MICHAEL–SIMON INEQUALITY FOR ANISOTROPIC ENERGIES CLOSE TO THE AREA VIA MULTILINEAR KAKEYA-TYPE BOUNDS

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ABSTRACT. Given an anisotropic integrand $F : \mathrm{Gr}_k(\mathbb{R}^n) \to (0, \infty)$, we can generalize the classical isotropic area by looking at the functional

$$\mathcal{F}(\Sigma^k) := \int_\Sigma F(T_x\Sigma)\, d\mathcal{H}^k.$$

While a monotonicity formula is not available for critical points [4], when $k = 2$ and $n = 3$ we show that the Michael–Simon inequality holds if F is convex and close to 1 (in C^1), meaning that $\mathcal{F}$ is close to the usual area.

Our argument is partly based on some key ideas of Almgren, who proved this result in an unpublished manuscript, but we largely simplify his original proof by showing a new functional inequality for vector fields on the plane, which can be seen as a quantitative version of Alberti's rank-one theorem.

As another byproduct, we also show Michael–Simon for another class of integrands which includes the ℓ^p norms for $p \in (1, \infty)$. For a general F satisfying the atomic condition [8], we also show that the validity of Michael–Simon is equivalent to compactness of rectifiable varifolds.

1. Introduction

1.1. **Setting and main result.** Geometric measure theory and, more broadly, a large part of calculus of variations and geometric analysis deal with the classical isotropic area. The study of its critical points $\Sigma^k \subset \mathbb{R}^n$ makes extensive use of the *monotonicity formula*, asserting that for all $p \in \mathbb{R}^n$ we have

$$\frac{\mathcal{H}^k(\Sigma \cap B_r(p))}{r^k} \le \frac{\mathcal{H}^k(\Sigma \cap B_s(p))}{s^k} \quad \text{for } 0 < r < s.$$

This fact has a number of useful consequences, best phrased in terms of varifolds (see, e.g., [14, Chapters 4 and 8]): among the fundamental ones, existence and upper semicontinuity of the density for stationary varifolds, upper semicontinuity of their support under varifold convergence, compactness of rectifiable and integral varifolds (either stationary or with local uniform bounds on the first variation), and existence and conical symmetry of blow-ups, the latter following from a more precise version of monotonicity.

There is a natural anisotropic generalization of the area functional, given by taking an integrand $F : \mathrm{Gr}_k(\mathbb{R}^n) \to (0, \infty)$ of class C^1 and defining

$$\mathcal{F}(\Sigma) := \int_\Sigma F(T_x\Sigma)\, d\mathcal{H}^k(x)$$

for a smoothly embedded $\Sigma^k \subset \mathbb{R}^n$, extending the definition to k-varifolds in the obvious way. In the anisotropic setting, it is known [4] that monotonicity fails (in the sense that if $\mathcal{F}$ satisfies an identity resembling too closely the quantitative version of monotonicity, then $\mathcal{F}$ is the isotropic area up to a linear change of coordinates). Thus, many basic tools break down at this level of generality.

However, a weaker (though arguably more robust) fact is believed to hold for appropriate classes of integrands F. Specifically, for $n \ge 3$ and $k \in \{2, \ldots, n-1\}$, the *Michael–Simon* *inequality* (first proved in [13] for the isotropic area, as a consequence of monotonicity) is conjectured to hold for appropriate F, leading to the following question.

Question 1.1. For which F does it hold that, given a rectifiable k-varifold V in $\mathbb{R}^n$ with $\Theta^k(|V|,x) \geq \theta_0 > 0$ for $|V|$-a.e. x, as well as finite total mass and first variation, i.e., $|V|(\mathbb{R}^n), |\delta^F V|(\mathbb{R}^n) < \infty$, we have

$$|V|(\mathbb{R}^n)^{k-1} \leq \frac{C(n,k,F)}{\theta_0}|\delta^F V|(\mathbb{R}^n)^k \ ? \tag{1}$$

While the range of applications of this bound would be far more limited compared to a monotonicity formula (e.g., it would not give upper density bounds or conicality of blow-ups), it would still have a number of key consequences, such as compactness of rectifiable and integral varifolds and, when the first variation is in L^p with $p > k$, a lower density bound of the form

$$|V|(B_r(p)) \geq cr^k \quad \text{for } p \in \operatorname{spt}(|V|),\ r < 1$$

(see (5) below) and upper semicontinuity of the support along converging sequences of varifolds.

Remark 1.2. It is clear that, by scaling and normalization of density, (1) is equivalent to its validity when $\theta_0 = 1$ and $|V|(\mathbb{R}^n) = 1$. Moreover, it is equivalent to the *functional* version of Michael–Simon (see Proposition 2.2 below): for any $f \in C^1_c(\mathbb{R}^n)$ we have

$$\Big[\int_{\mathbb{R}^n} |f|^{k/(k-1)}\, d|V|\Big]^{(k-1)/k} \leq \frac{C'(n,k,F)}{\theta_0^{1/k}}\Big[\int_{\mathbb{R}^n} |df|\, d|V| + \int_{\mathbb{R}^n} |f|\, d|\delta^F V|\Big]. \tag{2}$$

In codimension one (when $k = n-1$), conjecturally the appropriate assumption should be *strict convexity* of F, by virtue of the fact that it is equivalent to the atomic condition (AC), and in turn to have rectifiability under the assumption $\Theta^k(|V|,\cdot) > 0$ a.e. [8]. More precisely, once we identify $\mathrm{Gr}_{n-1}(\mathbb{R}^n)$ with $\mathbb{RP}^{n-1}$ (i.e., a hyperplane with its unit normal $\pm\nu$) and $F : \mathrm{Gr}_{n-1}(\mathbb{R}^n) \to (0,\infty)$ with an even function $F : \mathbb{S}^{n-1} \to (0,\infty)$, we extend the latter to a 1-homogeneous function $F : \mathbb{R}^n \to [0,\infty)$, which is then required to be convex, and actually strictly convex along lines not passing through 0.

One of the main results of the present paper is to answer Question 1.1 affirmatively for surfaces in $\mathbb{R}^3$, when F is convex and close enough to the area. This result was first proved in an unpublished manuscript of Almgren, from which we borrowed several key ideas, although we bypass a number of technical steps from his proof by leveraging some new functional inequalities presented below.

Theorem 1.3. If $n = 3$, $k = 2$, and F is a convex integrand close enough to the isotropic area in the C^1 topology (i.e., $\|F|_{\mathbb{S}^2} - 1\|_{C^1}$ is small enough), then the following holds. Given a rectifiable 2-varifold V in $\mathbb{R}^3$ with finite total mass and first variation, letting $\theta(x) := \Theta^2_*(|V|,x)$, we have the scale-invariant bound

$$|V|(\mathbb{R}^3) \leq C(F)\mathcal{H}^2(\{\theta > 0\})^{1/2} \cdot |\delta^F V|(\mathbb{R}^3). \tag{3}$$

In particular, (1) holds for a possibly different constant $C(F)$.

Note that the second conclusion follows from the first one, since we can bound the measure $\mathcal{H}^2(\{\theta > 0\}) = \mathcal{H}^2(\{\theta \geq \theta_0\})$ from above by $\theta_0^{-1}|V|(\mathbb{R}^3)$. It should be noted that the proof is not perturbative: in fact, it proceeds by singling out an explicit set of integrands F (open in the C^1 topology for $F|_{\mathbb{S}^2}$) which happens to contain the isotropic area. This result immediately implies the first assumption in the regularity lemma from [5, p. 25]. Hence, we can apply Allard's regularity theorem in the anisotropic setting [5, p. 27] and obtain the following.

Theorem 1.4. If $n = 3$, $k = 2$, and F is a convex smooth integrand close enough to the isotropic area in the C^1 topology, then the following holds for some universal $\varepsilon > 0$. Given an integer-rectifiable 2-varifold V in $B_{2r}(x_0) \subset \mathbb{R}^3$ and $x_0 \in \operatorname{spt}(|V|)$, if for some $\nu_0 \in \mathbb{S}^2$ we have

$$\frac{|V|(B_{2r}(x_0))}{\pi(2r)^2} \in \Big[\frac{1}{2}, \frac{3}{2}\Big],$$

as well as

$$|\delta^F V| \le \Lambda |V|$$

for some $\Lambda \in (0, \varepsilon/r)$, and

$$\int_{B_{2r}(x_0)} \langle x - x_0, \nu_0\rangle^2 \, d|V|(x) \le \varepsilon r^4,$$

then in the ball $B_r(x_0)$ the varifold V agrees with a $C^{1,\alpha}$ graph of multiplicity 1 over the plane $\nu_0^\perp$, for any $\alpha \in (0,1)$ (with a scale-invariant $C^{1,\alpha}$ bound, vanishing as $\varepsilon \to 0$).

Moreover, for the same dimensions, using similar ideas (but a simpler tool, namely Theorem 1.6 in place of Theorem 1.7), we also obtain the following. Note that the technical condition in its statement holds for any ℓ^p norm with $p \in (1, \infty)$, namely if

$$F(\nu) = (|\nu^x|^p + |\nu^y|^p + |\nu^z|^p)^{1/p}.$$

Theorem 1.5. Assume that $n = 3$, $k = 2$, and $F : \mathbb{R}^3 \to [0, \infty)$ is a strictly convex integrand and even in each component. Moreover, for any coordinate plane P, assume that $\frac{\pi_P(\nabla F(\nu))}{|\pi_P(\nabla F(\nu))|}$ depends only on $\frac{\pi_P(\nu)}{|\pi_P(\nu)|}$, for all $\nu \in \mathbb{S}^2 \setminus \pi_P^{-1}(0)$ (note that $\pi_P(\nabla F(\nu)) \neq 0$ for such ν). Then the bound (3) holds true.

1.2. **New variants of multilinear Kakeya in dimension two.** Almgren's argument is quite technical in that it involves studying the lengths of carefully chosen pieces of curves (corresponding to vector fields with summable divergence appearing as rows of the first variation matrix), consisting of points where a certain projected density is large enough and such vector fields are transverse enough. Our argument borrows some of his key ideas, such as Definition 3.8, and bakes them directly into a new standalone functional inequality (see (4)), from which Theorem 1.3 follows quite directly.

Before stating this key inequality, let us mention that one of the starting points in its proof was the following simpler version.

Theorem 1.6. Given $S, T \in W^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$, assume that $S^x, T^y \ge 0$ and $\det(S, T) \ge 0$ a.e. Then

$$\int_{\mathbb{R}^2} \det(S, T) \le \frac{1}{4}\Big(\int_{\mathbb{R}^2} |\operatorname{div} S|\Big)\Big(\int_{\mathbb{R}^2} |\operatorname{div} T|\Big).$$

In fact, this result was already obtained in [10], where other nonlinear constraints are studied. We will present nonetheless a short proof of it because, besides recovering the sharp constant, our proof uses (in spirit) Smirnov's decomposition of normal 1-currents as superpositions of curves [15], which may be thought as the intuitive reason why this holds (as discussed in Section 3), and the same technique can be used to prove the following instrumental bound, where crucially we drop the assumption $\det(S, T) \ge 0$.

Theorem 1.7. Given two vector fields $S, T \in W^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$, we define

$$S^P := (S^x - |S^y|)^+, \quad S^N := (S^x - |S^y|)^-,$$
$$T^P := (T^y - |T^x|)^+, \quad T^N := (T^y - |T^x|)^-.$$

Also, let $\chi:\mathbb{R}^2\to[0,1]$ be a Borel function supported in a bounded set. Assuming $S^x,T^y\ge 0$ then, for some universal constant $C>0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\chi\min\{S^P,T^P\}\le C\|\chi\|_{L^2}\Big[\int_{\mathbb{R}^2}(|S|+|\operatorname{div}S|)\Big]^{1/2}\Big[\int_{\mathbb{R}^2}(|T|+|\operatorname{div}T|)\Big]^{1/2}\\ +\int_{\mathbb{R}^2}(C|\operatorname{div}S|+C|\operatorname{div}T|+S^N+T^N).\end{aligned}\tag{4}$$

The same holds if $S^x,S^y,T^x,T^y,\operatorname{div}S,\operatorname{div}T$ are just real-valued measures on the plane with finite total variation (provided that $S^x,T^y\ge 0$).

Besides its own interest, this inequality can be viewed as a quantitative version of previous existing results, such as [2, Proposition 8.6] or Alberti's rank-one theorem, proved initially in [1] and re-obtained with different techniques in [9, 12].

Corollary 1.8. Given $u\in BV(\mathbb{R}^n,\mathbb{R}^m)$ and writing $(Du)^s=A|Du|^s$ (polar decomposition of the singular part of Du), we have $\operatorname{rk}(A)=1$ at $|Du|^s$-a.e. point.

We refer to Section 3 for the short argument used to deduce this from Theorem 1.7, giving yet another proof of Alberti's rank-one theorem.

1.3. **General results in arbitrary dimension.** Back to the general setting of arbitrary k, n, and F, in [8] the *atomic condition (AC)* for F was introduced. Namely, F satisfies (AC) if any average

$$\int_{\mathrm{Gr}_k(\mathbb{R}^n)}B_F(P)\,d\lambda(P),\quad \lambda\in\mathcal{P}(\mathrm{Gr}_k(\mathbb{R}^n))$$

has rank $\ge k$, with equality if and only if $\lambda=\delta_{P_0}$ is a Dirac mass; here $B_F(P)$ is the matrix naturally associated with $P\in\mathrm{Gr}_k(\mathbb{R}^n)$ in the computation of the first variation (see (7)). It was shown in [8] that (AC) holds true if and only if, for any varifold V with locally bounded first variation, the condition $\Theta^{k,*}(|V|,x)>0$ for $|V|$-a.e. x implies (and hence is equivalent to) the rectifiability of V.

Thus, it constitutes a very natural assumption that we will make throughout the rest of this introduction (specifically, in Proposition 1.9 and Corollary 1.11). In this paper, we will also prove the following general facts.

Proposition 1.9. Given $n\ge 3$, $k\in\{2,\dots,n-1\}$, and $F:\mathrm{Gr}_k(\mathbb{R}^n)\to(0,\infty)$ satisfying (AC), the following are equivalent.

(i) The Michael–Simon bound (1) holds true for some $C(n,k,F)>0$.

(ii) We have compactness of rectifiable varifolds: given a sequence $(V_i)_{i\in\mathbb{N}}$ of rectifiable k-varifolds, in $\mathbb{R}^n$ or (equivalently) in the torus $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, if
$$\Theta^k(|V_i|,x)\ge\theta_0>0 \text{ for } |V_i|\text{-a.e. } x,\quad \sup_{i\in\mathbb{N}}|\delta^F V_i|(K)<\infty$$
for any compact set K, and $V_i\rightharpoonup V$, then V is a rectifiable k-varifold with $\Theta^k(|V|,x)\ge\theta_0>0$ for $|V|$-a.e. x.

(iii) There is no sequence $(V_i)_{i\in\mathbb{N}}$ of rectifiable k-varifolds in $\mathbb{T}^n$ such that
$$|V_i|(\mathbb{T}^n)=1,\quad \Theta^k(|V_i|,x)\ge i \text{ for } |V_i|\text{-a.e. } x,\quad |\delta^F V_i|(\mathbb{T}^n)\to 0$$
and $V_i\rightharpoonup\mathcal{L}^n\otimes\lambda$ as measures on $\mathbb{T}^n\times\mathrm{Gr}_k(\mathbb{R}^n)$, for some probability measure $\lambda\in\mathcal{P}(\mathrm{Gr}_k(\mathbb{R}^n))$.

(iv) There is no sequence $(V_i)_{i\in\mathbb{N}}$ of rectifiable k-varifolds in $\mathbb{R}^n$, with $\operatorname{spt}(|V_i|)\subseteq[0,1]^n$, such that
$$|V_i|(\mathbb{R}^n)=1,\quad \Theta^k(|V_i|,x)\ge i \text{ for } |V_i|\text{-a.e. } x,\quad \sup_{i\in\mathbb{N}}|\delta^F V_i|(\mathbb{R}^n)<\infty$$
and $V_i\rightharpoonup(\mathcal{L}^n\llcorner[0,1]^n)\otimes\lambda$ as measures on $\mathbb{R}^n\times\mathrm{Gr}_k(\mathbb{R}^n)$, for some probability measure $\lambda\in\mathcal{P}(\mathrm{Gr}_k(\mathbb{R}^n))$.

In fact, in (ii), the fact that $\Theta^k(|V|,x) \geq \theta_0 > 0$ for $|V|$-a.e. x holds automatically if V is rectifiable.

This shows that, if compactness of rectifiable varifolds (or, equivalently, Michael–Simon) fails, then we can find a counterexample exhibiting a phenomenon called *diffuse concentration*: the measures $|V_i|$ are concentrated on Borel sets E_i with $\mathcal{H}^k(E_i)$ (and thus their projection $\pi_P(E_i)$ on any coordinate k-plane has $\mathcal{L}^k(\pi_P(E_i)) \to 0$, making $(\pi_P)_*|V_i|$ look more and more like a singular measure), but nonetheless their limit $|V|$ is the Lebesgue measure (whose projection $(\pi_P)_*|V|$ is Lebesgue).

Moreover, by an adaptation of Allard's strong constancy lemma [5, p. 3] (see also [8, 9]), we can show the following statement, in which $S(\mathcal{D})$ denotes the finite-sum set of $\mathcal{D} \subseteq \mathbb{R}$, i.e., the set of all possible finite, nonempty sums of elements of $\mathcal{D}$ (possibly with repetitions).

Proposition 1.10. Given a sequence of rectifiable k-varifolds, in $\mathbb{R}^n$ or in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, converging to a rectifiable k-varifold V, suppose that for each $i \in \mathbb{N}$ we have a set $\mathcal{D}_i \subseteq (0,\infty)$ with $\inf \mathcal{D}_i > 0$ and a Borel set $E_i \subseteq \mathbb{R}^n$ such that

$$\Theta^k(|V_i|,x) \in \mathcal{D}_i \text{ for } |V_i|\text{-a.e. } x \notin E_i,$$

as well as $|V_i|(E_i \cap K) \to 0$ and $\sup_i |\delta^F V_i|(K) < \infty$ for any compact set K. Then, up to a subsequence, for $|V|$-a.e. x we have

$$\Theta^k(|V|,x) = \lim_{i\to\infty} v_i$$

for suitable $v_i \in S(\mathcal{D}_i)$ depending on x.

The following is a direct consequence of the previous two facts.

Corollary 1.11. Given F satisfying (AC), if any of the equivalent conditions above holds, then we also have compactness of integer-rectifiable varifolds: given a sequence $(V_i)_{i\in\mathbb{N}}$ of rectifiable k-varifolds, in $\mathbb{R}^n$ or in $\mathbb{T}^n$, if

$$\Theta^k(|V_i|,x) \in \mathbb{N}\setminus\{0\} \text{ for } |V_i|\text{-a.e. } x, \quad \liminf_{i\to\infty} |\delta^F V_i|(K) < \infty$$

for any compact set K, and $V_i \rightharpoonup V$, then V is a rectifiable k-varifold with $\Theta^k(|V|,x) \in \mathbb{N}\setminus\{0\}$ for $|V|$-a.e. x.

Let us now state another simple consequence of (1). Recall that, given $p \in [1,\infty]$ and a k-varifold V, we say that its first variation is *locally in* L^p if V has locally bounded first variation and $|\delta^F V| = f|V|$ for some $f \in L^p_{loc}(|V|)$.

Corollary 1.12. If any of the equivalent conditions of Proposition 1.9 holds, then the following hold true as well.

(i) Given a rectifiable k-varifold V, assume that $\Theta^k(|V|,x) \geq \theta_0$ for $|V|$-a.e. x and that $\delta^F V$ is locally in L^p for some $p \in (k,\infty]$. Then, for all $x \in \operatorname{spt}(|V|)$, we have $\Theta^k_*(|V|,x) \geq c(n,k,F)\theta_0$. More precisely, if $|\delta^F V| = f|V|$ with $\|f\|_{L^p(B_1(x),|V|)} \leq \Lambda$, there exists $r_0(n,k,p,F,\Lambda) \in (0,1)$ such that

$$|V|(B_r(x)) \geq c(n,k,F)\theta_0 r^k \quad \text{for all } r \in (0,r_0). \tag{5}$$

(ii) Moreover, if $V_i \rightharpoonup V$ are as in (i) and $\limsup_{i\to\infty} \|f_i\|_{L^p(K,|V_i|)} < \infty$ (where $|\delta^F V_i| = f_i|V_i|$) for any compact set K, then

$$\operatorname{spt}(|V_i|) \to \operatorname{spt}(|V|)$$

in the Hausdorff sense.

Remark 1.13. While we study autonomous integrands $F(P)$, for simplicity and in order to have scale-invariant bounds when possible, in general one can consider non-autonomous ones, of the form $F(x, P)$. This generalization is necessary in order to look at anisotropic integrands on manifolds M^n, where F is defined on the Grassmannian bundle of k-planes and thus locally takes this form. Compared to the autonomous case, the general case just introduces an error term which is easily handled (and can be locally incorporated in a reduced form of $\delta^F V$, namely the right-hand side of (6)). Straightforward modifications show that all the stated consequences of Michael–Simon still hold true, assuming this bound in a local form such as

$$|V|(\mathbb{R}^n)^{k-1} \le \frac{C(n, k, F, K)}{\theta_0} |\delta^F V|(\mathbb{R}^n)^k \quad \text{whenever } \operatorname{spt}(|V|) \subseteq K,$$

where $K \subset \mathbb{R}^n$ (or $K \subseteq M^n$) is an arbitrary compact set.

Acknowledgements

As already mentioned, this paper uses many ideas due to Frederick J. Almgren Jr. The authors are very grateful to Jean Taylor, who shared with them the unpublished manuscript of Almgren. A.P.'s research is funded by the European Research Council (ERC) through StG 101165368 "MAGNETIC". G.D.P.'s research is funded by the European Research Council (ERC) through CoG 101169953 "RISE".[1]

2. Preliminaries and proof of Propositions 1.9, 1.10, and Corollary 1.12

Given two integers $n \ge 3$ and $k \in \{2, \dots, n-1\}$, let $\mathrm{Gr}_k(\mathbb{R}^n)$ be the Grassmannian of k-planes in $\mathbb{R}^n$ (without orientation). Recall that it admits a natural structure of compact manifold of dimension $k(n-k)$ and can be identified with the set of matrices

$$\{S \in \mathbb{R}^{n \times n} \,:\, S^T = S,\ S^2 = S,\ \operatorname{tr} S = k\},$$

with the smooth structure induced from $\mathbb{R}^{n \times n}$. Given a plane $P \in \mathrm{Gr}_k(\mathbb{R}^n)$ and a linear isomorphism $L \in \mathrm{GL}(n)$, we denote by $L[P] = L(P) \in \mathrm{Gr}_k(\mathbb{R}^n)$ the image of P through L.

Consider a *k-dimensional varifold* (or simply *k-varifold*) V in an open set $U \subseteq \mathbb{R}^n$, namely a nonnegative Radon measure on $U \times \mathrm{Gr}_k(\mathbb{R}^n)$. Letting $\pi : \mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n) \to \mathbb{R}^n$ denote the projection on the first factor, we will often write

$$|V| := \pi_* V,$$

called the *weight* of V (a Radon measure on U). A diffeomorphism $\varphi : U \to U$ induces a diffeomorphism $\widehat{\varphi}$ of $U \times \mathrm{Gr}_k(\mathbb{R}^n)$, mapping (x, P) to $(\varphi(x), d\varphi(x)[P])$. The *varifold pushforward* of V through φ is

$$\varphi_* V := \widehat{\varphi}_*(J_\varphi V), \quad J_\varphi(x, P) := \frac{\mathcal{H}^k(d\varphi(x)[P \cap B_1(0)])}{\mathcal{H}^k(P \cap B_1(0))} = \frac{\mathcal{H}^k(d\varphi(x)[P \cap B_1(0)])}{\omega_k},$$

where the Jacobian factor $J_\varphi : \mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n) \to (0, \infty)$ is the usual correction factor motivated by the area formula.

Let $F : \mathrm{Gr}_k(\mathbb{R}^n) \to (0, \infty)$ be a C^1 function, which is often called an *anisotropic integrand*. It induces a functional on k-varifolds in U given by

$$\mathcal{F}(V) := \int_{U \times \mathrm{Gr}_k(\mathbb{R}^n)} F(P) \, dV(x, P).$$

[1]Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council.

More generally, given $B \subseteq U$ Borel, we set

$$\mathcal{F}(V,B) := \int_{B\times \mathrm{Gr}_k(\mathbb{R}^n)} F(P)\, dV(x,P);$$

it is clear that for two constants $0 < c(n,k,F) \le C(n,k,F)$ we have

$$c|V|(B) \le \mathcal{F}(V,B) \le C|V|(B).$$

Given a vector field $X \in C^1_c(U)$, the *first variation of V along X* (with respect to F) is defined as

$$\langle \delta^F V, X\rangle := \frac{d}{dt}\Big|_{t=0} \mathcal{F}((\varphi^X_t)_* V, \mathrm{spt}(X)),$$

where $(\varphi^X_t)_{t\in\mathbb{R}}$ is the flow of X at time t. It can be shown (see, e.g., [8, Lemma A.2]) that, under the above identification, the previous derivative always exists and is given explicitly by

$$\langle \delta^F V, X\rangle = \int_{U\times \mathrm{Gr}_k(\mathbb{R}^n)} \langle B_F(S), dX(x)\rangle\, dV(x,S), \tag{6}$$

where we use the Hilbert–Schmidt inner product on matrices and $B_F(S) \in \mathbb{R}^{n\times n}$ is uniquely defined by

$$\langle B_F(S), L\rangle := F(S)\langle S, L\rangle + dF(S)[(I-S)LS + SL^T(I-S)] \tag{7}$$

for all $L \in \mathbb{R}^{n\times n}$ (note that $(I-S)LS + SL^T(I-S) \in T_S\mathrm{Gr}_k(\mathbb{R}^n)$).

As in the isotropic case (where $F \equiv 1$), we will say that V has *locally bounded first variation* if $\langle \delta^F V, X\rangle$ can be locally represented by integration of X against a vector-valued measure. In this case, $|\delta^F V|$ denotes the associated total variation measure.

We will eventually focus on the codimension-one case $k = n-1$ (in particular, when $k = 2$ and $n = 3$). In this case, we can identify $\mathrm{Gr}_{n-1}(\mathbb{R}^n) \cong \mathbb{RP}^{n-1}$, i.e., a hyperplane P is identified with $\pm\nu$, the unit vector perpendicular to P. We can then identify F with an even function

$$F : \mathbb{S}^{n-1} \to (0,\infty).$$

Taking the 1-homogeneous extension (still denoted by F), namely $F(\lambda\nu) := \lambda F(\nu)$ for all $\lambda \ge 0$, we obtain a function $F : \mathbb{R}^n \to [0,\infty)$. In this case, the *atomic condition (AC)* mentioned in the introduction is equivalent to require that F is strictly convex along all lines which do not pass through 0 [8, Theorem 1.3]. Also, we have the simpler formula

$$B_F(\nu) = F(\nu)I - \nu \otimes dF(\nu), \tag{8}$$

where $dF(\nu) \in (\mathbb{R}^n)^*$ is the differential of the 1-homogeneous extension at ν (note that $B_F(\nu) = B_F(-\nu)$, as expected).

We now turn to the proofs of the general facts stated in the introduction, starting with a well-known lemma.

Lemma 2.1. Given a k-varifold V in an open set U with locally bounded first variation, if $\overline{B}_r(p) \subset U$ and the derivative $\frac{d}{d\rho}|V|(B_\rho(p))|_{\rho=r}$ exists then $V' := \mathbf{1}_{B_r(p)}V$ has

$$|\delta^F V'|(\mathbb{R}^n) \le |\delta^F V|(B_r(p)) + C(n,k,F)\frac{d}{d\rho}|V|(B_\rho(p))\Big|_{\rho=r}.$$

Proof. This is easily seen by taking a cut-off function χ such that $\chi = 1$ on $B_{r-h}(p)$, $\chi = 0$ outside $B_r(p)$, and $|d\chi| \le 2/h$ (for a given $h \in (0,r)$), and noting that the varifold χV has

$$\langle \delta^F(\chi V), X\rangle = \langle \delta^F V, \chi X\rangle + O(\|d\chi\|_{C^0}\|X\|_{C^0}) \cdot |V|(B_r(p) \setminus B_{r-h}(p))$$

for any $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, so that

$$|\delta^F(\chi V)|(\mathbb{R}^n) \le |\delta^F V|(B_r(p)) + C(n,k,F)\frac{|V|(B_r(p)) - |V|(B_{r-h}(p))}{h},$$

which gives the claim in the limit $h \to 0$. □

Proof of Corollary 1.12. Let us prove (i). Letting $\mu(r) := |V|(B_r(x))$, since for a.e. $r > 0$ the truncated varifold $V' := \mathbf{1}_{B_r(x)} V$ has $|\delta^F V'|(\mathbb{R}^n) \le |\delta^F V|(B_r(x)) + C\mu'(r)$, we get

$$\mu(r)^{(k-1)/k} \le C|\delta^F V|(B_r(x)) + C\mu'(r) \le C_x \mu(r)^\alpha + C(n,k,F)\mu'(r)$$

for a.e. $r \in (0,1)$, where $\alpha := 1 - \frac{1}{p}$ and C_x depends on n, k, F and also on (an upper bound on) $\|f\|_{L^p(B_r(x),|V|)}$, where $|\delta^F V| = f|V|$. Since $x \in \operatorname{spt}(|V|)$, we have $\mu(r) > 0$ and hence

$$(\mu(r)^{1/k})' \ge \frac{c - C_x' \mu(r)^\beta}{k}$$

for a.e. $r \in (0,1)$, where $c = C(n,k,F)^{-1}$ and $\beta := \alpha - (1 - \frac{1}{k}) > 0$. As long as $C_x' r^{\beta k} < c/2$, we either have $\mu(r) \ge r^k$ or $\frac{c - C_x'\mu(\rho)^\beta}{k} \ge \frac{c}{2k}$ for all $\rho \in (0,r)$. In the latter case, we obtain $\mu(r)^{1/k} \ge \frac{cr}{2k}$, giving the claim in both cases (for a different $c > 0$). The upper semicontinuity of the support along converging sequences is a direct consequence. □

Proposition 2.2. The Michael–Simon bound (1) is equivalent to its functional version (2).

Proof. To see that (1) implies (2), note that a simple cut-off argument as in the previous proof shows that, for a.e. $t > 0$, $V' := \mathbf{1}_{\{|f|>t\}} V$ has

$$|\delta^F V'|(\mathbb{R}^n) \le |\delta^F V|(\{|f| > t\}) - Ch'(t), \quad h(t) := \int_{\{|f|>t\}} |df|\, d|V|,$$

for a possibly different C. Assuming for simplicity $\theta_0 = 1$ and applying (1) to V', we get

$$m(t)^{(k-1)/k} \le C|\delta^F V|(\{|f| > t\}) - Ch'(t), \quad m(t) := |V|(\{|f| > t\}),$$

which gives (2) thanks to the well-knwon bound $(\int_0^\infty m(t) t^{p-1}\, dt)^{1/p} \le p^{-1/p} \int_0^\infty m(t)^{1/p}\, dt$ for any decreasing $m(t)$ and $p \in [1,\infty)$ (we take $p := k/(k-1)$). □

Let us now show Proposition 1.10, from which Proposition 1.9 will follow quite easily.

Proof of Proposition 1.10. Up to a subsequence, we can assume that $|\delta^F V_i| \rightharpoonup \nu$ for a suitable Radon measure ν. For $|V|$-a.e. x, the rectifiable varifold V admits an approximate tangent plane and there exists $C > 0$, depending on x, such that[2]

$$\nu(B_r(x)) \le C|V|(B_r(x)) \quad \text{for all } r \in (0,1).$$

Take any such point x_0; up to a translation and a rotation, we can assume that $x_0 = 0$ and the tangent plane is θ times $P = \operatorname{span}\{e_1, \dots, e_k\}$, for some constant $\theta > 0$.

By a straightforward diagonal argument, we can find a sequence of radii $r_i \to 0$ such that, denoting by W_i the varifolds dilated by a factor r_i^{-1} and $\tilde{E}_i := r_i^{-1} E_i$, we have

$$W_i \rightharpoonup \theta P, \quad |W_i|(\tilde{E}_i) \to 0,$$

where P is identified with the corresponding multiplicity one varifold. To conclude, it suffices to show that $\theta = \lim_{i\to\infty} v_i$ for suitable $v_i \in S(\mathcal{D}_i)$. Clearly, it is enough to check that this holds along a subsequence.

Note that

$$(9) \qquad |\delta^F W_i|(B_1(0)) = r_i^{1-k} |\delta^F V_i|(B_{r_i}(0)) \le C r_i^{1-k} |V_i|(B_{r_i}(0)) = C r_i |W_i|(B_1(0)) \to 0.$$

[2]Recall that, for any two Radon measures ν and μ, we have

$$\limsup_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} < +\infty \quad \text{for } \mu\text{-a.e. } x.$$

Moreover, for each $i \in \mathbb{N}$ we can find $\rho_i \in (1/2, 3/4)$ such that $\frac{d}{d\rho}|W_i|(B_\rho(0))|_{\rho=\rho_i} \leq C|W_i|(B_1(0)) \leq C$, so that by Lemma 2.1 the varifold $W_i' := \mathbf{1}_{B_{\rho_i}(0)} W_i$ has $|\delta^F W_i'|(\mathbb{R}^n) \leq C$. We can now apply [8, Lemma 3.2] to the sequence (W_i') (with $F_i := F$) and deduce the strong convergence

$$(\pi_P)_*|W_i'| \to f\mathcal{H}^k \llcorner P,$$

for some $f \in L^1(P)$ supported in the unit ball. Moreover, since $W_j \rightharpoonup \theta P$, we have $f = \theta$ on $P \cap B_{1/2}(0)$.

Now, denoting by $T_x W_i' \in \mathrm{Gr}_k(\mathbb{R}^n)$ the (normalized) approximate tangent plane (which exists $|W_i'|$-a.e.), let $J_i(x) \in [0, 1]$ denote the Jacobian of the projection π_P along $T_x W_i'$, namely $J_i(x) = \omega_k^{-1}\mathcal{H}^k(\pi_P(B_1(0) \cap T_x W_i'))$. By the varifold convergence $W_i \rightharpoonup \theta P$, we have

$$\int_{B_1(0)} |J_i(x) - 1| \, d|W_i'| \to 0,$$

so that

$$(\pi_P)_*[J_i|W_i'| \llcorner (\mathbb{R}^n \setminus \tilde{E}_i)] \to f\mathcal{H}^k \llcorner P.$$

Since $\Theta^k(|W_i'|, x) \in \mathcal{D}_i$ for $|W_i'|$-a.e. $x \in \mathbb{R}^n \setminus \tilde{E}_i$, by the area formula we have

$$(\pi_P)_*[J_i|W_i'| \llcorner (\mathbb{R}^n \setminus \tilde{E}_i)] = f_i\mathcal{H}^k \llcorner P$$

for some $f_i \in L^1(P)$ taking values in $S(\mathcal{D}_i)$ a.e. (as $\inf \mathcal{D}_i > 0$ for all i). Since $f_i \to \theta$ strongly in $L^1(B_{1/2}(0))$, the claim follows. □

Finally, let us turn to the equivalence between Michael–Simon and compactness of rectifiable varifolds, namely Proposition 1.9.

Proof of Proposition 1.9. Let us first check that (i) implies (ii): let us then assume that $V_i \rightharpoonup V$ is a sequence as in (ii), with $\theta_0 = 1$. Given a point $p \in \mathrm{spt}(|V_i|)$, if $|\delta^F V_i|(B_r(p)) \leq \Lambda|V_i|(B_r(p))$ for all $r \in (0, 1/2)$ then we claim that (1) gives $c(n, k, F, \Lambda) > 0$ such that

$$|V_i|(B_r(p)) \geq cr^k \quad \text{for all } r \in (0, 1/2). \tag{10}$$

Indeed, letting $\mu(r) := |V_i|(B_r(p))$, whenever the classical derivative $\mu'(r)$ exists, by Lemma 2.1 the varifold $V_i' := \mathbf{1}_{B_r(p)} V_i$ has total first variation bounded by $\Lambda\mu(r) + C(n, k, F)\mu'(r)$. Now, using Theorem 1.3 for V_i' (which can be viewed as a varifold in $\mathbb{R}^n$ even when $M = \mathbb{T}^n$, as $B_{1/2}(p) \subset \mathbb{T}^n$ is isometric to $B_{1/2}(0) \subset \mathbb{R}^n$), we deduce that

$$\mu(r)^{(k-1)/k} \leq \Lambda\mu(r) + C\mu'(r) \quad \text{for a.e. } r \in (0, 1/2),$$

which easily gives (10) (see also the proof of Corollary 1.12).

We now repeat the first part of the proof of [14, Theorem 40.6]. Let $K := \overline{B}_R(0)$ and call $S_{i,\ell} \subseteq K$ the set of points $p \in \mathrm{spt}(|V_i|) \cap K$ such that $|\delta^F V_i|(B_r(p)) \leq \ell|V_i|(B_r(p))$ for all $0 < r < 1/2$. Since $\sup_i |\delta^F V_i|(B_{R+1}) < \infty$, using Besicovitch's covering lemma it is immediate to check that $|V_i|(K \setminus S_{i,\ell}) \to 0$ as $\ell \to \infty$, uniformly in i.

It follows that, letting S_ℓ denote the (relatively closed) set of points q in $B_R(0)$ such that $q = \lim_{j\to\infty} p_{i_j}$, for some sequence $p_{i_j} \in S_{i_j,\ell}$ (and some subsequence $i_j \to \infty$), we have

$$\lim_{\ell\to\infty} |V|(B_R(0) \setminus S_\ell) = 0.$$

Indeed, for any compact $K' \subseteq B_R(0) \setminus S_\ell$ there exists a small $\rho > 0$ such that $S_{i,\ell} \cap B_\rho(K') = \emptyset$ for i large enough, yielding $|V|(K') \leq \liminf_{i\to\infty} |V_i|(B_\rho(K') \setminus S_{i,\ell})$ and thus $|V|(B_R(0) \setminus S_\ell) \leq \liminf_{i\to\infty} |V_i|(K \setminus S_{i,\ell}) \to 0$ as $\ell \to \infty$.

On the other hand, given $q \in S_\ell$, (10) gives $|V|(\overline{B}_r(q)) \geq c(n, k, F, \ell)r^k$ for all $r \in (0, 1/2)$, and thus

$$\Theta^k_*(|V|, q) > 0 \quad \text{for all } q \in \bigcup_\ell S_\ell.$$

Since the complement of $\bigcup_\ell S_\ell$ is $|V|$-negligible, by [8, Theorem 1.2] the varifold V is rectifiable. Moreover, by Proposition 1.10 it has density ≥ 1 at $|V|$-a.e. point, as desired.

It is obvious that (ii) implies (iv). Moreover, we claim that (iv) implies (iii): if we have a bad sequence as in (iii), then by averaging we can find $c_1^i, \dots, c_n^i \in \mathbb{R}/\mathbb{Z}$ such that $|V_i|(\pi_j^{-1}(B_r(c_j^i))) \leq 4r$ for any r small enough (depending on i, j), where $\pi_j : \mathbb{T}^n \to \mathbb{T}^1$ is the projection to the j-th coordinate. Up to a translation, we can assume that $c_j^i = 0$. Then, lifting V_i to a periodic varifold $\tilde{V}_i$ in $\mathbb{R}^n$, the truncated varifolds $\mathbf{1}_{(0,1)^n}\tilde{V}_i$ have uniformly bounded first variation (by the same argument of Lemma 2.1) and satisfy all the other conclusions of a bad sequence in (iv).

Finally, let us see that (iii) implies (i). Assuming by contradiction that (1) fails, we will construct a bad sequence as in (iii). Fix a finite $\alpha > \frac{n+1}{k}$; since (1) is scale-invariant, if it fails then there exists a sequence (V_i) of rectifiable k-varifolds with $\Theta^k(|V_i|, x) \geq 1$ for $|V_i|$-a.e. x and

$$|V_i|(\mathbb{R}^n) = 1, \quad |\delta^F V_i|(\mathbb{R}^n) \leq \varepsilon_i^\alpha,$$

for a vanishing sequence $0 < \varepsilon_i \leq 1/i$. Dilating V_i by a factor $\varepsilon_i^{(n+1)/k} < 1$ and multiplying the resulting varifold by ε_i^{-1}, we obtain a new sequence of varifolds W_i satisfying

$$|W_i|(\mathbb{R}^n) = \varepsilon_i^n, \quad |\delta^F W_i|(\mathbb{R}^n) \leq \varepsilon_i^\beta, \quad \Theta^k(|W_i|, x) \geq \varepsilon_i^{-1} \geq i \text{ for } |W_i|\text{-a.e. } x,$$

where $\beta := \alpha + \frac{(n+1)(k-1)}{k} - 1 > n$. Finally, assuming without loss of generality that $\varepsilon_i = \frac{1}{\ell_i}$ for some integer $\ell_i > 1$, we define

$$\tilde{Z}_i := \sum_{c \in \mathbb{Z}^n} (W_i + \varepsilon_i c)$$

where, with a slight abuse of notation, we denote by $V_i + a$ the translation of V_i by $a \in \mathbb{R}^n$. It is easy to check that finite partial sums have locally uniformly bounded mass, so that the series does indeed define a k-varifold in $\mathbb{R}^n$. This varifold is $\mathbb{Z}^n$-periodic and thus is the lift of a varifold V_i on $\mathbb{T}^n$, which has

$$|Z_i|(\mathbb{T}^n) = \varepsilon_i^{-n} |W_i|(\mathbb{R}^n), \quad |\delta^F Z_i|(\mathbb{T}^n) \leq \varepsilon_i^{-n} |\delta^F W_i|(\mathbb{R}^n)$$

and hence

$$|Z_i|(\mathbb{T}^n) = 1, \quad |\delta^F Z_i|(\mathbb{T}^n) \leq \varepsilon_i^\gamma, \quad \gamma := \beta - n > 0,$$

as well as $\Theta^k(|Z_i|, x) \geq i$ at $|Z_i|$-a.e. x. Since in fact Z_i is $\varepsilon_i\mathbb{Z}^n$-periodic, it is clear that it converges to a varifold of the form $Z = \mathcal{L}^n \otimes \lambda$, with $\lambda \in \mathcal{P}(\mathrm{Gr}_k(\mathbb{R}^n))$, along a subsequence. □

3. Nonlinear inequalities for vector fields on the plane

The main goal of this section is to prove the following new nonlinear inequality for vector fields on the plane, stated again for the reader's convenience.

Theorem 3.1. Given two vector fields $S, T \in W^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$, we define

$$\begin{aligned} S^P &:= (S^x - |S^y|)^+, \quad S^N := (S^x - |S^y|)^-, \\ T^P &:= (T^y - |T^x|)^+, \quad T^N := (T^y - |T^x|)^-. \end{aligned} \tag{11}$$

Also, let $\chi : \mathbb{R}^2 \to [0, 1]$ be a Borel function supported in a bounded set. Assuming $S^x, T^y \geq 0$ then, for some universal constant $C > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \chi \min\{S^P, T^P\} \leq C \|\chi\|_{L^2} \Big[\int_{\mathbb{R}^2} (|S| + |\operatorname{div} S|) \Big]^{1/2} \Big[\int_{\mathbb{R}^2} (|T| + |\operatorname{div} T|) \Big]^{1/2} \\ + \int_{\mathbb{R}^2} (C|\operatorname{div} S| + C|\operatorname{div} T| + S^N + T^N). \end{aligned} \tag{12}$$

The same holds if $S^x, S^y, T^x, T^y, \operatorname{div} S, \operatorname{div} T$ are just real-valued measures on the plane with finite total variation (provided that $S^x, T^y \geq 0$).

Remark 3.2. In the general case of measures, quantities such as $\int_{\mathbb{R}^2} |S|$ should be interpreted as $|S|(\mathbb{R}^2)$ and S^P, S^N, T^P, T^N are defined by the same formula (equivalently, writing $S = \sigma|S|$ for a unit-valued σ, we have $S^P = (\sigma^x - |\sigma^y|)^+|S|$). Recall also that $\min\{\mu, \nu\} := (\mu + \nu - |\mu - \nu|)/2$ for two real-valued measures μ, ν (equivalently, writing $\mu = f(|\mu| + |\nu|)$ and $\nu = g(|\mu| + |\nu|)$, we have $\min\{\mu, \nu\} = \min\{f, g\}(|\mu| + |\nu|)$).

While (12) is not scale-invariant, taking an arbitrary χ such that $\chi = 0$ $\mathcal{L}^2$-a.e. and applying the bound to all rescalings of S, T, we immediately obtain the following, which can be seen as a special case of [2, Proposition 8.6].

Corollary 3.3. Under the same assumptions, denoting by $\mathfrak{S}, \mathfrak{T}$ the singular parts of the measures S, T, we have

$$\int_{\mathbb{R}^2} \min\{\mathfrak{S}^P, \mathfrak{T}^P\} \leq C \int_{\mathbb{R}^2} (S^N + T^N).$$

In particular, if $S^N = T^N = 0$ then the two measures $\mathfrak{S}^P$ and $\mathfrak{T}^P$ are mutually singular.

Thus, in the last corollary, the assumption that $\operatorname{div} S$ and $\operatorname{div} T$ are finite measures is used only qualitatively. Given $u \in BV(\mathbb{R}^2, \mathbb{R}^2)$ and taking

$$S := (\partial_y u^y, -\partial_x u^y), \quad T := (-\partial_y u^x, \partial_x u^x),$$

we have $\operatorname{div} S = \operatorname{div} T = 0$. Taking φ_r supported in a ball $B_r(p)$ and applying the previous bound to $\varphi_r S, \varphi_r T$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} \varphi_r \min\{(A^x_x - |A^x_y|)^+, (A^x_x - |A^x_y|)^+\}\, d|Du|^s \\ &\leq C \int_{\mathbb{R}^2} \varphi_r[(A^x_x - |A^x_y|)^- + (A^x_x - |A^x_y|)^-]\, d|Du|,\end{aligned}$$

where A (with rows A^x, A^y) is given by the polar decomposition $Du = A|Du|$ and $|Du|^s$ denotes the singular part. In particular, if p is an approximate continuity point for A and a point of density one for $|Du|^s$ (with respect to $|Du|$), taking $r \to 0$ we see that $A(p)$ must be far from I. Since this must also hold if we compose u with linear transformations, this immediately implies the following.

Corollary 3.4 (Alberti's rank-one theorem [1, 9, 12])**.** Given $u \in BV(\mathbb{R}^2, \mathbb{R}^2)$ and writing $(Du)^s = A|Du|^s$, we have $\operatorname{rk}(A) = 1$ at $|Du|^s$-a.e. point. By a well-known slicing argument (see, e.g., [7, Proposition 1.3]), this implies that the same holds for a function $u \in BV(\mathbb{R}^n, \mathbb{R}^m)$ with any $m, n \geq 1$.

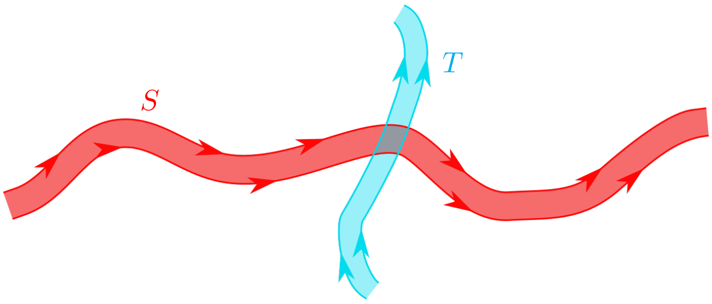
Before turning to Theorem 3.1, we will first obtain a simpler and perhaps more intuitive bound, for vector fields obeying a certain nonlinear constraint. Namely, we will derive the following sharp estimate, which was also proved in [10, Theorem A] (although with non-sharp constant). We will give a full proof of it since our techniques are different and more readily adaptable to give also Theorem 3.1 above.

Theorem 3.5. Given $S, T \in W^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$, assume that $S^x, T^y \geq 0$ and $\det(S, T) \geq 0$ a.e. Then

$$\int_{\mathbb{R}^2} \det(S, T) \leq \frac{1}{4}\Big(\int_{\mathbb{R}^2} |\operatorname{div} S|\Big)\Big(\int_{\mathbb{R}^2} |\operatorname{div} T|\Big).$$

The intuition behind this result is that, by Smirnov's decomposition theorem for 1-dimensional currents [15], the current associated to S is a (weighted) superposition of curves γ. The boundary of each curve contributes 2 to the total mass of the boundary, which is $\int_{\mathbb{R}^2} |\operatorname{div} S|$, so that (informally) the weighted number of curves building up S is

$\frac{1}{2}\int_{\mathbb{R}^2}|\operatorname{div} S|$, and similarly for T. On the other hand, the condition $\det(S,T)\ge 0$ forces a curve in S and a curve in T to meet at most once, and the integral of $\det(S,T)$ counts the (weighted) total number of intersections. This is best seen by looking at an example where $S,T\in BV(\mathbb{R}^2,\mathbb{R}^2)$ are supported on ε-fattened curves, for ε small, and point along the curve; this example also shows that the bound is sharp, with equality achieved quite often. With this intuition in mind, this bound can be seen as a functional version of the (trivial) planar case of the multilinear Kakeya inequality [6, 11], extended to a situation where the tubes are not necessarily straight.



The following is an immediate corollary, which will be used to prove Theorem 1.5. In its statement, we consider the class $\mathcal{M}^+$ of matrix-valued measures M (on $\mathbb{R}^2$, with values in $\mathbb{R}^{2\times 2}$) such that, writing $M=A|M|$ for some unit-valued Borel $A:\mathbb{R}^2\to\mathbb{R}^{2\times 2}$, we have $A^x_x, A^y_y, \det(A)\ge 0$ ($|M|$-a.e.). We let $\sqrt{\det(M)}:=\sqrt{\det(A)}|M|$, which is a well-defined nonnegative measure.

Corollary 3.6. Given two vector fields $S,T\in W^{1,1}(\mathbb{R}^2,\mathbb{R}^2)$, assume that $S^x,T^y\ge 0$ and $\det(S,T)\ge 0$ a.e. Also, let $\chi:\mathbb{R}^2\to[0,1]$ be a Borel function supported in a bounded set. Then we have

$$\int_{\mathbb{R}^2}\chi\sqrt{\det(S,T)}\le\frac{1}{2}\|\chi\|_{L^2}\Big[\int_{\mathbb{R}^2}|\operatorname{div} S|\Big]^{1/2}\Big[\int_{\mathbb{R}^2}|\operatorname{div} T|\Big]^{1/2}. \tag{13}$$

The same holds if $S^x,S^y,T^x,T^y,\operatorname{div} S,\operatorname{div} T$ are just real-valued measures on the plane with finite total variation, provided that $(S,T)\in\mathcal{M}^+$ (where we view S,T as rows of a matrix-valued measure).

Indeed, by approximation, it is enough to check this result when $\chi\in C^0_c(\mathbb{R}^2)$. If $S,T\in W^{1,1}(\mathbb{R}^2,\mathbb{R}^2)$ then the statement follows from Theorem 3.5 and Cauchy–Schwarz. In general, it follows by approximating S,T by $S_\varepsilon,T_\varepsilon\in C^\infty_c(\mathbb{R}^2,\mathbb{R}^2)$ in such a way that (viewing $S_\varepsilon,T_\varepsilon$ as measures) we have the tight convergence $|S_\varepsilon|,|T_\varepsilon|\rightharpoonup|S|,|T|$, as well as $|\operatorname{div} S_\varepsilon|,|\operatorname{div} T_\varepsilon|\rightharpoonup|\operatorname{div} S|,|\operatorname{div} T|$. Indeed, by Reshetnyak's continuity principle, we will also have

$$\sqrt{\det(S_\varepsilon,T_\varepsilon)}\rightharpoonup\sqrt{\det(S,T)}$$

as measures, since the assignment $A\mapsto\sqrt{\det(A)^+}$ is 1-homogeneous.

3.1. **Proof of the simpler Theorem 3.5.** *Step 1.* In order to prove Theorem 3.5, we first notice that we can reduce to the case of smooth, compactly supported vector fields. Indeed, we can assume that both S,T are supported in a ball $B_R(0)$ and $|S|^2+|T|^2\le\Lambda^2$ for some $R,\Lambda>0$. Let us fix a nonnegative cut-off function $\psi\in C^\infty_c(\mathbb{R}^2)$ such that $\psi=1$ on $B_{R+1}(0)$. Replacing S^x and T^y with $S^x+\varepsilon|S^y|+\varepsilon\psi$ and $T^y+\varepsilon|T_x|+\varepsilon\psi$ (for $\varepsilon>0$ small), respectively, we obtain perturbed vector fields (still denoted by S,T) such that

$$S^x\ge\varepsilon|S|,\quad T^y\ge\varepsilon|T|,$$

the angle between S and T is $\ge\varepsilon'>0$ (at points where $S,T\ne 0$), and

$$S^x,T^y\ge\varepsilon\quad\text{on } B_{R+1}(0),$$

while S, T are smooth on the complement of $B_R(0)$. In particular, we have $\det(S,T) \geq \varepsilon'' > 0$ on $B_{R+1}(0)$. Calling A_x, A_y the two rows of a generic 2×2 matrix A, we select $\lambda, \mu \in (0, \varepsilon)$ and $\nu \in (0, \varepsilon'')$ giving a regular value for the function $A \mapsto (A_x^x, A_y^y, \det(A))$, so that

$$\mathcal{M} := \{A : A_x^x \geq \lambda,\ A_y^y \geq \mu,\ \det(A) \geq \nu\}$$

admits a $(1 + C\delta)$-Lipschitz projection $\pi : B_\delta(\mathcal{M}) \cap B_{2\Lambda}(0) \to \mathcal{M}$ for $\delta > 0$ small; also, viewing S, T as rows, we have $(S, T) \in \mathcal{M}$ on $B_{R+1}(0)$. Next, we can apply the standard Schoen–Uhlenbeck trick: approximating the pair (S, T) by a mollification with variable radius (leaving (S, T) unchanged on the complement of $B_{R+1}(0)$), by the embedding $W^{1,2} \hookrightarrow VMO$ the resulting pair (S', T') belongs to the domain of π. We can then take $(S'', T'') := \pi(S', T')$ as a smooth perturbation of (S, T) still satisfying the assumptions.

Step 2. Assuming henceforth that $S, T \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$, we claim that, up to another small perturbation, we can assume $S = \alpha Z$ and $T = \beta W$, for smooth, complete vector fields Z, W and coefficients $\alpha, \beta \in C_c^\infty(\mathbb{R}^2)$ such that

$$\alpha \geq 0, \quad \beta \geq 0, \quad \operatorname{div} Z = \operatorname{div} W = 0, \quad Z^x > 0, \quad W^y > 0, \quad \det(Z, W) > 0. \tag{14}$$

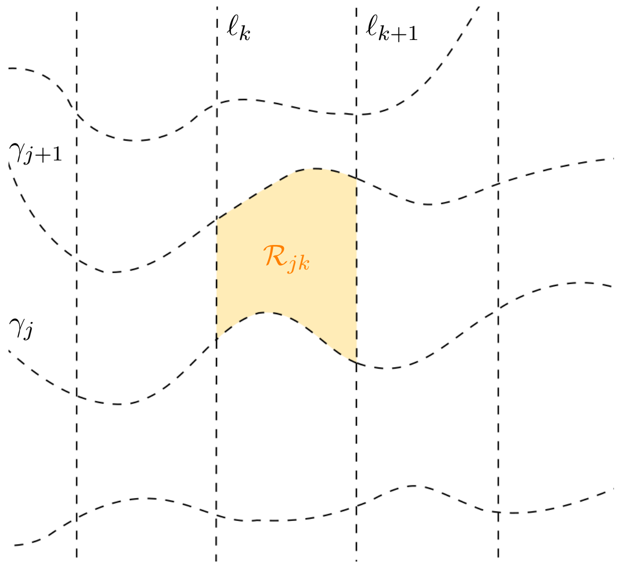
Indeed, given $\varepsilon > 0$ and a nonnegative cut-off function $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi = 1$ near the supports of S and T, we can let $S' := \varphi\tilde{S}$ and $T' := \varphi\tilde{T}$, with $\tilde{S} := S + \varepsilon\partial_x$ and $T' := T + \varepsilon\partial_y$. Since S', T' are arbitrarily close to S, T in the smooth topology, it is enough to prove the inequality for these new vector fields.

Also, we can write $\tilde{S} = \tilde{\alpha}Z$ for a vector field Z with $\operatorname{div} Z = 0$ and $\tilde{\alpha}$ a smooth positive function. Indeed, the plane is foliated by the integral curves of $\tilde{S}$ starting on the vertical axis $\{0\} \times \mathbb{R}$, since $\tilde{S}^x > 0$ and, outside of a compact set, $\tilde{S} = \varepsilon\partial_x$. Hence, we can let $\alpha' := 1$ on the vertical axis $\{0\} \times \mathbb{R}$ and solve the equation $\operatorname{div}(\tilde{\alpha}^{-1}\tilde{S}) = 0$ along all the integral curves $\gamma : \mathbb{R} \to \mathbb{R}^2$, where it becomes the ordinary differential equation

$$\frac{d}{dt}(\tilde{\alpha} \circ \gamma)(t) = (\tilde{\alpha} \operatorname{div} \tilde{S}) \circ \gamma(t).$$

Now, letting $\alpha := \varphi\tilde{\alpha}$ and $\beta := \varphi\tilde{\beta}$, we arrive at $S' = \alpha Z$ and $T' = \beta W$, as desired. Note that (14) also holds. In the sequel, we replace S and T with the approximations S' and T'.

Step 3. It is convenient to further approximate S and T with piecewise divergence-free vector fields, as follows. For $\tau > 0$, let $\gamma_j : \mathbb{R} \to \mathbb{R}^2$ be the integral curve of Z with $\gamma_j(0) = (0, j\tau)$, for $j \in \mathbb{Z}$. Together with the vertical lines $\ell_k := \{k\tau\} \times \mathbb{R}$, these curves split the plane into a family of (open) regions $\mathcal{P}_\tau := (\mathcal{R}_{jk})_{j,k\in\mathbb{Z}}$ diffeomorphic to the unit square, where $\mathcal{R}_{jk}$ is bounded by γ_j, γ_{j+1}, ℓ_k, and ℓ_{k+1}.



For each region $\mathcal{R} = \mathcal{R}_{jk} \in \mathcal{P}_\tau$, we denote $\partial_L \mathcal{R} := \overline{\mathcal{R}} \cap \ell_k$ and $\partial_R \mathcal{R} := \overline{\mathcal{R}} \cap \ell_{k+1}$ the left and right sides of the boundary of $\mathcal{R}$. We then let

$$S_\tau := \sum_{\mathcal{R} \in \mathcal{P}_\tau} \alpha_{\mathcal{R}} \mathbf{1}_{\mathcal{R}} Z,$$

with $\alpha_{\mathcal{R}} \geq 0$ a constant chosen so that the flow of $\alpha_{\mathcal{R}} Z|_{\mathcal{R}}$ across $\partial_L \mathcal{R}$ equals the flow of $S|_{\mathcal{R}}$ across the same segment. Note that S_τ belongs to $BV(\mathbb{R}^2, \mathbb{R}^2)$ and that its divergence is a measure supported on the vertical lines $\bigcup_k \ell_k$, since Z is divergence-free and parallel to the curves γ_j.

Step 4. Let us show a simple fact.

Lemma 3.7. The total variation of $\operatorname{div} S_\tau$ on $\mathbb{R}^2$ is bounded by $\int_{\mathbb{R}^2} |\operatorname{div} S|$.

Proof. Given two adjacent regions $\mathcal{R}$ and $\mathcal{R}'$ with $\partial_R \mathcal{R} = \partial_L \mathcal{R}'$, note that $\alpha_{\mathcal{R}'}$ is chosen so that

$$\int_{\partial_L \mathcal{R}'} \alpha_{\mathcal{R}'} Z^x = \int_{\partial_L \mathcal{R}'} S^x = \int_{\partial_R \mathcal{R}} S^x.$$

Hence,

$$\begin{aligned} (\alpha_{\mathcal{R}'} - \alpha_{\mathcal{R}}) \int_{\partial_L \mathcal{R}'} Z^x &= \int_{\partial_R \mathcal{R}} S^x - \alpha_{\mathcal{R}} \int_{\partial_R \mathcal{R}} Z^x \\ &= \int_{\partial_R \mathcal{R}} S^x - \alpha_{\mathcal{R}} \int_{\partial_L \mathcal{R}} Z^x \\ &= \int_{\partial_R \mathcal{R}} S^x - \int_{\partial_L \mathcal{R}} S^x, \end{aligned}$$

where we used $\operatorname{div} Z = 0$ in the second equality. Recalling that $Z^x > 0$, we get

$$|\alpha_{\mathcal{R}'} - \alpha_{\mathcal{R}}| \int_{\partial_L \mathcal{R}'} Z^x = \left| \int_{\partial_R \mathcal{R}} S^x - \int_{\partial_L \mathcal{R}} S^x \right| \leq \int_{\mathcal{R}} |\operatorname{div} S|.$$

Since the total variation of $\operatorname{div} S_\tau$ is the sum of the quantity in the left-hand side over all pairs of regions $\mathcal{R}, \mathcal{R}'$ with $\partial_R \mathcal{R} = \partial_L \mathcal{R}'$, the claim follows. □

By the previous lemma, since $S_\tau \to S$ pointwise as $\tau \to 0$, it is enough to prove Theorem 3.5 for S_τ and T_τ, where T_τ is obtained in a similar way (interchanging x and y), relative to a partition $\mathcal{Q}_\tau$ of the plane.

Step 5. To motivate what follows, we make the following formal observation: since the coefficients in the approximation $S_\tau = \sum_{\mathcal{R}} \alpha_{\mathcal{R}} \mathbf{1}_{\mathcal{R}} Z$ are nonnegative, S_τ can be viewed as a superposition (with nonnegative coefficients) of indicator functions times Z, in such a way that the divergence of these terms sums up to $\operatorname{div} S_\tau$ *without cancellation.*

We now proceed to make this rigorous. Note that $\alpha_{\mathcal{R}} = 0$ for all but finitely many regions. Given $\mu > 0$ different from all the (finitely many) values $\alpha_{\mathcal{R}}$, for each $j \in \mathbb{Z}$ consider a maximal chain of consecutive regions $\mathcal{R}_{jk}, \dots, \mathcal{R}_{jk'}$ such that the associated constants satisfy $\alpha_{\mathcal{R}} > \mu$, and let $\widehat{\mathcal{R}}$ be their union, which coincides with the region bounded by γ_j, γ_{j+1}, ℓ_k, and $\ell_{k'+1}$ (up to negligible sets).

It is easy to check that the sum $s(\mu)$ of the total variations $|\operatorname{div}(\mathbf{1}_{\widehat{\mathcal{R}}} Z)|(\mathbb{R}^2)$, as $\widehat{\mathcal{R}}$ (and j) vary, equals the sum of $\int_{\partial_L \mathcal{R}'} Z^x$ for all couples of adjacent regions $\mathcal{R}, \mathcal{R}'$ such that μ lies between $\alpha_{\mathcal{R}}$ and $\alpha_{\mathcal{R}'}$ (regardless of the order). Hence,

$$\int_0^\infty s(\mu)\, d\mu = \sum_{\mathcal{R},\mathcal{R}' : \partial_R \mathcal{R} = \partial_L \mathcal{R}} |\alpha_{\mathcal{R}} - \alpha_{\mathcal{R}'}| \int_{\partial_L \mathcal{R}'} Z^x = |\operatorname{div} S_\tau|(\mathbb{R}^2).$$

The analogous sums $t(\nu)$ coincide for the vector field W, for a given value $\nu > 0$.

Step 6. If we prove that

$$\int_{\widehat{\mathcal{R}} \cap \widehat{\mathcal{S}}} \det(Z, W) \le \frac{1}{4} |\operatorname{div}(\mathbf{1}_{\widehat{\mathcal{R}}} Z)|(\mathbb{R}^2)\, |\operatorname{div}(\mathbf{1}_{\widehat{\mathcal{S}}} Z)|(\mathbb{R}^2) \tag{15}$$

for two chains of regions $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{S}}$ as above (relative to Z and W, respectively), then summing over all chains $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{S}}$ (for a given choice of μ and ν) we get

$$\int_{\mathbb{R}^2} \det\Big(\sum_{\alpha_{\mathcal{R}} > \mu} \mathbf{1}_{\mathcal{R}} Z, \sum_{\alpha_{\mathcal{S}} > \nu} \mathbf{1}_{\mathcal{S}} W \Big) \le \frac{1}{4} s(\mu) t(\nu),$$

and Theorem 3.5 follows by integrating in μ and ν.

Step 7. In order to prove (15), we first find two smooth functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$Z = \nabla^\perp f, \quad W = \nabla^\perp g,$$

where $\nabla^\perp = (-\partial_y, \partial_x)$. The function f is just any primitive of the closed form $Z^y\, dx - Z^x\, dy$, and similarly for g. Note that the level sets of f and g are precisely the (maximal) integral curves of Z and W.

The key observation is that, since $\det(Z, W) > 0$, an integral curve of Z meets an integral curve of W only once (since $\det(Z, W) = -df(W)$, actually f decreases along an integral curve of W). Hence, the map $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is injective. Since $\det(Z, W) \ge 0$ is the Jacobian determinant of this map, by the area formula the integral of $\det(Z, W)$ is bounded by the area of the image:

$$\int_{\widehat{\mathcal{R}} \cap \widehat{\mathcal{S}}} \det(Z, W) \le \mathcal{L}^2((f, g)(\widehat{\mathcal{R}} \cap \widehat{\mathcal{S}})).$$

Since $\widehat{\mathcal{R}}$ is foliated by level sets of f starting on the left side $\partial_L \widehat{\mathcal{R}} = \partial_L \mathcal{R}_{jk}$, the oscillation of f on $\widehat{\mathcal{R}}$, namely $\sup_{\widehat{\mathcal{R}}} f - \inf_{\widehat{\mathcal{R}}} f$, is bounded by (and in fact equal to)

$$\int_{\partial_L \widehat{\mathcal{R}}} |\partial_y f| = \int_{\partial_L \widehat{\mathcal{R}}} Z^x = \frac{1}{2} |\operatorname{div}(\mathbf{1}_{\widehat{\mathcal{R}}} Z)|(\mathbb{R}^2),$$

where we used the fact that $\int_{\partial_L \widehat{\mathcal{R}}} Z^x = \int_{\partial_R \widehat{\mathcal{R}}} Z^x$ in the last equality. Hence, the image of $(f, g)|_{\widehat{\mathcal{R}} \cap \widehat{\mathcal{S}}}$ is included in a rectangle of area

$$\frac{1}{4} |\operatorname{div}(\mathbf{1}_{\widehat{\mathcal{R}}} Z)|(\mathbb{R}^2)\, |\operatorname{div}(\mathbf{1}_{\widehat{\mathcal{S}}} W)|(\mathbb{R}^2),$$

which proves (15) and thus Theorem 3.5.

3.2. **A more general bound.** We present here an intermediate version, which will be instrumental in obtaining Theorem 3.1.

For simplicity, assume again that $S = \alpha Z$ and $T = \beta W$, for (smooth, complete) divergence-free vector fields Z, W with $Z^x, W^y > 0$, such that all their integral curves are graphs over the entire horizontal and vertical axes, respectively. We assume that $\alpha, \beta \in C_c^\infty(\mathbb{R}^2)$ are nonnegative, but we drop the assumption that $\det(S,T) \geq 0$.

Given $j, k \in \mathbb{Z}$, we let $\mathcal{U}_j^x$ be the *vertical* stripe $[j, j+1] \times \mathbb{R}$, while $\mathcal{U}_k^y$ will denote the *horizontal* stripe $\mathbb{R} \times [k, k+1]$.

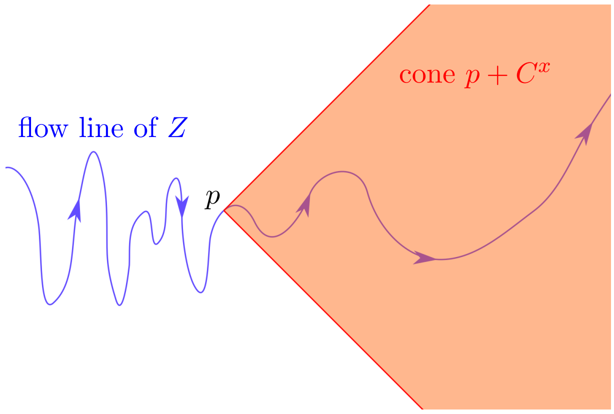
Definition 3.8. We define the cones

$$C^x := \{(x,y) : |y| < x\}, \quad C^y := \{(x,y) : |x| < y\}.$$

Also, given $p \in \mathcal{U}_j^x$, consider the integral curve $\gamma_p : \mathbb{R} \to \mathbb{R}^2$ of Z with initial condition $\gamma_p(0) = p$ and let $\tau_p \geq 0$ be such that $\gamma_p^x(\tau_p) = j+1$ (so that γ_p leaves $\mathcal{U}_j^x$ after τ_p). We then define $G_j^x \subseteq \mathcal{U}_j^x$ to be the (Borel) set of points p such that $\gamma_p(t) \in p + C^x$ for $t \in (0, \tau_p]$. The set $G_k^y \subseteq \mathcal{U}_k^y$ is defined analogously (with W and C^y in place of Z and C^x). Finally, we let

$$G_{jk} := G_j^x \cap G_k^y.$$

The following picture illustrates a typical point $p \in G_j^x$.



Remark 3.9. The key property of this set is that, whenever $p, q \in G_{jk}$, the two points cannot belong to the same integral curves for Z and W, unless $p = q$. Indeed, if they were distinct points on the same integral curve of Z, then either $q \in p + C^x$ or $p \in q + C^x$; in both cases, it would follow that $|p^y - q^y| < |p^x - q^x|$. The same argument for W would give the reverse inequality and thus a contradiction.

We will prove that, for modified vector fields S_f and T_f, the inequality

$$\int_{G_{jk}} \det(S_f, T_f) \leq \int_{\mathcal{U}_j^x} (S^x + |\operatorname{div} S|) \int_{\mathcal{U}_k^y} (T^y + |\operatorname{div} T|) \tag{16}$$

holds. In order to reach this inequality, we need to localize the proof of Theorem 3.5. Bounding the integral of $\det(S_f, T_f)$ on the set G_{jk}, contained in the square $\mathcal{U}_j^x \cap \mathcal{U}_k^y$, essentially corresponds to a bound for $\int_{\mathcal{R} \cap \mathcal{S}} \det(Z, W)$ in the previous proof, for two regions $\mathcal{R}$ and $\mathcal{S}$, rather than two chains $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{S}}$.

We modify the proof from the previous subsection as follows. Given any $p \in \mathcal{U}_j^x$, we can write $p = \gamma(t)$ for a unique integral curve $\gamma : [0, T] \to \mathcal{U}_j^x$ of Z starting on the left side of $\mathcal{U}_j^x$ and ending on the right side (namely, $\gamma^x(0) = j$ and $\gamma^x(T) = j+1$). We then let

$$\alpha_f(p) := \min_{[0,T]} (\alpha \circ \gamma), \quad S_f := \alpha_f Z. \tag{17}$$

Similarly we define β_f and T_f on $\mathcal{U}_k^y$.

Since α_f is constant along integral curves of Z, the (continuous) vector field $\alpha_f Z$ is still divergence-free on $\mathcal{U}_j^x$. As in the previous subsection, we claim that

$$\int_{G_{jk}} \det(S_f, T_f) \le \Big(\int_{\{j\}\times\mathbb{R}} S^x \, d\mathcal{H}^1\Big)\Big(\int_{\mathbb{R}\times\{k\}} T^y \, d\mathcal{H}^1\Big). \tag{18}$$

Indeed, since S_f is divergence-free, we can write $S_f = \nabla^\perp\varphi$ and $T_f = \nabla^\perp\psi$ for two C^1 functions φ and ψ (on $\mathcal{U}_j^x$ and $\mathcal{U}_k^y$, respectively). The integral curves of Z where $\alpha_f > 0$ correspond precisely to the regular level sets of φ.

By Remark 3.9, we can then assert that the map (φ, ψ) is injective on $G_{jk} \cap \{\alpha_f\beta_f > 0\}$. Hence, by the area formula, $\int_{G_{jk}} \det(S_f, T_f) = \int_{G_{jk}} \det(\alpha_f Z, \beta_f W)$ is bounded by the area of the image of this map.

The oscillation of φ (namely, $\sup_{\mathcal{U}_j^x} \varphi - \inf_{\mathcal{U}_j^x} \varphi$) is bounded by (and actually equal to)

$$\int_{\{j\}\times\mathbb{R}} |\partial_y \varphi| = \int_{\{j\}\times\mathbb{R}} \alpha_f Z^x \le \int_{\{j\}\times\mathbb{R}} \alpha Z^x = \int_{\{j\}\times\mathbb{R}} S^x,$$

thanks to the fact that $\alpha_f \le \alpha$. This, together with the same bound for the oscillation of ψ, gives our claim (18).

Further, note that

$$\int_{\{j\}\times\mathbb{R}} S^x = \int_{\{s\}\times\mathbb{R}} S^x - \int_{[j,s]\times\mathbb{R}} \operatorname{div} S$$

for all $s \in [j, j+1]$. Averaging over this interval, we obtain

$$\int_{\{j\}\times\mathbb{R}} S^x \le \int_{\mathcal{U}_j^x} (S^x + |\operatorname{div} S|).$$

This proves (16).

3.3. **Complementary inequalities.** The vector field $S_f = \alpha_f Z$ admits a useful estimate on the complement $\mathcal{U}_j^x \setminus G_j^x$, as we now show.

Proposition 3.10. We have the bound

$$\int_{\mathcal{U}_j^x \setminus G_j^x} (S_f^x - |S_f^y|)^+ \le \int_{\mathcal{U}_j^x} (S_f^x - |S_f^y|)^-,$$

as well as the analogous one for T_f on $\mathcal{U}_k^y$ (with x and y interchanged).

This fact will be a direct consequence of the next elementary lemma.

Lemma 3.11. Given $\xi \in W^{1,1}([0,L], \mathbb{R})$ (continuous), define the Borel set

$$E := \{s \in [0, L] : \xi(t) > \xi(s) \text{ for all } t > s\}.$$

Then we have

$$\int_{[0,L]\setminus E} \dot{\xi}^+ \le \int_{[0,L]} \dot{\xi}^-.$$

Proof. Let $I := [0, L]$. If $\xi(L) \le \xi(0)$ then $\int_I \dot{\xi} \le 0$, hence $\int_I \dot{\xi}^+ \le \int_I \dot{\xi}^-$, and the claim follows in this case.

Assume now $\xi(L) > \xi(0)$. The image $\xi(E)$ includes $[\xi(0), \xi(L)]$ since, for any $\xi(0) \le \lambda \le \xi(L)$, we have $\max f^{-1}(\lambda) \in E$. Since $E \subseteq \{\dot{\xi} \ge 0\}$ up to negligible sets, the area formula gives

$$\int_E \dot{\xi}^+ = \int_E |\dot{\xi}| \ge \xi(L) - \xi(0) = \int_I \dot{\xi}^+ - \int_I \dot{\xi}^-. \qquad \square$$

Proof of Proposition 3.10. As above, we write $S_f = \nabla^\perp\varphi$. Given a regular level set of φ (where $\alpha_f > 0$), we can parametrize it with an integral curve $\gamma : [0, L] \to \mathcal{U}_j^x$ of $Z/|Z|$, with $\gamma^x(0) = j$ and $\gamma^x(L) = j + 1$. Note that γ has unit speed and that

$$\dot\gamma = \frac{(-\partial_y\varphi, \partial_x\varphi)}{|d\varphi|} \circ \gamma.$$

We apply the previous lemma with $\xi(t) := \int_0^t (\dot\gamma^x - |\dot\gamma^y|)$. With E as in the statement of the lemma, if $s \in E$ then for all $t > s$ we have

$$0 < \xi(t) - \xi(s) = \gamma^x(t) - \gamma^x(s) - \int_s^t |\dot\gamma^y| \le \gamma^x(t) - \gamma^x(s) - |\gamma^y(t) - \gamma^y(s)|.$$

This means that $\gamma(t) - \gamma(s) \in C^x$, hence $\gamma(s) \in G_j^x$.

Since $\dot\xi = \frac{-\partial_y\varphi - |\partial_x\varphi|}{|d\varphi|} \circ \gamma$, the lemma gives

$$\int_{[0,L]\setminus E} \frac{(-\partial_y\varphi - |\partial_x\varphi|)^+}{|d\varphi|} \circ \gamma \le \int_0^L \frac{(-\partial_y\varphi - |\partial_x\varphi|)^-}{|d\varphi|} \circ \gamma.$$

Integrating over all regular level sets of φ and using the coarea formula, we get

$$\int_{\mathcal{U}_j^x \setminus G_j^x} (-\partial_y\varphi - |\partial_x\varphi|)^+ \le \int_{\mathcal{U}_j^x} (-\partial_y\varphi - |\partial_x\varphi|)^-.$$

Since $\nabla^\perp\varphi = S_f$, the claim follows. □

We now turn to estimate

$$S_d := S - S_f = \alpha_d Z, \quad \alpha_d := \alpha - \alpha_f. \tag{19}$$

Recall that $0 \le \alpha_f \le \alpha$, which gives $0 \le \alpha_d \le \alpha$.

Proposition 3.12. For any $\Lambda > 2$, we have

$$\int_{\mathcal{U}_j^x} (S_d^x - |S_d^y|)^+ \le \Lambda \int_{\mathcal{U}_j^x} |\operatorname{div} S| + \frac{1}{\Lambda - 2} \int_{\mathcal{U}_j^x} (S_d^x - |S_d^y|)^-.$$

Proof. Writing $Z = \nabla^\perp f$, consider a level set $\gamma : [0, L] \to \mathcal{U}_j^x$ of f, parametrized by arclength. We have

$$\int_0^L \alpha_d(\dot\gamma^x - |\dot\gamma^y|)^+ \, dt = \int_0^{\max \alpha_d} \int_{\{\alpha_d > s\}} (\dot\gamma^x - |\dot\gamma^y|)^+ \, dt \, ds,$$

where we write α_d in place of $\alpha_d \circ \gamma$ for simplicity. We distinguish two cases: if the length $\mathcal{H}^1(\gamma \cap \{\alpha_d > s\}) \le \Lambda$, we just bound the inner integral by Λ.

Otherwise, note that, since $\dot\gamma^x + |\dot\gamma^y| \ge 1$ and $\int_{\{\alpha_d > s\}} \dot\gamma^x \le \int_0^L \dot\gamma^x = 1$ (as $\dot\gamma^x > 0$), we must have

$$\int_{\{\alpha_d > s\}} (\dot\gamma^x - |\dot\gamma^y|)^- \ge \int_{\{\alpha_d > s\}} (|\dot\gamma^y| - \dot\gamma^x) \ge \int_{\{\alpha_d > s\}} (1 - 2\dot\gamma^x) \ge \Lambda - 2,$$

which gives

$$\int_{\{\alpha_d > s\}} (\dot\gamma^x - |\dot\gamma^y|)^+ \le \int_{\{\alpha_d > s\}} \dot\gamma^x \le 1 \le \frac{1}{\Lambda - 2} \int_{\{\alpha_d > s\}} (\dot\gamma^x - |\dot\gamma^y|)^-.$$

Also, the maximum of α_d along γ is bounded by $\int_0^L |\dot\alpha_d|$. Since $\dot\gamma = \frac{Z}{|Z|}$ and $\operatorname{div} Z = 0$, we have $|\dot\alpha_d| = \frac{|\operatorname{div} S_d|}{|Z|} = \frac{|\operatorname{div} S|}{|df|}$, where we omit composition with γ. To sum up, recalling that $\dot\gamma = \frac{(-\partial_y f, \partial_x f)}{|df|}$, we get

$$\int_0^L \alpha_d \frac{(-\partial_y f - |\partial_x f|)^+}{|df|} \circ \gamma \le \Lambda \max \alpha_d + \frac{1}{\Lambda - 2} \int_0^\infty \int_{\{\alpha_d > s\}} \frac{(-\partial_y f - |\partial_x f|)^-}{|df|} \circ \gamma(t)\, dt\, ds$$
$$\le \Lambda \int_0^L \frac{|\operatorname{div} S|}{|df|} \circ \gamma + \frac{1}{\Lambda - 2} \int_0^L \alpha_d \frac{(-\partial_y f - |\partial_x f|)^-}{|df|} \circ \gamma.$$

The claim follows using the coarea formula for f. □

3.4. **Proof of Theorem 3.1.** By approximation, it is enough to show the claim assuming that $\chi \in C^0_c(\mathbb{R}^2)$ and (given such χ) that $S, T \in C^\infty_c(\mathbb{R}^2, \mathbb{R}^2)$. Indeed, if $S_\varepsilon, T_\varepsilon \rightharpoonup S, T$ as measures, with $|S_\varepsilon|, |T_\varepsilon| \rightharpoonup |S|, |T|$ and $|\operatorname{div} S_\varepsilon|, |\operatorname{div} T_\varepsilon| \rightharpoonup |\operatorname{div} S|, |\operatorname{div} T|$ tightly, then by Reshetnyak's continuity principle we also have

$$\int_{\mathbb{R}^2} S_\varepsilon^N \to \int_{\mathbb{R}^2} S^N, \quad \int_{\mathbb{R}^2} T_\varepsilon^N \to \int_{\mathbb{R}^2} T^N,$$

and similarly (since $|(S_\varepsilon, T_\varepsilon)| \rightharpoonup |(S,T)|$) also

$$\int_{\mathbb{R}^2} \min\{S_\varepsilon^P, T_\varepsilon^P\} \rightharpoonup \int_{\mathbb{R}^2} \min\{S^P, T^P\}.$$

As in the previous proofs, we can also suppose that

$$S = \alpha Z, \quad T = \beta W,$$

for two smooth, complete, divergence-free vector fields Z, W satisfying $Z^x, W^y > 0$ (for nonnegative coefficients $\alpha, \beta \in C^\infty_c(\mathbb{R}^2)$). Also, we can assume that maximal integral curves of Z are graphs over the (entire) horizontal axis, while those of W are graphs over the vertical axis.

As in the previous subsection, we define the bands $\mathcal{U}^x_j := [j, j+1] \times \mathbb{R}$ and $\mathcal{U}^y_k := \mathbb{R} \times [k, k+1]$, and we let the sets G^x_j, G^y_k, and $G_{jk} = G^x_j \cap G^y_k$ be as in Definition 3.8.

Let us split again $S = S_f + S_d$ (see (17) and (19)) on each band $\mathcal{U}^x_j$ and define $S^P_f, S^N_f, S^P_d, S^N_d$ as in (11) (for instance, $S^P_f := (S^x_f - |S^y_f|)^+$), as well as the analogous objects for T. Proposition 3.10 and Proposition 3.12 show that

$$\int_{\mathcal{U}^x_j \setminus G^x_j} S^P_f \le \int_{\mathcal{U}^x_j} S^N_f$$

and

$$\int_{\mathcal{U}^x_j} S^P_d \le \Lambda \int_{\mathcal{U}^x_j} |\operatorname{div} S| + \frac{1}{\Lambda - 2} \int_{\mathcal{U}^x_j} S^N_d.$$

Setting $G^x := \bigcup_{j \in \mathbb{Z}} G^x_j$ and summing the previous bounds over j, we then get

$$\int_{\mathbb{R}^2 \setminus G^x} S^P_f + \int_{\mathbb{R}^2} S^P_d \le \Lambda \int_{\mathbb{R}^2} |\operatorname{div} S| + \int_{\mathbb{R}^2} S^N_f + \frac{1}{\Lambda - 2} \int_{\mathbb{R}^2} S^N_d.$$

To continue, recall that S_f and S_d are nonnegative multiples of Z. This implies that $S^P = S^P_f + S^P_d$ and $S^N = S^N_f + S^N_d$. Hence, fixing any $\Lambda \ge 3$ we get

$$\int_{\mathbb{R}^2 \setminus G^x} S^P_f + \int_{\mathbb{R}^2} S^P_d \le \Lambda \int_{\mathbb{R}^2} |\operatorname{div} S| + \int_{\mathbb{R}^2} S^N. \tag{20}$$

Also, we can bound

$$\min\{S^P, T^P\} \le \min\{S^P_f, T^P_f\} + S^P_d + T^P_d \le \sqrt{S^P_f T^P_f} + S^P_d + T^P_d.$$

With $G := G^x \cap G^y$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^2} \chi \min\{S^P, T^P\} &\le \int_G \chi \min\{S_f^P, T_f^P\} + \int_{\mathbb{R}^2\setminus G} \min\{S_f^P, T_f^P\} + \int_{\mathbb{R}^2} (S_d^P + T_d^P) \\ &\le \int_G \chi \sqrt{S_f^P T_f^P} + \int_{\mathbb{R}^2\setminus G^x} S_f^P + \int_{\mathbb{R}^2\setminus G^y} T_f^P + \int_{\mathbb{R}^2} (S_d^P + T_d^P) \\ &\le \int_G \chi \sqrt{S_f^P T_f^P} + \Lambda \int_{\mathbb{R}^2} (|\operatorname{div} S| + |\operatorname{div} T|) + \int_{\mathbb{R}^2} (S^N + T^N),\end{aligned}$$

where we used (20) and its analogue for T in the last inequality.

Finally, we have $S_f^P T_f^P \le C \det(S_f, T_f)$ on G by the elementary bound

$$(v^x - |v^y|)(w^y - |w^x|) \le C \det(v, w)$$

for vectors $v \in \overline{C^x}$ and $w \in \overline{C^y}$ (note that $S_f \in \overline{C^x}$ on G, except possibly on the negligible set $\mathbb{Z} \times \mathbb{R}$, and similarly $T_f \in \overline{C^y}$). In turn, this bound follows from the fact that, assuming $|v| = |w| = 1$, $\det(v, w)$ is comparable with $\min\{|v - w|, |v + w|\}$, while $v^x - |v^y|$ is comparable with $\operatorname{dist}(v, \partial C^x)$, and similarly $w^x - |w^y|$ is comparable with $\operatorname{dist}(w, \partial C^y)$.

From this remark and Cauchy–Schwarz it follows that

$$\int_G \chi \sqrt{S_f^P T_f^P} \le C\Big(\int_{\mathbb{R}^2} \chi^2\Big)^{1/2}\Big(\int_G \det(S_f, T_f)\Big)^{1/2}.$$

However, since $G = \bigcup_{j,k} G_{jk}$, by (16) we have

$$\begin{aligned}\int_G \det(S_f, T_f) &\le \sum_{j,k} \int_{\mathcal{U}_j^x} (S^x + |\operatorname{div} S|) \int_{\mathcal{U}_k^y} (T^y + |\operatorname{div} T|) \\ &\le \int_{\mathbb{R}^2} (|S| + |\operatorname{div} S|) \int_{\mathbb{R}^2} (|T| + |\operatorname{div} T|).\end{aligned}$$

This concludes the proof of Theorem 3.1.

4. Proof of Michael–Simon for anisotropies close to the area

In this section we deduce Theorem 1.3 from the nonlinear inequality stated in Theorem 3.1. Given a rectifiable 2-varifold V with finite total mass and first variation, we identify it with a measure on $\mathbb{R}^3 \times \mathbb{S}^2$. We can require that such measure is invariant under $(x, \nu) \mapsto (x, -\nu)$ in order to have a unique identification, although this is not really necessary. In the sequel, we let

$$\Pi := \pi_{x,y} \circ \pi : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^2, \quad \text{i.e.,} \quad \Pi((x, y, z), P) := (x, y).$$

Taking $\psi \in C_c^1(\mathbb{R}^2)$ (viewed as a function of x, y and identified with a map in $C^1(\mathbb{R}^3)$ by $(x, y, z) \mapsto \psi(x, y)$), we formally have

$$\langle \delta^F V, \psi \partial_x \rangle = \int_{\mathbb{R}^2} \partial_x \psi \, d\mathcal{A}_x^x + \int_{\mathbb{R}^2} \partial_y \psi \, d\mathcal{A}_x^y, \tag{21}$$

where, thanks to (8), the measures $\mathcal{A}_x^x$ and $\mathcal{A}_x^y$ are given by

$$\mathcal{A}_x^x = \Pi_*[(F(\nu) - \nu^x \partial_x F(\nu))V(\cdot, \nu)] = \Pi_*[(\nu^y \partial_y F(\nu) + \nu^z \partial_z F(\nu))V(\cdot, \nu)] \tag{22}$$

(we used the fact that $F(\nu) = dF(\nu)[\nu]$ by 1-homogeneity of F) and

$$\mathcal{A}_x^y = \Pi_*[-\nu^x \partial_y F(\nu) V(\cdot, \nu)]. \tag{23}$$

In fact, a straightforward cut-off argument (using the fact that V has finite mass) shows that the right-hand side of (21) is bounded by $|\delta^F V|(\mathbb{R}^3)\|\psi\|_{C^0}$, even if we are using a vector field which is not compactly supported.

Similarly, we have

$$\langle \delta^F V, \psi \partial_y \rangle = \int_{\mathbb{R}^2} \partial_x \psi \, d\mathcal{A}_y^x + \int_{\mathbb{R}^2} \partial_y \psi \, d\mathcal{A}_y^y,$$

with

$$\begin{aligned} \mathcal{A}_y^x &= \Pi_*[-\nu^y \partial_x F(\nu) V(\cdot,\nu)], \\ \mathcal{A}_y^y &= \Pi_*[(\nu^x \partial_x F(\nu) + \nu^z \partial_z F(\nu)) V(\cdot,\nu)]. \end{aligned} \tag{24}$$

For the area we have $F(\nu) = |\nu|$, so that

$$\begin{pmatrix} \mathcal{A}_x^x & \mathcal{A}_x^y \\ \mathcal{A}_y^x & \mathcal{A}_y^y \end{pmatrix} = \Pi_* \left[\begin{pmatrix} (\nu^y)^2 + (\nu^z)^2 & -\nu^x \nu^y \\ -\nu^x \nu^y & (\nu^x)^2 + (\nu^z)^2 \end{pmatrix} V(\cdot,\nu) \right] \tag{25}$$

is symmetric and positive semidefinite. Note carefully that the same matrix $\mathcal{A}$ given by (22)–(24) is not symmetric for a general F, nor it has (formally) nonnegative determinant in general. However, we now show that one can always reduce to the case where the diagonal entries are nonnegative, assuming that F is convex. In fact, the following holds in arbitrary dimension (when $k = n-1$), with the same proof.

Proposition 4.1. Assume that F is convex (which holds if F satisfies (AC)). Given a varifold W, there exists a linear isomorphism $L \in \mathrm{SL}(3)$, depending only on F, such that the matrix-valued measure $\mathcal{A}$, associated with L_*F and the varifold $V := L_*W$, satisfies

$$\mathcal{A}_x^x, \mathcal{A}_y^y \geq 0. \tag{26}$$

Also, up to composing L with a permutation of the coordinates, we can assume that

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} (\nu^z)^2 \, dV(p,\nu) \geq \frac{1}{3} |V|(\mathbb{R}^3), \tag{27}$$

while trivially $|\delta^{L_*F} V|(\mathbb{R}^3) \leq C(F) |\delta^F W|(\mathbb{R}^3)$.

In this statement, the integrand L_*F is defined in such a way that, denoting by $L_*\mathcal{F}$ the corresponding anisotropic area, the formula $L_*\mathcal{F}(L_*W) = \mathcal{F}(W)$ holds. Namely, noting that if $\nu \perp P \in \mathrm{Gr}_2(\mathbb{R}^3)$ then $L^T\nu \perp L^{-1}(P)$ and $|L^T \nu|$ is precisely $J_{L^{-1}}(P)$ (the Jacobian of L^{-1} along P), we let

$$L_*F(\nu) := |L^T \nu| F\Big(\frac{L^T \nu}{|L^T \nu|}\Big) = F(L^T \nu)$$

for every $\nu \in \mathbb{S}^2$. Using the definition of first variation, it is easy to check that

$$\langle \delta^{L_*F}(V), X \rangle = \langle \delta^F(W), Y \rangle, \quad Y(p) := L^{-1} X(Lp),$$

for any vector field $X \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$, which implies the last part of the previous statement.

Proof. The desired condition $\mathcal{A}_x^x, \mathcal{A}_y^y \geq 0$ holds for $G = L_*F$ provided that, using indices in $\{1,2,3\}$ (rather than $\{x,y,z\}$), we have

$$dG(\nu)[\nu - \nu^1 e_1] \geq 0, \quad dG(\nu)[\nu - \nu^2 e_2] \geq 0 \tag{28}$$

for all $\nu = (\nu^1, \nu^2, \nu^3) \in \mathbb{R}^3$. We claim that the last property holds if the convex set

$$K := \{P \,:\, G(P) \leq 1\}$$

has the property that $e_1^\perp$ is a supporting hyperplane at the point $p_1 := K \cap \mathbb{R}^+ e_1$, and the same holds for e_2.

To check this, note that $G = 1$ on ∂K. Hence, assuming the last geometric condition, the gradient $\nabla G(p_1)$ is parallel to e_1. Thus, the first inequality in (28) is actually an equality at p_1, and also at $-p_1$ by symmetry. Given any $\nu \in \partial K \setminus \{\pm p_1\}$, let us write

$\nu = ap_1 + w$ with $w \perp e_1$. Again, $\nabla G(\nu)^\perp$ is a supporting hyperplane for K at ν, meaning that

$$\langle \nabla G(\nu), \nu - p\rangle \geq 0 \quad \text{for all } p \in K.$$

By the geometric condition we have $|a| \leq 1$ (since K lies between the two affine planes $-p_1 + e_1^\perp$ and $p_1 + e_1^\perp$). Hence, we can take $p := ap_1 \in K$ and deduce that

$$\langle \nabla G(\nu), \nu - \nu^1 e_1\rangle = \langle \nabla G(\nu), w\rangle = \langle \nabla G(\nu), \nu - p\rangle \geq 0.$$

By 1-homogeneity, the same is true for all $\nu \in \mathbb{R}^3$. The same holds for e_2, giving (28).

To conclude, we just need to find a linear (orientation-preserving) transformation T such that the image of K has the desired geometric property (which in fact will hold for all e_i). Once this is done, up to a permutation of the coordinates we can also guarantee (27), since $(\nu^x)^2 + (\nu^y)^2 + (\nu^z)^2 = 1$.

In order to find T, we maximize the volume $\mathcal{L}^3(T(K))$ over all $T \in GL^+(\mathbb{R}^3)$ such that $T(K)$ is a subset of $Q := [-1,1]^3$. This is equivalent to maximize $\det(T)$ under the same constraint, and it is easy to check that the maximum is indeed achieved. Letting $K' := T(K)$, we claim that K' contains $\pm e_i$ for $i = 1,2,3$, which implies (together with $K' \subseteq Q$) that K' has the desired property. Indeed, if for instance we have $e_1 \notin K'$, then by Hahn–Banach we can find a linear functional λ such that $\lambda(e_1) > 1$ and $\lambda(p) < 1$ for all $p \in K'$. By symmetry, we have

$$K' \subseteq \{|\lambda| \leq 1\} \cap \{|e_2^*| \leq 1\} \cap \{|e_3^*| \leq 1\} =: Q'.$$

However, Q' is strictly contained in the set where the first constraint is replaced by $\{|\lambda| \leq \lambda(e_1)\}$, which has the same volume as Q (since its intersection with any line parallel to e_1 has the same length as the intersection with Q). Hence, taking a linear map $S \in GL^+(\mathbb{R}^3)$ such that $S(Q') = Q$, we must have $\det(S) > 1$ and $ST(K) \subseteq S(Q') = Q$, contradicting the maximality of $\det(T)$ (the map S is found by requiring that $\lambda S^{-1} = e_1^*$, $e_2^* S = e_2^*$, and $e_3^* S = e_3^*$, i.e., it is the matrix with rows λ, e_2^*, e_3^*). □

We assume that the linear change of coordinates given by Proposition 4.1 has already been applied, but we keep denoting by F the transformed integrand.

Remark 4.2. Note that, if F is close to the area (i.e., $\|F|_{\mathbb{S}^2} - 1\|_{C^1}$ is small), then L_*F is still close to it. Indeed, it is enough to check that the map T from the proof of Proposition 4.1 is close to a rotation. The map T was obtained by maximizing the volume of $T(K)$ (under the constraint that $T(K) \subseteq [-1,1]^3$), where $K := \{|F| \leq 1\}$. If K is the unit ball, then the constraint $\pm e_i \in T(K)$ obtained along the proof forces T to be a rotation. Hence, T must be close to a rotation by a straightforward compactness argument.

The vector fields

$$S := \mathcal{A}_x^x \partial_x + \mathcal{A}_x^y \partial_y, \quad T := \mathcal{A}_y^x \partial_x + \mathcal{A}_y^y \partial_y$$

are measures with total mass bounded by $C(F)|V|(\mathbb{R}^3)$ and have $S^x, T^y \geq 0$. Since $\delta^F V$ has finite total variation, by (21) the divergence of S is a measure with

$$|\operatorname{div} S|(\mathbb{R}^2) \leq |\delta^F V|(\mathbb{R}^3),$$

and the same holds for T.

Since the statement of Theorem 1.3 is scale-invariant, up to a dilation we can assume that $\mathcal{H}^2(\{\theta > 0\}) = \eta_0$, for a fixed small constant $\eta_0 > 0$ to be chosen later.

Before applying Theorem 3.1, let us first show two elementary inequalities for real numbers.

Lemma 4.3. Given $a, b, c \in \mathbb{R}$, we have

$$\min\{(b^2+c^2-ab)^+, (a^2+c^2-ab)^+\} \geq c^2 - \frac{a^2+b^2}{4}$$

and there exists $\gamma > 4$ (independent of a, b, c) such that

$$(b^2+c^2-ab)^- + (a^2+c^2-ab)^- \leq \frac{a^2+b^2}{\gamma}.$$

Proof. We can assume that $a^2+b^2+c^2 = 1$. The inequality $b^2+c^2-ab \geq c^2 - \frac{a^2}{4}$ gives immediately the first claim. Also, it shows that if $|c| \geq \frac{1}{2}$ then $b^2+c^2-ab \geq \frac{1-a^2}{4} \geq 0$, and similarly $a^2+c^2-ab \geq 0$, so that the second conclusion is trivial in this case.

Assuming $|c| \leq \frac{1}{2}$, from the same inequality we get $(b^2+c^2-ab)^- \leq \frac{a^2}{4}$. Similarly we have $(a^2+c^2-ab)^- \leq \frac{b^2}{4}$, and we deduce that

$$(b^2+c^2-ab)^- + (a^2+c^2-ab)^- \leq \frac{a^2+b^2}{4}. \tag{29}$$

We claim that equality can never happen, which implies the second conclusion. Indeed, in order to have equality in (29) we must have $(c^2 - \frac{a^2}{4})^- = \frac{a^2}{4}$ (hence, either $c = 0$ or $a = 0$) and $(c^2 - \frac{b^2}{4})^- = \frac{b^2}{4}$ (hence, either $c = 0$ or $b = 0$); thus, we must have $c = 0$ or $a = b = 0$. Since we are assuming $a^2+b^2+c^2 = 1$ and $|c| \leq \frac{1}{2}$, this forces $c = 0$, as well as equality in $b^2 - ab \geq -\frac{a^2}{4}$ and $a^2 - ab \geq -\frac{b^2}{4}$ (unless a or b vanish). Hence, $b = \frac{a}{2}$ and $a = \frac{b}{2}$, or equivalently $a = b = 0$ (we reach the same conclusion if $a = 0$ or $b = 0$). This however contradicts the assumption $a^2+b^2+c^2 = 1$. ☐

Since the assignment $(x, y) \mapsto (x - |y|)^-$ is subadditive, from the previous lemma (applied with $a := \nu^x$, $b := \nu^y$, and $c := \nu^z$) and (25) we deduce that

$$S^N + T^N = (\mathcal{A}^x_x - |\mathcal{A}^y_x|)^- + (\mathcal{A}^y_y - |\mathcal{A}^x_y|)^- \leq \Pi_*\Big[\frac{(\nu^x)^2 + (\nu^y)^2}{\gamma} V(\cdot, \nu)\Big]$$

when F is the area, which implies that

$$S^N + T^N \leq \Pi_*\Big[\Big(\frac{(\nu^x)^2 + (\nu^y)^2}{\gamma} + \varepsilon\Big) V(\cdot, \nu)\Big]$$

for an arbitrarily small $\varepsilon > 0$, if $F|_{\mathbb{S}^2}$ is close enough to the area in the C^1 topology.

Also, the previous lemma implies that

$$\min\{S^P, T^P\} \geq \Pi_*\Big[\Big((\nu^z)^2 - \frac{(\nu^x)^2 + (\nu^y)^2}{4} - \varepsilon\Big) V(\cdot, \nu)\Big]$$

for F close to the area.

Finally, we set $E := \pi_{x,y}(\{\theta > 0\})$, which is an analytic set; as such, we can find a Borel set $E' \supseteq E$ such that $\mathcal{L}^2(E' \setminus E) = 0$. Observing that S, T are concentrated on E' and applying Theorem 3.1 with $\chi := \mathbf{1}_E$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^3\times \mathrm{Gr}_2(\mathbb{R}^3)} \Big((\nu^z)^2 - \frac{(\nu^x)^2+(\nu^y)^2}{4} - \varepsilon\Big)\, dV(p,\nu) \\
&\le \int_E \min\{S^P, T^P\} \\
&\le C\mathcal{L}^2(E)^{1/2}\int_{\mathbb{R}^2}(|S|+|T|+|\mathrm{div}\, S|+|\mathrm{div}\, T|) \\
&\quad + \int_{\mathbb{R}^2}(C|\mathrm{div}\, S| + C|\mathrm{div}\, T| + S^N + T^N) \\
&\le C\mathcal{L}^2(E)^{1/2}|V|(\mathbb{R}^3) + C(\mathcal{L}^2(E)^{1/2}+1)|\delta^F V|(\mathbb{R}^3) \\
&\quad + \int_{\mathbb{R}^3\times \mathrm{Gr}_2(\mathbb{R}^3)} \Big(\frac{(\nu^x)^2+(\nu^y)^2}{\gamma} + \varepsilon\Big)\, dV(p,\nu).
\end{aligned}$$

Recalling that $(\nu^x)^2 + (\nu^y)^2 = 1 - (\nu^z)^2$ and (27), the fact that $\gamma > 4$ implies that $(\nu^z)^2 - (\frac{1}{4} + \frac{1}{\gamma})(1 - (\nu^z)^2)$ is at least $\frac{1}{3} - (\frac{1}{4} + \frac{1}{\gamma})\frac{2}{3} =: 2c > 0$ in average, and hence

$$c|V|(\mathbb{R}^3) \le C\mathcal{L}^2(E)^{1/2}|V|(\mathbb{R}^3) + (C\mathcal{L}^2(E)^{1/2}+1)|\delta^F V|(\mathbb{R}^3),$$

provided that $\varepsilon < c$. However, since $\mathcal{L}^2(E) \le \mathcal{H}^2(\{\theta > 0\}) = \eta_0$, we can now choose η_0 such that $C\sqrt{\eta_0} \le \frac{c}{2}$ and deduce that $|V|(\mathbb{R}^3) \le C|\delta^F V|(\mathbb{R}^3)$ (for a possibly different C). This completes the proof of Theorem 1.3.

5. Proof of Theorem 1.5

First of all, note that we have

$$\nu^x\partial_x F(\nu) \ge 0, \quad \nu^y\partial_y F(\nu) \ge 0, \quad \nu^z\partial_z F(\nu) \ge 0,$$

since F is convex and symmetric with respect to the coordinate planes. Moreover, the last assumption on F is easily checked to be equivalent to

$$(\nu^i\tilde\nu^j - \nu^j\tilde\nu^j)(\partial_i F(\nu)\partial_j F(\tilde\nu) - \partial_j F(\nu)\partial_i F(\tilde\nu)) \ge 0 \tag{30}$$

for every pair of indices $\{i,j\} \subset \{1,2,3\}$.

Given a rectifiable varifold V, since $\nu^x\partial_x F(\nu) + \nu^y\partial_y F(\nu) + \nu^z\partial_z F(\nu) = F(\nu)$, up to a permutation of the coordinates we can assume that

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} \nu^z\partial_z F(\nu)\, dV(p,\nu) \ge \frac{\mathcal{F}(V)}{3} \ge c|V|(\mathbb{R}^3).$$

Taking $\mathcal{A}$ as in the proof of Theorem 1.3, we claim that $\mathcal{A} \in \mathcal{M}^+$. Indeed, let us disintegrate V with respect to the projection Π. Writing

$$V(x,y,z,\nu) = \Pi_* V(x,y) \otimes \lambda_{x,y}(z,\nu)$$

for a family of probability measures $\lambda_{x,y}$ on $\mathbb{R}\times\mathbb{S}^2$, we have $\mathcal{A} = \Lambda\Pi_* V$, where

$$\Lambda(x,y) = \int_{\mathbb{R}\times\mathbb{S}^2} \begin{pmatrix} \nu^y\partial_y F(\nu) + \nu^z\partial_z F(\nu) & -\nu^x\partial_y F(\nu) \\ -\nu^y\partial_x F(\nu) & \nu^x\partial_x F(\nu) + \nu^z\partial_z F(\nu) \end{pmatrix} d\lambda_{x,y}(z,\nu).$$

This gives immediately $\mathcal{A}^x_x, \mathcal{A}^y_y \ge 0$. Finally, we have $\det(\Lambda) \ge 0$ pointwise since

$$\begin{aligned}
\det(\Lambda(x,y)) = \iint_{(\mathbb{R}\times\mathbb{S}^2)^2} &[(\nu^y\partial_y F(\nu) + \nu^z\partial_z F(\nu))(\tilde\nu^x\partial_x F(\tilde\nu) + \tilde\nu^z\partial_z F(\tilde\nu)) \\
&- \nu^x\partial_y F(\nu)\tilde\nu^y\partial_x F(\tilde\nu)]\, d\lambda_{x,y}(z,\nu)\, d\lambda_{x,y}(\tilde z,\tilde\nu);
\end{aligned}$$

after symmetrizing the integrand, namely summing the same expression after interchanging the roles of $\nu, \tilde{\nu}$ (and dividing by 2), using again the fact that $\nu^x \partial_x F(\nu) + \nu^y \partial_y F(\nu) + \nu^z \partial_z F(\nu) = F(\nu)$ we obtain that it equals

$$\begin{aligned}
&\frac{1}{2}[F(\nu)\tilde{\nu}^z \partial_z F(\tilde{\nu}) + F(\tilde{\nu})\nu^z \partial_z F(\nu)] + \frac{1}{2}[\nu^y \partial_y F(\nu)\tilde{\nu}^x \partial_x F(\tilde{\nu}) + \tilde{\nu}^y \partial_y F(\tilde{\nu})\nu^x \partial_x F(\nu) \\
&\quad - \nu^x \partial_y F(\nu)\tilde{\nu}^y \partial_x F(\tilde{\nu}) - \tilde{\nu}^x \partial_y F(\tilde{\nu})\nu^y \partial_x F(\nu)] \\
&\geq \frac{1}{2}[F(\nu)\tilde{\nu}^z \partial_z F(\tilde{\nu}) + F(\tilde{\nu})\nu^z \partial_z F(\nu)] \\
&\geq \nu^z \partial_z F(\nu)\tilde{\nu}^z \partial_z F(\tilde{\nu}),
\end{aligned}$$

thanks to (30). Hence, we obtain

$$\sqrt{\det(\Lambda(x,y))} \geq \int_{(\mathbb{R}\times\mathbb{S}^2)^2} \nu^z \partial_z F(\nu)\, d\lambda_{x,y}(z,\nu),$$

giving

$$\sqrt{\det(\mathcal{A})}(\mathbb{R}^3) = \int_{\mathbb{R}^2} \sqrt{\det(\Lambda)}\, d\Pi_* V \geq \int_{\mathbb{R}^3\times\mathbb{S}^2} \nu^z \partial_z F(\nu)\, dV(p,\nu).$$

We now conclude as in the proof of Theorem 1.3, using Corollary 3.6 in place of Theorem 3.1.

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