SCALE-INVARIANT BOUNDS ON THIN SETS FOR MEASURES SATISFYING A FIRST-ORDER PDE AND THE ANISOTROPIC MICHAEL–SIMON INEQUALITY

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ABSTRACT. We prove mass estimates for PDE-constrained measures on sets of σ-finite $\mathcal{H}^m$-measure, assuming an appropriate rank condition. As an application, we derive a Michael–Simon inequality for varifolds with bounded anisotropic first variation with respect to an integrand that satisfies the natural atomic condition (AC1).

1. INTRODUCTION

1.1. **Setting.** The understanding of the interplay between PDE constraints and possible patterns of oscillation and concentration of (sequences of) functions is a central topic in PDE and the Calculus of Variations and dates back to the work of Tartar on compensated compactness [21]. In the last decade there has been a renewed interest in it, especially in the form of the singular part of PDE-constrained measures and in the behavior of sequences of functions. In particular, in the work [12] the first and fourth named authors showed that the singular part of a measure satisfying a PDE constraint is forced to lie in the *wave cone* associated with that PDE operator. More precisely, specializing to the case of first-order operators, it was shown in [12, Theorem 1.1] that if V, W are finite-dimensional vector spaces, $\mathcal{A} : C^\infty(\mathbb{R}^d; V) \to C^\infty(\mathbb{R}^d; W)$ is a linear differential operator,

$$\mathcal{A} := \sum_{j=1}^{d} \mathcal{A}_j \partial_j, \quad \mathcal{A}_j \in \mathrm{Lin}(V, W),$$

and $\sigma \in \mathcal{M}(\mathbb{R}^d; V)$ is a Radon measure such that

$$\mathcal{A}\sigma = \nu \in \mathcal{M}(\mathbb{R}^d; W),$$

and E is such that $\mathcal{L}^d(E) = 0$, then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}|\sigma|}(x) \in \Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1} \mathrm{Ker}\, \mathbb{A}(\xi), \quad \mathbb{A}(\xi) := \sum_{j=1}^{d} \xi_j \mathcal{A}_j \in \mathrm{Lin}(V, W)$$

for $|\sigma|$-a.e. $x \in E$. This work was later extended in [5], where it was shown that further restrictions are imposed on the polar $\frac{\mathrm{d}\sigma}{\mathrm{d}|\sigma|}(x)$ on the part of σ which is singular with respect to lower-dimensional Hausdorff measures: If E is such that $\mathcal{H}^m(E) = 0$, then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}|\sigma|}(x) \in \Lambda^m_{\mathcal{A}} := \bigcap_{\pi\ m\text{-plane}} \bigcup_{\xi \in \pi \setminus \{0\}} \mathrm{Ker}\, \mathbb{A}(\xi).$$

In fact, in this statement one could replace the m-dimensional Hausdorff measure with the m-dimensional integral-geometric measure. In particular, by combining this with the Besicovitch–Federer projection theorem, one can recover various rectifiability results: see the surveys [13, 8] and the lecture notes [9]. These ideas were also behind the main result in [10], where the authors identified a necessary and sufficient condition on the anisotropic integrand for the validity of Allard's rectifiability theorem for varifolds with controlled anisotropic first variation: see Section 1.3 below for precise statements.

Besides these qualitative results, it is a natural question whether they can be quantified. This was done in a simple perturbative regime in [6]; see also [19] and the recent work [7]. Apart from its intrinsic interest, a quantitative enhancement of the results in [12] and [5] would have implications in recovering a priori bounds on concentration measures for various PDEs. In

particular, it was clear that these improvements would be a key step in proving a Michael–Simon-type estimate for varifolds with controlled first variation with respect to an anisotropic energy. This connection was used in [11] by the first and third named authors, building upon some unpublished ideas of Almgren: there, the validity of a Michael–Simon-type inequality for 2-dimensional varifolds in $\mathbb{R}^3$ was established under the additional assumption that the anisotropic energy is sufficiently close to the area. The same strategy was later extended to 2-dimensional varifolds in any codimension by Firester and Tsiamis in [17].

Recently, the second and fourth named authors proved (a generalized version of) Bouchitté's Vanishing Mass Conjecture [18]. Among the several consequences of the main result in [18] is a quantitative improvement of the results in [12], namely [18, Theorem 1.9]. Soon after the appearance of [18], Firester and Tsiamis applied the main result of [18] to prove a Michael–Simon inequality for stationary varifolds in various situations, covering in particular all convex anisotropies in codimension one [16].

The goal of this work is to provide a quantitative bound for measures that satisfy a first-order PDE constraint, based on the techniques introduced in [18]. It can be thought of as a first step towards a quantification of the result in [5]. As a main application, we show how this result implies the Michael–Simon inequality for varifolds with bounded first variation with respect to anisotropic energies satisfying a suitable (weak) atomic condition in all dimensions and codimensions.

1.2. **Statements of the main results.** We first state our main result for the PDE operator $\mathcal{A} = \operatorname{div}$, since it is the case that directly applies to the setting of varifolds with bounded anisotropic first variation.

Theorem 1.1. *Let $0 < m \le d \le D$ be integers and let $\mathcal{K} \subset \mathbb{R}^{D\times d}$ be a (nonempty) compact and convex set, with*

$$\lambda := \min_{X\in\mathcal{K}} \sigma_m(X) > 0, \quad \Lambda := \max_{X\in\mathcal{K}} \sigma_1(X) < \infty,$$

where $\sigma_1(X) \ge \cdots \ge \sigma_d(X) \ge 0$ denote the singular values of X. Let $M \subset \mathbb{R}^d$ be a Borel set with $\mathcal{H}^m(M) < \infty$ and let $\mu = \theta\mathcal{H}^m \llcorner M$ be a finite positive measure on $\mathbb{R}^d$. Suppose that $\sigma := X\mu$, where $X : M \to \mathcal{K}$, and that $\operatorname{div}\sigma$ (the row-wise divergence) is a finite measure. Then

$$\mu(\mathbb{R}^d) \le C(d,\lambda,\Lambda)\mathcal{H}^m(M)^{1/m}|\operatorname{div}\sigma|(\mathbb{R}^d). \tag{1.1}$$

Remark 1.2. *Thanks to* [5, Proposition 3.1], *the measure μ and hence the set M (up to replacing it with $M \cap \{\theta > 0\}$) are automatically m-rectifiable.*

Since, with the notation of the introduction, we have

$$\Lambda^m_{\operatorname{div}} = \{X \in \mathbb{R}^{D\times d} : \operatorname{rank} X < m\} = \{X : \sigma_m(X) = 0\},$$

the above result can be thought of as a quantitative enhancement of [5]. Note also that the condition that the matrix field X is valued in $\mathcal{K}$, with constants $0 < \lambda \le \Lambda < \infty$, is an ellipticity condition and enforces that the polar (and any of its averages) of the measure σ lies in a set that has positive distance from the m-dimensional wave cone. In particular, the convexity of $\mathcal{K}$ is as essential here as it is for [18, Theorem 1.9], because it entails that averages retain the same constraint.

The following generalization to arbitrary first-order operators $\mathcal{A}$ also holds and, in fact, it is easily seen to be equivalent to the previous statement.

Theorem 1.3. *Let $0 < m \le d$ be integers and consider a (nonempty) compact and convex set $\mathcal{K} \subset V$, as well as a first-order operator $\mathcal{A} = \sum_{j=1}^d \mathcal{A}_j\partial_j$, where $\mathcal{A}_j \in \operatorname{Lin}(V,W)$ are linear maps between Euclidean spaces. For $T_\kappa : \mathbb{R}^d \to W$ given by*

$$T_\kappa(\xi) := \sum_j \xi_j\mathcal{A}_j\kappa, \quad \kappa \in \mathcal{K},$$

assume that

$$\lambda := \min_{\kappa\in\mathcal{K}} \sigma_m(T_\kappa) > 0, \quad \Lambda := \max_{\kappa\in\mathcal{K}} \sigma_1(T_\kappa) < \infty.$$

Letting $M \subset \mathbb{R}^d$ and μ be as above, and $X : M \to \mathcal{K}$, $\sigma := X\mu$, suppose that $\mathcal{A}\sigma$ is a finite measure. Then

$$\mu(\mathbb{R}^d) \le C(\mathcal{A}, d, \lambda, \Lambda)\mathcal{H}^m(M)^{1/m}|\mathcal{A}\sigma|(\mathbb{R}^d). \tag{1.2}$$

To see that Theorem 1.3 follows from Theorem 1.1, identify $W = \mathbb{R}^D$ and let $\mathrm{d}\tilde{\sigma}(x) := T_{X(x)}\,\mathrm{d}\mu(x)$; equivalently, $\tilde{\sigma}$ is the matrix-valued measure with columns $\mathcal{A}_j\sigma$. Since $\operatorname{div}\tilde{\sigma} = \mathcal{A}\sigma$, one concludes that the statement of Theorem 1.1 in this case is Theorem 1.3.

The above results can be improved to deduce a higher integrability property for the density of μ, namely the following.

Theorem 1.4. *Assume that μ is a finite positive measure on $\mathbb{R}^d$ and that $\sigma := X\mu$, where $X : \mathbb{R}^d \to \mathcal{K}$ is as in Theorem 1.3. There exist $p \in (1, \infty)$ and $C > 0$, depending on d, λ, Λ, such that the following holds: If $\mathcal{A}\sigma$ is a finite measure and M is a Borel set with σ-finite $\mathcal{H}^m$-measure then, writing $\mu \llcorner M = \theta\mathcal{H}^m \llcorner M$, we have*

$$\|\theta\|_{L^{p,\infty}(\mathcal{H}^m\llcorner M)} \le \frac{C}{r^{m(p-1)/p}}\left[\mu(\mathbb{R}^d) + r|\mathcal{A}\sigma|(\mathbb{R}^d)\right]^{1/p}\mu(\mathbb{R}^d)^{1-1/p}$$

for all $r > 0$.

Remark 1.5. *By a standard layer-cake argument we also get uniform bounds on $\|\theta\|_{L^q(\mathcal{H}^m\llcorner M)}$ for all $q < p$. Whether the $L^{p,\infty}$ bounds extend up to the endpoint $p = m/(m-1)$ is left as an open question.*

1.3. **Anisotropic integrands and Michael–Simon inequality.** As an application of Theorem 1.1 we will extend the Michael–Simon inequality in [20] to varifolds with bounded anisotropic first variation.

The Michael–Simon inequality is a fundamental geometric inequality generalizing the classical isoperimetric inequalities to m-dimensional submanifolds of $\mathbb{R}^d$. In its simplest geometric form, it asserts that for an m-dimensional smooth submanifold M one has

$$\left(\mathcal{H}^m(M)\right)^{1/m} \le C(m,d)\left(\int_M |H_M| + \mathcal{H}^{m-1}(\partial M)\right)^{1/(m-1)}.$$

The natural setting for this inequality is that of rectifiable varifolds, as shown by Allard in his seminal paper [2]. In this setting, assuming density $\Theta_V(x) \ge 1$ for $|V|$-a.e. x, it reads as

$$\left(|V|(\mathbb{R}^d)\right)^{(m-1)/m} \le C(m,d)|\delta V|(\mathbb{R}^d).$$

Several geometric consequences can be obtained from this inequality, the most remarkable one being a lower density bound for minimal surfaces and stationary varifolds: If V is a stationary, rectifiable varifold in $B_r(x)$ with density at least 1 a.e. and $x \in \operatorname{spt} V$, then

$$|V|(B_\rho(x)) \gtrsim \rho^m \quad \text{for all } 0 < \rho < r.$$

This lower density bound turns out to be a key ingredient in establishing compactness of the class of rectifiable varifolds with bounded first variation and in the proof of the ε-regularity theorem [2]. Actually, as shown in [11], the validity of the Michael–Simon inequality is *equivalent* to the compactness theorem.

The original proof of the Michael–Simon inequality was based on the use of the monotonicity formula for the area integrand [20]. However, the validity of a monotonicity formula essentially characterizes the area integrand [3], and this has long prevented the extension of the Michael–Simon inequality to varifolds whose first variation is bounded with respect to anisotropic energy. As explained for instance in [11], this was essentially the only ingredient missing to extend the compactness and regularity theory to the anisotropic setting; see the survey [14] for a detailed account and also [11, 15].

To state our main result, let us recall some notation: Let $\mathrm{Gr}_m(\mathbb{R}^d)$ be the Grassmannian of m-planes in $\mathbb{R}^d$ and consider an autonomous integrand $F : \mathrm{Gr}_m(\mathbb{R}^d) \to (0, \infty)$ of class C^1. Define

$$\mathcal{F}(V) := \int_{\mathbb{R}^d} F(T_xV)\,\mathrm{d}|V|(x), \tag{1.3}$$

where V is an m-rectifiable varifold and T_xV is its tangent m-plane at $|V|$-a.e. point x. The first variation of $\mathcal{F}$ is given by

$$\langle \delta^F V, Y\rangle := \int B_F(T_xV) : DY(x)\,\mathrm{d}|V|(x)$$

for any vector field $Y \in \mathrm{C}^1_c(\mathbb{R}^d;\mathbb{R}^d)$, where $B_F : \mathrm{Gr}_m(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ is explicitly computable in terms of F [10, Lemma A.2]. Note that, in the isotropic case of the usual area, where $F \equiv 1$, $B_F(P)$ is simply the orthogonal projection onto P. In general, $B_F(P)$ has rank m and its transpose, $B_F(P)^T$, takes values in P (while $\mathrm{Ker}\, B_F(P) = P^\perp$), but it need not be symmetric or positive semidefinite.

An energy F is said to satisfy the *atomic condition (AC)* if

(i) the convex hull $\mathcal{K}_F := \mathrm{co}(\{B_F(P) \mid P \in \mathrm{Gr}_m(\mathbb{R}^d)\})$ consists of matrices of rank at least m;
(ii) non-extremal points of $\mathcal{K}_F$ have rank $> m$.

By itself, Part (i) is referred to as the *(AC1) condition*. The atomic condition was introduced in [10] as a very natural ellipticity condition, in that the analogue of Allard's rectifiability criterion for varifolds (i.e., that varifolds V with bounded anisotropic first variation are rectifiable on the set $\{x : \Theta^{m,*}(|V|,x) > 0\}$) holds if and only if (AC) is in place (see [10, Theorem 1.2] and [5, Proposition 3.1] for the improvement to positive upper density). We also note that, in codimension 1, (AC1) is equivalent to convexity of the integrand (once one identifies $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ with $\mathbb{S}^{d-1}/\{\pm 1\}$ and extends the resulting even function $F : \mathbb{S}^{d-1} \to (0,\infty)$ one-homogeneously).

We then have the following result.

Theorem 1.6. *Let F be an anisotropic integrand that satisfies (AC1). Then the Michael–Simon inequality*

$$|V|(\mathbb{R}^d) \le C(F)\mathcal{H}^m(\{\Theta_V \ge 1\})^{1/m}|\delta^F V|(\mathbb{R}^d) \tag{1.4}$$

holds for all rectifiable varifolds V with density $\Theta_V \ge 1$ $|V|$-a.e. (with both $|V|$ and $|\delta^F V|$ finite measures). As a consequence, for $m > 1$ we have

$$|V|(\mathbb{R}^d)^{(m-1)/m} \le C(F)|\delta^F V|(\mathbb{R}^d). \tag{1.5}$$

Proof. The rectifiable measure $\mathrm{d}\sigma(x) := B_F(T_xV)\,\mathrm{d}|V|(x)$ satisfies $\mathrm{div}\,\sigma = -\delta^F V$ and $B_F(T_xV)$ takes values in the compact convex set $\mathcal{K}_F$. By (AC1), every matrix in $\mathcal{K}_F$ has rank at least m and thus, since $\mathcal{K}_F$ is compact, we have $\lambda = \min_{X\in\mathcal{K}_F} \sigma_m(X) > 0$.

If $\Theta_V \ge 1$ $|V|$-a.e., we may take $M := \{\Theta_V \ge 1\}$ and apply Theorem 1.1, which gives (1.4). Noting that $\mathcal{H}^m(M) \le |V|(\mathbb{R}^d)$ can be absorbed into the left-hand side, we also get (1.5). □

As already mentioned, recent progress on similar results to Theorem 1.6 was made by the first and third named authors in [11] following unpublished work of Almgren, where such an inequality was obtained in the case $m = 2$ and $d = 3$ (hence codimension 1) for integrands F that are C^1-close to the area functional. Subsequently, Firester and Tsiamis in [17] extended this approach to 2-varifolds in any ambient dimension d. The same authors, exploiting the resolution of the Vanishing Mass Conjecture by the second and fourth named authors in [18], were able to extend inequality (1.5) to arbitrary dimension and codimension, retaining the assumption of C^1-closeness to the area, and to any convex anisotropy in codimension 1 [16, Theorem 1.1]. Such results have been more recently exploited to prove an anisotropic Allard regularity theorem in [15]. Theorem 1.6 settles the validity of an anisotropic Michael–Simon inequality (and thus the compactness problem) in full generality, by proving it in arbitrary dimension and codimension without asking for closeness to the area functional, and under the natural (AC1) condition.

1.4. **Strategy of proof.** For simplicity, we only discuss the strategy of proof of Theorem 1.1, which is strongly inspired by that of [18, Proposition 3.1]; see also the AI disclosure statement. A tentative strategy to prove Theorem 1.1 would be to restrict the measure σ to pieces of C^1 graphs and try to obtain good estimates on these pieces. Unfortunately, this is not a straightforward operation, as restricting the field σ to Borel sets does not preserve the PDE constraint (i.e., the bound on the divergence) and leads to additional difficulties (much like in the original proof of the singular density theorem [12], or [5]). Nevertheless, in this spirit, we restrict the "weight" μ

while keeping the same "polar" X: namely, we introduce a better-controlled positive measure μ_0 such that $\mu_0 \leq \mu$, where μ is the one in the statement of Theorem 1.1; it is actually given by $\mu_0 := \mu \llcorner E$, where E is an m-dimensional set with finite Minkowski content. The point is that we are still able to obtain useful bounds for such a μ_0, even if $X\mu_0$ does not have finite divergence, exploiting the tame m-dimensional nature of μ_0 and the fact that $\sigma = X\mu$ obeys a PDE constraint. Once this is done, a simple approximation argument allows us to infer a bound for the original measure μ.

The crucial part of the argument is to obtain bounds on the *entropy dissipation* for the "heated" μ_0. One can indeed define for $t \in [0, T)$ the density

$$\rho_t := \mathcal{P}_{T-t}\mu_0$$

and estimate the rate of growth of its entropy (which increases in t under the conventions used here). This rate is expressed in terms of the *Fisher information* (which is the "derivative" of the entropy functional). Roughly speaking, the control on the divergence of σ allows for an integral bound on the Fisher information

$$I(\rho_t) := -\int_{\mathbb{R}^d} \rho_t \operatorname{tr} \nabla^2 \log \rho_t \, \mathrm{d}x.$$

The point is that, after using a trivial upper bound for $-\nabla^2 \log \rho_t$ along the $d - m$ directions on which we have no control, we are able to get effective bounds along the m tangent directions by exploiting the PDE.

Finally, we can reinterpret the previous integral bound as an upper bound for the total gap

$$\limsup_{t \to T^-} \tilde{H}(\rho_t) - \tilde{H}(\rho_0),$$

where

$$\tilde{H}(\rho_t) := H(\rho_t) + \frac{(d-m)\mu_0(\mathbb{R}^d)}{2} \log(T - t)$$

is a renormalized entropy adapted to the diffusion of m-dimensional measures. While for $\tilde{H}(\rho_0)$ we can use a trivial estimate, the final part of the argument involves obtaining a lower bound on $\limsup_{t \to T^-} \tilde{H}(\rho_t)$ (roughly speaking, the renormalized entropy of μ_0) using the size of the support of μ_0 (together with the Minkowski content estimate).

Acknowledgements. The authors' research is funded by the European Research Council (ERC) and the UK Research and Innovation (UKRI) agency: GDP acknowledges support from CoG 101169953 "RISE", LG and FR acknowledge ERC/UKRI grant EP/Z000297/1 "CONCENTRATE", and AP acknowledges StG 101165368 "MAGNETIC".

Disclosure on AI usage. An initial proof of Theorem 1.1 was provided to the authors by a commercial AI model, in a conversation where it was initially asked if the techniques in [18] could be used to prove a sequential version of the result in [5] (in the spirit of the Vanishing Mass Conjecture). The model could not obtain that result but, when asked to show Theorem 1.1 instead, it provided an essentially complete proof, by combining one of the main new analytic ingredients of [18], namely [18, Lemma 3.2], with a log-Sobolev-type inequality. We later improved the result by also showing higher integrability of the density. For completeness, we report in [1] the draft provided by the AI model. All the proofs in the paper have been checked by the authors and they take full responsibility for them. AI was also used in the final stage to check for typos.

2. Notation

We write $|\nu|$ for the total variation measure associated with a (Borel) vector-valued measure ν on $\mathbb{R}^d$. Similarly, $|V|$ denotes the weight of a varifold V, i.e., the projection of V onto the base space $\mathbb{R}^d$. The divergence of a matrix-valued measure is taken row-wise, i.e., $(\operatorname{div} \nu)_i := \sum_{j=1}^d \partial_j \nu_{ij}$ as distributions.

For rectangular matrices of the same size, we use the Hilbert–Schmidt scalar product, namely

$$\langle A, B \rangle = A : B := \operatorname{tr}(A^T B), \quad |A|^2 = A : A.$$

For sets $E \subseteq \mathbb{R}^d$, we will write $B_r(E) := \{x : \mathrm{dist}(x, E) < r\}$ to denote the r-neighborhood. Given a measure space (X, μ) and $p \in [1, \infty)$, let us recall that the weak-L^p quasinorm of a measurable function $f : X \to \mathbb{R}$ is given by

$$\|f\|_{L^{p,\infty}(\mu)} := \sup_{u>0} u\mu(\{|f| > u\})^{1/p}.$$

We will often apply the heat flow semigroup to compactly supported measures ν: letting

$$p_s : \mathbb{R}^d \to (0, \infty), \quad p_s(x) := (4\pi s)^{-d/2} e^{-|x|^2/(4s)}$$

denote the usual fundamental solution of the heat equation, for any $s > 0$ we define

$$\mathcal{P}_s \nu := p_s * \nu.$$

Given a smooth $f : \mathbb{R}^d \to (0, \infty)$ with sufficiently fast decay at infinity (e.g., faster than $|x|^{-d-\varepsilon}$), we define its *entropy* to be

$$H(f) := \int_{\mathbb{R}^d} f \log f \, \mathrm{d}x.$$

Note that we use the convention making entropy increase along the *backward* heat flow. We will compute entropies of functions that do not necessarily satisfy $\int_{\mathbb{R}^d} f \, \mathrm{d}x = 1$.

3. Heat estimates

In this section we explain how to adapt the bounds from [18, Section 3] for our purposes, while keeping the exposition self-contained. For simplicity, let us assume that $|X| \leq \Lambda$ (up to enlarging Λ). Moreover, we assume throughout the section that μ has compact support and $\mu(\mathbb{R}^d) = 1$. Let us set $\delta := |\operatorname{div} \sigma|(\mathbb{R}^d)$ and fix $T > 0$. For $0 \leq t < T$, using the forward heat semigroup $(\mathcal{P}_s)_{s>0}$, we define

$$U_t := \mathcal{P}_{T-t}\mu, \quad V_t := \mathcal{P}_{T-t}\sigma, \quad R_t := \mathcal{P}_{T-t}|\operatorname{div} \sigma|, \quad S_t := U_t + \sqrt{T-t}\, R_t, \tag{3.1}$$

as well as

$$B_t := \frac{V_t}{S_t}, \quad \zeta_t := \nabla \log S_t = \frac{\nabla S_t}{S_t}, \quad \eta_t := \frac{\mathcal{P}_{T-t} \operatorname{div} \sigma}{S_t}, \quad \kappa_t := \frac{R_t}{2\sqrt{T-t}\, S_t}. \tag{3.2}$$

We will often drop the subscript t.

In particular, we have

$$\mathcal{H} S = -\kappa S, \quad \mathcal{H} V = 0, \quad \int_{\mathbb{R}^d} S_t \, \mathrm{d}x = 1 + \sqrt{T-t}\, \delta, \tag{3.3}$$

where $\mathcal{H} := \partial_t + \Delta$, while by convexity

$$\frac{V}{U} \in \mathcal{K}, \quad |B| \leq \Lambda. \tag{3.4}$$

For a (nonzero) submeasure $0 \leq \mu_0 \leq \mu$, set

$$\rho_t := \mathcal{P}_{T-t}\mu_0, \quad \gamma := \mu_0(\mathbb{R}^d),$$

and note that $0 < \gamma \leq 1$, as well as $0 \leq \rho \leq U \leq S$.

The following lemma replaces [18, Lemma 3.2]; compared to it, besides taking $a \equiv 1$, we use ρ in place of S. Moreover, as opposed to the argument in [18, Section 3], where ρ gets dissipated on the region where $q(B) \geq -\varepsilon$, here ρ simply evolves through the backward heat semigroup.

Lemma 3.1. *There exists a constant $C(d, \Lambda)$ such that*

$$\int_0^T \int_{\mathbb{R}^d} \rho\big(|\nabla B|^2 + |B \nabla \log \rho|^2\big) \, \mathrm{d}x \, \mathrm{d}t \leq C\gamma + C\gamma \log\left(\frac{1 + \sqrt{T}\, \delta}{\gamma}\right). \tag{3.5}$$

Proof. For any smooth φ we first compute that

$$\mathcal{H}(S\varphi) = S\mathcal{H}_\zeta\varphi - \kappa S\varphi, \tag{3.6}$$

where $\mathcal{H}_\zeta := \mathcal{H} + 2\zeta\cdot\nabla$, so that $\mathcal{H}_\zeta B = \kappa B$.

Letting

$$J(t) := \int_{\mathbb{R}^d} \rho_t \log(\rho_t/S_t)\,\mathrm{d}x,$$

using Jensen's inequality for $s\mapsto s\log s$, and recalling that $\rho\le S$, we get

$$\gamma\log\left(\frac{\gamma}{1+\sqrt{T-t}\,\delta}\right) \le J(t) \le 0. \tag{3.7}$$

Recalling that $\mathcal{H}\rho = 0$ and using (3.6) with $\varphi = q := \rho/S$ we get again $\mathcal{H}_\zeta q = \kappa q$. Hence, by the chain rule, $\Phi(q) := q\log q$ is such that

$$\mathcal{H}_\zeta\Phi(q) = \Phi'(q)\mathcal{H}_\zeta q + \Phi''(q)|\nabla q|^2 = (1+\log q)\kappa q + \frac{|\nabla q|^2}{q},$$

so that

$$\mathcal{H}(\rho\log(\rho/S)) = \mathcal{H}(S\Phi(q)) = S\mathcal{H}_\zeta\Phi(q) - \kappa S\Phi(q) = \kappa\rho + \rho|\nabla\log q|^2$$

and thus

$$J'(t) = \int_{\mathbb{R}^d}\rho_t\left|\nabla\log\frac{\rho_t}{S_t}\right|^2\mathrm{d}x + \int_{\mathbb{R}^d}\kappa_t\rho_t\,\mathrm{d}x. \tag{3.8}$$

It follows that

$$\int_0^T\int_{\mathbb{R}^d}\rho\left|\nabla\log\frac{\rho}{S}\right|^2\mathrm{d}x\,\mathrm{d}t \le -J(0) \le \gamma\log\left(\frac{1+\sqrt{T}\,\delta}{\gamma}\right). \tag{3.9}$$

We now compute that

$$\mathcal{H}(\rho|B|^2) = 2\rho|\nabla B|^2 + 2\langle\nabla\rho - \rho\zeta, \nabla|B|^2\rangle + 2\kappa\rho|B|^2.$$

Integrating in space and using $|\nabla|B|^2| \le 2\Lambda|\nabla B|$ (recall that $|B|\le\Lambda$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho_t|B_t|^2\,\mathrm{d}x \ge \int_{\mathbb{R}^d}2\rho_t|\nabla B_t|^2\,\mathrm{d}x - \int_{\mathbb{R}^d}4\Lambda\rho_t|\nabla\log(\rho_t/S_t)||\nabla B_t|\,\mathrm{d}x.$$

By Young's inequality, since $\int_{\mathbb{R}^d}\rho_t|B_t|^2\,\mathrm{d}x \le \Lambda^2\int_{\mathbb{R}^d}\rho_t\,\mathrm{d}x = \Lambda^2\gamma$ we deduce that

$$\int_0^T\int_{\mathbb{R}^d}\rho|\nabla B|^2\,\mathrm{d}x\,\mathrm{d}t \le \Lambda^2\gamma + C\int_0^T\int_{\mathbb{R}^d}\rho|\nabla\log(\rho/S)|^2\,\mathrm{d}x\,\mathrm{d}t,$$

which gives

$$\int_0^T\int_{\mathbb{R}^d}\rho|\nabla B|^2\,\mathrm{d}x\,\mathrm{d}t \le C\gamma + C\gamma\log\left(\frac{1+\sqrt{T}\,\delta}{\gamma}\right) \tag{3.10}$$

thanks to (3.9).

Next, by construction we have $|\eta_t| \le \frac{1}{\sqrt{T-t}}$, so that

$$\int_{\mathbb{R}^d}\rho_t|\eta_t|^2\,\mathrm{d}x \le \frac{1}{T-t}\int_{\mathbb{R}^d}\rho_t\,\mathrm{d}x = \frac{\gamma}{T-t}$$

and

$$\int_{\mathbb{R}^d}\rho_t|\eta_t|^2\,\mathrm{d}x \le \int_{\mathbb{R}^d}S_t|\eta_t|^2\,\mathrm{d}x \le \frac{1}{\sqrt{T-t}}\int_{\mathbb{R}^d}|\mathcal{P}_{T-t}\operatorname{div}\sigma|\,\mathrm{d}x \le \frac{\delta}{\sqrt{T-t}}.$$

If $\sqrt{T} > \gamma/\delta$ then, letting $s := T-t$, we can estimate

$$\int_0^T\int_{\mathbb{R}^d}\rho|\eta|^2\,\mathrm{d}x\,\mathrm{d}s \le \int_0^{\gamma^2/\delta^2}\frac{\delta}{\sqrt{s}}\,\mathrm{d}s + \int_{\gamma^2/\delta^2}^T\frac{\gamma}{s}\,\mathrm{d}s = 2\gamma + 2\gamma\log\left(\frac{\sqrt{T}\,\delta}{\gamma}\right),$$

while if $\sqrt{T} \le \gamma/\delta$ then the second bound gives directly that

$$\int_0^T \int_{\mathbb{R}^d} \rho|\eta|^2 \,\mathrm{d}x\,\mathrm{d}s \le 2\sqrt{T}\,\delta,$$

so that in both cases we have

$$\int_0^T \int_{\mathbb{R}^d} \rho|\eta|^2 \,\mathrm{d}x\,\mathrm{d}t \le 4\gamma \log\left(1 + \frac{\sqrt{T}\,\delta}{\gamma}\right). \tag{3.11}$$

Also, [18, eq. (3.9)] is unchanged and reads $\operatorname{div} B + B\zeta = \eta$. Together with (3.10) and (3.11), this gives

$$\int_0^T \int_{\mathbb{R}^d} \rho|B\zeta|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\gamma + C\gamma \log\left(\frac{1+\sqrt{T}\,\delta}{\gamma}\right). \tag{3.12}$$

Since $|B| \le \Lambda$ and $B\nabla \log\rho = B\zeta + B\nabla\log(\rho/S)$, combining (3.12) with (3.9) we also get the desired integral bound for $\rho|B\nabla\log\rho|^2$. □

We now apply the previous bounds in order to obtain an upper bound on (a renormalized version of) the Fisher information, which in turn bounds the entropy growth along the evolution of ρ_t.

Lemma 3.2. *For every $0 < \tau < T$, writing $I(\rho_t) := \int_{\mathbb{R}^d} \rho_t|\nabla\log\rho_t|^2 \,\mathrm{d}x$, we have*

$$\int_0^\tau \left[I(\rho_t) - \frac{(d-m)\gamma}{2(T-t)}\right] \mathrm{d}t \le C\gamma + C\gamma\log\left(\frac{1+\sqrt{T}\,\delta}{\gamma}\right) \tag{3.13}$$

for some constant $C(d,\lambda,\Lambda)$.

Proof. Choose a smooth $\chi : [0,\infty) \to [0,1]$, equal to zero on $[0,\lambda^2/16]$ and to one on $[\lambda^2/4,\infty)$. Let $Q := \chi(B^TB)$ be defined by spectral calculus as

$$Q := O\,\mathrm{diag}(\chi(\sigma_1^2(B)),\dots,\chi(\sigma_d^2(B)))O^T,$$

where we write $B^TB = O\,\mathrm{diag}(\sigma_1^2(B),\dots,\sigma_d^2(B))O^T$ using the spectral theorem. Moreover, set $\Pi := Q^2$, which plays the role of a projection matrix. We have

$$0 \le \Pi \le I_d, \quad |Qz|^2 \le C|Bz|^2, \quad |\nabla Q| \le C|\nabla B|. \tag{3.14}$$

On $\{U_t \ge \sqrt{T-t}\,R_t\}$, (3.1) and (3.4) imply $\sigma_m(B_t) \ge \lambda/2$. To see this, observe that $S_t \le 2U_t$ gives $U_t/S_t \ge 1/2$, and thanks to (3.4) we can write

$$\sigma_m(B_t) = \frac{U_t}{S_t}\sigma_m(V_t/U_t) \ge \lambda/2.$$

Hence, $\operatorname{tr}\Pi_t \ge m$ by construction of χ. The complementary set E_t has

$$\int_{E_t} \rho_t \,\mathrm{d}x \le \int_{E_t} U_t \,\mathrm{d}x \le \int_{E_t} \sqrt{T-t}\,R_t \,\mathrm{d}x \le \sqrt{T-t}\,\delta.$$

Therefore,

$$\int_{\mathbb{R}^d} \rho_t \operatorname{tr}(I_d - \Pi_t) \,\mathrm{d}x \le (d-m)\gamma + d\min\{\gamma, \sqrt{T-t}\,\delta\}. \tag{3.15}$$

Differentiating the convolution with the Gaussian p_s (where $s := T-t$) and noting that

$$\nabla^2 p_s(x-y) = \left[\frac{(x-y)\otimes(x-y)}{4s^2} - \frac{I_d}{2s}\right] p_s(x-y),$$

we get

$$\nabla^2 \log\rho_t \ge -\frac{I_d}{2(T-t)}. \tag{3.16}$$

This is the well-known log semi-convexity improvement of the heat flow, and can be proved by observing that the difference of the two terms equals the covariance of y under the probability

measure $p_s(x-y)\,\mathrm{d}\mu_0(y)/\rho_t(x)$, divided by $4s^2$. Using $I(\rho_t) = -\int_{\mathbb{R}^d} \rho_t \operatorname{tr}(\nabla^2 \log \rho_t)\,\mathrm{d}x$, (3.15), and (3.16), we furthermore obtain

$$\begin{aligned} I(\rho_t) &= -\int_{\mathbb{R}^d} \rho_t \operatorname{tr}((I_d - \Pi_t)\nabla^2 \log \rho_t)\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_t \operatorname{tr}(\Pi_t \nabla^2 \log \rho_t)\,\mathrm{d}x \\ &\le \frac{(d-m)\gamma}{2(T-t)} + \min\left[\frac{d\gamma}{2(T-t)}, \frac{d\delta}{2\sqrt{T-t}}\right] - \sum_i \int_{\mathbb{R}^d} \rho_t [e_i^T (\nabla^2 \log \rho_t) e_i]\,\mathrm{d}x, \end{aligned}$$

where $e_i = e_{i,t}$ denotes the i-th column of Q_t.

Now, for every smooth vector field e with compact support, integration by parts gives

$$-\int_{\mathbb{R}^d} \rho_t [e^T (\nabla^2 \log \rho_t) e]\,\mathrm{d}x = \int_{\mathbb{R}^d} \rho_t [(\operatorname{div} e + e \cdot \nabla \log \rho_t)^2 - \operatorname{tr}((De)^2)]\,\mathrm{d}x, \tag{3.17}$$

where $(De)_{jk} := \partial_k e_j$. Indeed, writing $b_t := \operatorname{div} e + e \cdot \nabla \log \rho_t = \rho_t^{-1} \operatorname{div}(\rho_t e)$, we have $\int_{\mathbb{R}^d} \rho_t b_t^2\,\mathrm{d}x = -\int_{\mathbb{R}^d} \rho_t e \cdot \nabla b_t\,\mathrm{d}x$ and

$$e \cdot \nabla b_t = e \cdot \nabla \operatorname{div} e + ((e \cdot \nabla) e) \cdot \nabla \log \rho_t + e^T (\nabla^2 \log \rho_t) e,$$

and, once we integrate, the first two terms on the right-hand side combine to give the trace term $-\int_{\mathbb{R}^d} \rho_t \operatorname{tr}((De)^2)\,\mathrm{d}x$. By a standard cut-off argument, (3.17) also applies to the columns $e_{i,t}$ of Q_t. Summing over i and recalling (3.14), this gives

$$I(\rho_t) \le \frac{(d-m)\gamma}{2(T-t)} + \min\left[\frac{d\gamma}{2(T-t)}, \frac{d\delta}{2\sqrt{T-t}}\right] + C\int_{\mathbb{R}^d} \rho_t (|\nabla B_t|^2 + |B_t \nabla \log \rho_t|^2)\,\mathrm{d}x. \tag{3.18}$$

Integrating in t, applying Lemma 3.1, and bounding the integral of the second term as in the previous proof, we reach the claim. □

4. Proofs of the main theorems

Proof of Theorem 1.1. Assume again that μ has compact support, $\mu(\mathbb{R}^d) = 1$, $|\operatorname{div} \sigma|(\mathbb{R}^d) = \delta$, and $0 \le \mu_0 \le \mu$ with $\gamma := \mu_0(\mathbb{R}^d) \in (0,1]$; the submeasure μ_0 will be chosen in a moment.

The idea now is to look at the entropy $H(\rho_t)$ as time passes. More specifically, we look at a renormalized version of it, adapted to codimension $d-m$: let

$$\tilde{H}(\rho_t) := H(\rho_t) + \frac{(d-m)\gamma}{2}\log(T-t) = \int_{\mathbb{R}^d} \rho_t \log \rho_t\,\mathrm{d}x + \frac{(d-m)\gamma}{2}\log(T-t).$$

Since $\mathcal{H}\rho = 0$, the derivative $\frac{\mathrm{d}}{\mathrm{d}t} H(\rho_t) = I(\rho_t) \ge 0$ is the Fisher information (recall that ρ_t evolves through the backward heat equation), and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}(\rho_t) = I(\rho_t) - \frac{(d-m)\gamma}{2(T-t)}.$$

The previous lemma then gives

$$\limsup_{t \to T^-} \tilde{H}(\rho_t) - \tilde{H}(\rho_0) \le C\gamma + C\gamma \log\left(\frac{1+\sqrt{T}\,\delta}{\gamma}\right).$$

Moreover, since $\rho_0 = \mathcal{P}_T \mu_0$, we have $\rho_0 \le \frac{\gamma}{(4\pi T)^{d/2}}$, and hence

$$H(\rho_0) \le \int_{\mathbb{R}^d} \rho_0 \log\left(\frac{\gamma}{(4\pi T)^{d/2}}\right)\mathrm{d}x \le \gamma \log \gamma - \frac{d\gamma}{2}\log(4\pi T),$$

giving in particular

$$\tilde{H}(\rho_0) \le -\frac{m\gamma}{2}\log T.$$

Thus, for some constant $C(d, \lambda, \Lambda)$, we get

$$\limsup_{t \to T^-} \tilde{H}(\rho_t) \le -\frac{m\gamma}{2}\log T + C\gamma + C\gamma \log\left(\frac{1+\sqrt{T}\,\delta}{\gamma}\right). \tag{4.1}$$

Next, we observe that μ is a rectifiable measure, by [5, Proposition 3.1]. We can then take $E \subseteq M$ to be a finite disjoint union of compact sets, each of which is included in a 1-Lipschitz graph (up to a rotation), and $\mu_0 := \mu \llcorner E$. We claim a lower bound on the left-hand side, namely

$$\liminf_{t \to T^-} \tilde{H}(\rho_t) \geq \gamma \log\left(\frac{\gamma}{\mathcal{H}^m(E)}\right) - C(d)\gamma. \tag{4.2}$$

Note that the geometry of E is not controlled, namely E could start resembling an m-plane only at very small scales, but the key point is that in the limit $t \to T$ only the picture at smaller and smaller scales matters. More precisely, we will just need a bound on the Minkowski content at (non-uniformly) small scales.

To establish (4.2), we first compare the probability density $\varphi_t := \rho_t/\gamma$ with a more explicit one, namely

$$\psi_s := \gamma_s^{-1} e^{-\operatorname{dist}(x,E)^2/(4s)}, \quad \gamma_s := \int_{\mathbb{R}^d} e^{-\operatorname{dist}(x,E)^2/(4s)} \, \mathrm{d}x,$$

where we set $s := T - t$ for simplicity. Since the relative entropy $\int_{\mathbb{R}^d} \varphi_t(\log \varphi_t - \log \psi_t) \, \mathrm{d}x \geq 0$, after a little rearrangement we get

$$H(\rho_t) \geq \gamma(\log \gamma - \log \gamma_s) - \frac{1}{4s} \int_{\mathbb{R}^d} \operatorname{dist}(x,E)^2 \rho_t(x) \, \mathrm{d}x.$$

Since $\operatorname{dist}(x,E) \leq |x-y|$ for all $y \in E$ and $\rho_t = \mathcal{P}_s \mu_0$, the last integral is bounded by

$$\int_E \int_{\mathbb{R}^d} |x-y|^2 p_s(x-y) \, \mathrm{d}x \, \mathrm{d}\mu_0(y) = 2d\gamma s,$$

so that

$$H(\rho_t) \geq \gamma(\log \gamma - \log \gamma_s) - \frac{d\gamma}{2}.$$

Finally, by construction of E, it is straightforward to check that there exists $r_0 > 0$ such that $|B_r(E)| \leq C(d) r^{d-m} \mathcal{H}^m(E)$ for all radii $0 < r < r_0$, giving

$$\begin{aligned}
\gamma_s &= \int_0^\infty \frac{r}{2s} e^{-r^2/(4s)} |B_r(E)| \, \mathrm{d}r \\
&\leq C(d) \int_0^{r_0} \frac{r^{1+d-m}}{2s} e^{-r^2/(4s)} \mathcal{H}^m(E) \, \mathrm{d}r + C(E) e^{-r_0^2/(8s)} \\
&\leq C(d) s^{(d-m)/2} \mathcal{H}^m(E)
\end{aligned}$$

for s small enough, proving (4.2).

Combining (4.1) with (4.2), we arrive at

$$\log\left(\frac{\gamma}{\mathcal{H}^m(E)}\right) \leq -\frac{m}{2} \log T + C + C \log\left(\frac{1 + \sqrt{T}\,\delta}{\gamma}\right). \tag{4.3}$$

This implies that $\delta > 0$, since otherwise we would reach a contradiction for T large enough. Noting that we can take E as above such that $\gamma = \mu_0(\mathbb{R}^d) = \mu(E) \in [1/2, 1]$, we can then take $T := \delta^{-2}$, which yields

$$\log \mathcal{H}^m(E) + \log \delta^m \geq -C$$

for some constant $C(d, \lambda, \Lambda)$, and hence

$$\mu(\mathbb{R}^d) = 1 \leq C \mathcal{H}^m(E)^{1/m} \delta = C \mathcal{H}^m(E)^{1/m} |\operatorname{div} \sigma|(\mathbb{R}^d),$$

as desired.

By homogeneity, this proves the theorem for a compactly supported μ, while the general case follows by a standard cut-off argument. □

Proof of Theorem 1.4. Without loss of generality, we can assume that $\mathcal{A} = \operatorname{div}$, as in the statement of Theorem 1.1. Moreover, as in the previous proof, we can assume that μ is compactly supported and $\mu(\mathbb{R}^d) = 1$ (by homogeneity).

We take again $\mu_0 := \mu \llcorner E$ with $E \subseteq M$ a finite disjoint union of compact pieces of 1-Lipschitz graphs. We observe that, taking $T := r^2$, (4.3) rearranges to

$$0 \leq \log\left(\frac{\mathcal{H}^m(E)}{r^m}\right) + C' + C' \log\left(\frac{1+r\delta}{\gamma}\right)$$

for some $C'(d, \lambda, \Lambda)$; since $\gamma = \mu(E)$, taking $\varepsilon := 1/C'$ this is the same as

$$\frac{\mu(E)}{1+r\delta} \leq C\left(\frac{\mathcal{H}^m(E)}{r^m}\right)^{\varepsilon}.$$

By approximation, this holds for any $E \subseteq M$ with $\mathcal{H}^m(E) \in (0, \infty)$ (assuming without loss of generality that M is m-rectifiable). Given $u > 0$, taking $E_u := \{x \in M : \theta(x) > u\}$ this yields

$$u\mathcal{H}^m(E_u) \leq \int_{E_u} \theta \, \mathrm{d}\mathcal{H}^m \leq \mu(E_u) \leq C\frac{u^{-\varepsilon}}{r^{m\varepsilon}}(1+r\delta),$$

where we used the fact that $\mathcal{H}^m(E_u) \leq u^{-1}\mu(E_u) \leq u^{-1}$. Taking $p := 1 + \varepsilon$, this is the same as

$$u^p\mathcal{H}^m(\{x \in M : \theta(x) > u\}) \leq \frac{C}{r^{m(p-1)}}(1+r\delta)$$

for any given $u > 0$, as desired. □

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