho}(\xi) \sum_{j=0}^{N} \overline{\widehat{\varrho_j}(\xi)}\, d\xi = \sum_{j=0}^{\infty} \int ((-\Delta)^{1/2}\rho)(\varrho_j c),$$

the second equality being true since $\sum_{j=0}^N \overline{\widehat{\varrho_j}(\xi)}$ approximates the Dirac mass δ_0 as $N \to \infty$. □

Lemma E.5. Assume $w \in \dot H^{1/2}(\mathbb{R})$ is supported outside $B(x_0, 2)$, for some $x_0 \in \mathbb{R}$. Then the distribution $(-\Delta)^{1/2} w$ restricts to a C^∞ function on $B(x_0, 1)$.

Proof. We can assume $x_0 = 0$. For $\rho \in C^\infty_c(B(0,1))$ and $k \geq 0$ integer, Lemma E.4 gives

$$\begin{aligned}\left\langle (-\Delta)^{1/2} w, (-\Delta)^k \rho\right\rangle &= \sum_{j\geq 1} \int ((-\Delta)^{k+1/2}\rho)(\varrho_j w)\\ &\lesssim \sum_{j\geq 1} 2^{-(2k+2)j} \|\rho\|_{L^1} \|w\|_{L^1(B(0,2^{j+1})\setminus B(0,2^{j-1}))} \lesssim \sum_{j\geq 1} 2^{-(2k+2)j} \|\rho\|_{L^2} \cdot (j+1)2^j \lesssim \|\rho\|_{L^2} ,\end{aligned}$$

where the inequalities follow from [**Gra14C**] and Lemma B.1. So, calling f the restriction of $(-\Delta)^{1/2} w$ to $B(0,1)$, we have $(-\Delta)^k f \in L^2(B(0,1))$. Equivalently, $\frac{d^{2k} f}{dx^{2k}} \in L^2(B(0,1))$. This implies that $\frac{d^{2k-1} f}{dx^{2k-1}} \in C^0(B(0,1))$ for all $k \geq 0$, hence $f \in C^\infty(B(0,1))$. □

Proof of Theorem 1.4. We fix $x_0 \in \mathbb{R}$ and we take a cut-off function $\eta \in C^\infty_c(B_1(x_0))$ satisfying $\eta = 1$ in a neighborhood of x_0. Recall from Lemma D.2 that

$$P^T(w)(-\Delta)^{1/2} w = h, \quad P^N(w)\nabla w = 0,$$

with $h = -\frac{2}{1+x^2} P^T(w) \left(R_\ell((f_j)_{j=1}^k) \circ \Pi^{-1}\right) \in L^1 \cap L^\infty(\mathbb{R})$. Therefore we have

$$\begin{aligned}\eta\nabla w =& \eta P^T(w)\nabla w = -\eta P^T(w)\mathcal{R}(-\Delta)^{1/2} w\\ =& \mathcal{R}(\eta P^T(w)(-\Delta)^{1/2} w) - \eta P^T(w)\mathcal{R}(-\Delta)^{1/2} w - \mathcal{R}(\eta h).\end{aligned} \tag{37}$$

We remark that $w \in H^{1/2,p}_{loc}(\mathbb{R}, \mathbb{R}^m)$: by Lemma B.5

$$\left|(-\Delta)^{1/4}(\psi w) - \psi(-\Delta)^{1/4} w\right|(x) \lesssim \int \frac{|\psi(x) - \psi(y)|}{|x-y|^{3/2}} |w(y)|\, dy \lesssim g * |w|\,(x) \in L^\infty(\mathbb{R})$$

with $g(x) := \min(|x|^{-1/2}, |x|^{-3/2}) \in L^1(\mathbb{R})$, for any $\psi \in C_c^\infty(\mathbb{R})$. Hence $(-\Delta)^{1/4}(\psi w)$ lies both in $L^2(\mathbb{R})$ and in $L^p(\mathbb{R}) + L^\infty(\mathbb{R})$ and thus it lies in $L^p(\mathbb{R})$, as well (which can be checked using the formula $\|f\|_{L^r}^r = \int_0^\infty r\lambda^{r-1}\mathcal{L}^1(\{|f| > \lambda\})\, d\lambda$ for $1 \le r < \infty$). Since trivially $\psi w \in L^p(\mathbb{R})$, [**Tri83**] gives $\psi w \in F^{1/2}_{p,2}(\mathbb{R}) = H^{1/2,p}(\mathbb{R})$.

Thus $\eta P^T(w) \in H^{1/2,p}(\mathbb{R})$ and, using again [**Tri83**] (with multipliers $|\xi|^{1/2}(1+|\xi|^2)^{-1/4}(\varrho_{j-1} + \varrho_j + \varrho_{j+1})$ for $j \in \mathbb{Z}$) and [**Gra14C**], we infer that $\eta P^T(w)$ and $(-\Delta)^{1/2}w$ satisfy the hypotheses of Lemma E.1. So, in view of (37), we get $\eta\nabla w \in L^{2p/(p+2)}(\mathbb{R})$, i.e. $w \in H^{1,\widetilde{p}}_{loc}(\mathbb{R})$ with $\widetilde{p} = 2p/(p+2)$.

We now fix another cut-off function $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi = 1$ on $B(x_0, 2)$ and we set

$$w_1 := \phi w, \quad w_2 := (1-\phi)w.$$

Lemma E.5 yields that $(-\Delta)^{1/2}w_2 \in C^\infty(B(x_0,1))$. Now assume that we already know $w \in H^{s,\widetilde{p}}_{loc}(\mathbb{R})$ for some real $s \ge 1$: by Lemma E.3 we get $h \in H^{\min(s,k-1),\widetilde{p}}_{loc}(\mathbb{R})$, so $\widetilde{h} := P^T(w)(-\Delta)^{1/2}w_1 = -P^T(w)(-\Delta)^{1/2}w_2 + h$ restricts to a function in $H^{\min(s,k-1),\widetilde{p}}_{loc}(B(x_0,1))$. We rewrite (37) as

$$\begin{aligned}\eta\nabla w =& \eta P^T(w)\nabla w_1 = -\eta P^T(w)\mathcal{R}(-\Delta)^{1/2}w_1 \\ =& \mathcal{R}(\eta P^T(w)(-\Delta)^{1/2}w_1) - \eta P^T(w)\mathcal{R}(-\Delta)^{1/2}w_1 - \mathcal{R}(\eta\widetilde{h}).\end{aligned}$$

The commutator on the right-hand side belongs to $H^{\min(s,k-1)-\gamma,\widetilde{p}}(\mathbb{R})$, for any $\gamma > \frac{1}{\widetilde{p}}$, thanks to Lemma E.2 (applied with $p = q := \widetilde{p}$). Therefore $\eta\nabla w \in H^{\min(s,k-1)-\gamma,\widetilde{p}}(\mathbb{R})$, which implies $w \in H^{\min(s+1,k)-\gamma,\widetilde{p}}_{loc}(\mathbb{R})$. Iterating this procedure we eventually get

$$w \in \bigcap_{\gamma > 1/\widetilde{p}} H^{k-\gamma,\widetilde{p}}_{loc}(\mathbb{R}).$$

We now show that, for any fixed $1 < p < \infty$,

$$w \in \bigcap_{\gamma > 1/p} H^{k-\gamma,p}_{loc}(\mathbb{R}).$$

Since $k \ge 2$, we know that $w \in H^{1,q}_{loc}(\mathbb{R})$ for all $q < \frac{\widetilde{p}}{2-\widetilde{p}}$ (because $H^{2-\gamma,\widetilde{p}}_{loc}(\mathbb{R}) \subseteq H^{1,q}_{loc}(\mathbb{R})$ with $\frac{1}{q} = \gamma + \frac{1}{\widetilde{p}} - 1$ whenever $\gamma < 1$, see [**Tri83**]). Proceeding as above we obtain

$$w \in \bigcap_{\gamma > 1/q} H^{k-\gamma,q}_{loc}(\mathbb{R})$$

for all $q < \frac{\widetilde{p}}{2-\widetilde{p}}$ (notice that $\frac{\widetilde{p}}{2-\widetilde{p}} > \widetilde{p}$). Iterating this with q in place of $\widetilde{p}$, we will eventually reach an exponent $q \ge 2$ and hence, as $\bigcap_{\gamma>1/q} H^{2-\gamma,q}_{loc}(\mathbb{R}) \subseteq \bigcap_{1<r<\infty} H^{1,r}_{loc}(\mathbb{R})$, all exponents in $(1,\infty)$. This proves the assertion. Finally, applying Corollary B.7,

$$w \in \bigcap_{1<p<\infty} \bigcap_{\epsilon>0} H^{k-1/p-\epsilon,p}_{loc}(\mathbb{R}) \subseteq \bigcap_{0<\delta<1} C^{k-1,\delta}_{loc}(\mathbb{R}).$$

So $u \in C^{k-1,\delta}(\partial S)$ for all $0 < \delta < 1$. In particular, if $\mathcal{N}$ is C^∞-smooth then u is C^∞ as well. □