GAUSSIAN RIGIDITY AND WIDTH OF MEASURES, WITH AN APPLICATION TO PREISS'S THEOREM

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ABSTRACT. Given $0 \leq m < n$ and a positive Radon measure μ on $\mathbb{R}^n$ with at most polynomial growth, assume that it has finite m-dimensional *Gaussian discrepancy*, defined as

$$\mathcal{D}(\mu) := \limsup_{R\to\infty} \Big[R^2 \sup_{x\in \operatorname{spt}\mu} |G^\mu_R(x) - 1| \Big],$$

where $G^\mu_R(x) := (4\pi R^2)^{-m/2} \int_{\mathbb{R}^n} e^{-|z-x|^2/(4R^2)}\, d\mu(z)$. Provided that μ admits a flat blow-down $\mathcal{H}^m \llcorner V$ (an assumption that cannot be dropped entirely), we prove that there exists a finite (smallest) $0 \leq \delta < \infty$, called the *width* of μ, such that

$$\operatorname{spt}\mu \subseteq \overline{B}_\delta(V + x_0), \quad \delta^2 \leq C(m,n)\mathcal{D}(\mu),$$

where $\overline{B}_\delta(V + x_0)$ denotes the closed δ-neighborhood of a suitable translate of V. Note that the exponent is optimal because of scale invariance.

In particular, when $\mathcal{D}(\mu) = 0$, the support of μ is included in an m-dimensional affine plane. We use this fact to give a simpler proof of Preiss's rectifiability theorem [9, Theorem 5.6]: while still based on Gaussian averages of μ, this new proof avoids showing uniqueness (and hence conicality) of blow-downs of m-uniform measures, as well as studying Gaussian-weighted moments beyond second order.

The strategy of proof consists of an anisotropic blow-down of μ, followed by an analysis of the resulting Young measure, using a rigidity lemma involving the heat semigroup.

1. INTRODUCTION

Given an integer $0 \leq m < n$ and a positive Radon measure μ on $\mathbb{R}^n$, we consider its Gaussian averages $G^\mu_R(x)$, given at each center $x \in \mathbb{R}^n$ and scale $R > 0$ by

$$G^\mu_R(x) := (4\pi R^2)^{-m/2} \int_{\mathbb{R}^n} e^{-|z-x|^2/(4R^2)}\, d\mu(z), \tag{1.1}$$

and assume that $G^\mu_R(x) < \infty$ for all $x \in \operatorname{spt}\mu$ and $R > 0$. The normalization factor gives $G^\mu_R(x) = 1$ for all $x \in \operatorname{spt}\mu$ if μ is *flat*, namely $\mu = \mathcal{H}^m \llcorner V$ for a suitable affine m-plane V, or more generally if μ is *m-uniform*, meaning that $\mu(B_r(x)) = \omega_m r^m$ for all $x \in \operatorname{spt}\mu$ and $r > 0$.

We then define the *Gaussian discrepancy* to be

$$\mathcal{D}(\mu) := \limsup_{R\to\infty} R^2 \sup_{x\in\operatorname{spt}\mu} |G^\mu_R(x) - 1|. \tag{1.2}$$

Using the notation

$$S_{x,r}\mu := r^{-m}(z \mapsto (z-x)/r)_*\mu, \quad S_r\mu := S_{0,r}\mu$$

for the rescalings of μ, recall that a *blow-down* is a weak-* limit $\lim_{j\to\infty} S_{R_j}\mu$, along a sequence of scales $R_j \to \infty$ (as usual, weak-* convergence is understood in duality with $C^0_c(\mathbb{R}^n)$).

In this article, we first prove the following rigidity result.

Theorem 1.1. If μ has a flat blow-down $\mathcal{H}^m \llcorner V$ and $\mathcal{D}(\mu) = 0$, namely

$$\sup_{x\in\operatorname{spt}\mu} |G^\mu_R(x) - 1| = o(R^{-2}) \quad \text{as } R \to \infty, \tag{1.3}$$

then $\operatorname{spt}\mu$ is contained in an affine m-plane parallel to V.

Remark 1.2. As a preliminary step in the proof (Lemma 2.1) will show, we also have $R_j^{-1} \operatorname{spt}\mu \to V$ in the local Hausdorff sense, and thus $0 \in V$. The assumption of a flat blow-down is necessary, at least in arbitrary dimension: in $\mathbb{R}^4$, the Kowalski–Preiss cone

$$\mu := \mathcal{H}^3 \llcorner \mathcal{C}, \quad \mathcal{C} := \{x \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$$

is a 3-uniform measure [3, Proposition 3.8], and hence has $\mathcal{D}(\mu) = 0$.

Example 1.3. Note that Theorem 1.1 can also be applied to lower-dimensional measures, such as atomic measures: for instance, letting $\mu := \sum_{a \in \mathbb{Z}^m \times \{0\}} \delta_a$, it is easy to check that $\mathcal{D}(\mu) = 0$, and indeed $\operatorname{spt} \mu \subset \mathbb{R}^m \times \{0\}$.

Generalizing Theorem 1.1, for an m-plane $V \ni 0$, we let

$$w_V(\mu) := \inf_{c \in V^\perp} \sup_{x \in \operatorname{spt} \mu} |\pi_{V^\perp} x - c|, \tag{1.4}$$

which we call the *width* of μ. The following quantitative version of Theorem 1.1 holds.

Theorem 1.4. For every $0 \leq m < n$ there exists $C(m,n) > 0$ such that, whenever μ has a flat blow-down $\mathcal{H}^m \llcorner V$ and $\mathcal{D}(\mu) < \infty$, we have

$$w_V(\mu)^2 \leq C \mathcal{D}(\mu). \tag{1.5}$$

Remark 1.5. While the statement is trivial when $m = n$, the reader might wonder what can be said when $m \in [0, \infty) \setminus \{0, 1, \dots, n\}$. In fact, in this case the assumption $\mathcal{D}(\mu) < \infty$ implies $\mu = 0$: if we had $\mu \neq 0$, then any blow-down of μ would be a nontrivial m-uniform measure (as shown by the proof of Lemma 2.1), which is impossible since such measures do not exist (see, e.g., the proof of Proposition 4.2).

Example 1.6. Given an m-plane $V \ni 0$ and $a \in V^\perp \setminus \{0\}$, consider $\mu := \frac{1}{2}\mathcal{H}^m \llcorner (V + a) + \frac{1}{2}\mathcal{H}^m \llcorner (V - a)$, for which the blow-down is $\mathcal{H}^m \llcorner V$. Then $w_V(\mu) = |a|$, while at every $x \in \operatorname{spt} \mu$ we have

$$G^\mu_R(x) = \frac{1 + e^{-|a|^2/R^2}}{2} = 1 - \frac{|a|^2}{2R^2} + O_a(R^{-4}),$$

so that $\mathcal{D}(\mu) = |a|^2/2$.

Example 1.7. In $\mathbb{R}^3$, for $m = 2$, let $V := \mathbb{R}^2 \times \{0\}$ and

$$\mu := \mathcal{H}^2 \llcorner V + \gamma \delta_{\lambda e_3} + \gamma \delta_{-\lambda e_3},$$

for arbitrary $\gamma, \lambda > 0$. Its blow-down is $\mathcal{H}^2 \llcorner V$ and its width is $w_V(\mu) = \lambda$. For any $p \in \mathbb{R}^2$ we can compute that

$$G^\mu_R(p, 0) = 1 + \frac{\gamma}{2\pi R^2} e^{-(|p|^2 + \lambda^2)/(4R^2)},$$

while

$$G^\mu_R(\pm \lambda e_3) = e^{-\lambda^2/(4R^2)} + \frac{\gamma}{4\pi R^2}(1 + e^{-\lambda^2/R^2}) = 1 - \frac{(\lambda^2/4) - (\gamma/(2\pi))}{R^2} + O_{\gamma,\lambda}(R^{-4}).$$

Consequently, we have $\mathcal{D}(\mu) = \max\{\frac{\gamma}{2\pi}, |\frac{\lambda^2}{4} - \frac{\gamma}{2\pi}|\}$. In particular, taking $\gamma := \pi\lambda^2/4$ gives $\mathcal{D}(\mu) = \lambda^2/8$, showing that $C(m,n) \geq 8$ for every $m \geq 2$ and $n \geq m + 1$.

As already mentioned, Theorem 1.1 applies to m-uniform measures, in which case it gives the classical assertion that an m-uniform measure with a flat blow-down is flat [9, Corollary 3.16]. This constitutes the main rigidity step in the proof of the following rectifiability theorem of Preiss [9, Theorem 5.6]. The original proof of this step was perhaps the most technical part of the proof of Preiss's result, involving a careful analysis of Gaussian-weighted moments of μ up to order four.

Theorem 1.8 (Preiss)**.** Let $\mu \neq 0$ be a positive Radon measure on $\mathbb{R}^n$ and let $\alpha \geq 0$. Assume that $\Theta^\alpha(\mu, x) := \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}$ exists and belongs to $(0, \infty)$, at μ-a.e. x. Then $\alpha = m \in \{0, \dots, n\}$ and μ is m-rectifiable.

The aforementioned assertion was also a key tool in the proof of other results, such as those in [10]. Theorem 1.8 was recently extended to codimension-one measures in Heisenberg groups in [8]. In fact, with slightly more work, one can also get the following stronger statement, which first appeared in [9, Corollary 5.5].

Theorem 1.9 (Preiss)**.** The same holds if, rather than assuming the density $\Theta^\alpha(\mu,x)$ to exist, we just assume

$$0 < \Theta^{\alpha,*}(\mu,x) \le (1+\varepsilon_0)\Theta^\alpha_*(\mu,x) < \infty \quad \text{for } \mu\text{-a.e. } x, \tag{1.6}$$

for a suitable threshold $\varepsilon_0(\alpha,n) > 0$.

It should be mentioned that the use of Gaussian averages to study measures already appeared several times, particularly in the study of uniform measures (see, e.g., [2, 9]).

1.1. **Strategy of proof.** Besides coarse growth bounds, in Section 2 we show that $\pi_{V^\perp}\operatorname{spt}\mu$ is bounded whenever $\mathcal{D}(\mu) < \infty$ and μ has a flat blow-down $\mathcal{H}^m \llcorner V$, by means of a simple Taylor expansion. Further, in the case $\mathcal{D}(\mu) = 0$, we also show that $\operatorname{spt}\mu$ is contained in the shell $\{x \in \mathbb{R}^n : |\pi_{V^\perp}x| = |b|\}$ (for some $b \in V^\perp$, up to a translation such that $-b \in \operatorname{spt}\mu$).

In Section 3 we prove Theorem 1.1. To do this, we identify $V = \mathbb{R}^m \times \{0\}$ and perform an anisotropic blow-down, shrinking the horizontal components (in $\mathbb{R}^m$) and leaving the vertical ones (in $\mathbb{R}^{n-m}$) unchanged. It is easy to see that such a blow-down disintegrates as $dx' \otimes \kappa_{x'}$, where x' denotes points in $\mathbb{R}^m$, while $(\kappa_{x'})_{x'\in\mathbb{R}^m}$ is a Young measure retaining information on the vertical oscillation of the initial μ. Each $\kappa_{x'}$ is a probability measure on the sphere $\{q \in V^\perp : |q| = |b|\}$.

We then study the center-of-mass function $h(x') := \int_{V^\perp} q\, d\kappa_{x'}(q)$ and its Gaussian averages $\mathcal{P}_r h$. It turns out that h satisfies several identities, such as

$$|b|^2 - h\cdot\mathcal{P}_r h = 2r^2 A_r \ge 0$$

among others, where $(A_r)_{r>0}$ is a family of auxiliary functions (given by $\lim_{L\to\infty} L^2(\mathcal{P}_r\rho_L - 1)$, where $\rho_L := S_{0,L}(\pi_V)_*\mu$, which is the projection of the anisotropic rescaling at scale L).

Since we can easily show that $r^2A_r \to 0$ in L^1_{loc} as $r \to 0$, we have $|b|^2 - h\cdot\mathcal{P}_r h \to 0$, which (together with the trivial bound $|h| \le |b|$) forces $|h| = |b|$, so that a.e. $\kappa_{x'}$ must be a Dirac mass $\delta_{h(x')}$. The aforementioned identities then give $h \in H^1_{loc}$, $\mathcal{P}_r h(0) = 0$, $\mathcal{P}_r(|Dh|^2)(0) \le |b|^2/r^2$, and $\mathcal{P}_r(h\cdot\mathcal{P}_s h)(0) = |b|^2r^2/(r^2+s^2)$, which can be shown to imply $h \equiv 0$, i.e., $|b| = 0$, giving $\operatorname{spt}\mu \subseteq V$.

In Section 4, we summarize the main arguments needed to deduce Theorems 1.8 and 1.9 from Theorem 1.1, for the sake of completeness; all the content of this section is actually well known, up to minor adaptations.

Finally, Section 5 is devoted to Theorem 1.4. Given μ as in its statement, since we already know that the width $w_V(\mu) < \infty$ and the claimed bound is scale-invariant, we can assume $w_V(\mu) = 1$ and reduce ourselves to showing that $\mathcal{D}(\mu)$ cannot be arbitrarily small. It might be tempting to use a direct compactness argument, but the issue is that the width does not pass to the limit. On the other hand, we can perform the same anisotropic blow-down used before: in this way, we get a Young measure $(\kappa_{x'})_{x'\in\mathbb{R}^m}$ where each $\kappa_{x'}$ is supported in an annular region $\{q \in V^\perp : |q| \in [1 - C\sqrt{\mathcal{D}(\mu)}, 1]\}$. We also obtain approximate versions of the previous identities, which then allow us to use a compactness argument reducing to the previous situation (in the limit $\mathcal{D}(\mu) \to 0$), for a suitable limiting h with $|h| = 1$, contradicting the conclusion that $h \equiv 0$.

1.2. **Usage of AI tools.** The statement and proof of Theorem 1.1 were found by a commercial AI model, once the author asked it to try to simplify the proof of the step "flatness at infinity implies flatness" in Preiss's proof using anisotropic blow-downs and Young measures. Besides rewriting the output, the author then obtained the quantitative version of it, namely Theorem 1.4. The AI model was also used to generate the examples above and in the proofreading stage. The author takes full responsibility for all the contents of the paper.

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2. Preliminary facts for measures with finite discrepancy

Throughout the section, we will assume that μ is a positive Radon measure on $\mathbb{R}^n$ such that $G^\mu_R(x) < \infty$ for every $x \in \operatorname{spt}\mu$ and $R > 0$, as well as

$$\mathcal{D}(\mu) < \infty,$$

where we recall that G^μ_R and the Gaussian discrepancy $\mathcal{D}(\mu)$ were introduced in (1.1)–(1.2). We first establish a few consequences of the assumption. Up to a translation, without loss of generality we will also assume that $0 \in \Sigma := \operatorname{spt}\mu$.

Lemma 2.1. There exist $R_0(\mu) \geq 1$ and constants $c(n), C(n) > 0$ such that

$$cr^m \leq \mu(B_r(x)) \leq Cr^m \quad \text{for } x \in \Sigma,\ r \geq R_0, \tag{2.1}$$

as well as

$$\mu(B_r(x)) \leq Cr^m \quad \text{for } x \in \mathbb{R}^n,\ r \geq R_0. \tag{2.2}$$

Further, every sequence $S_{x_j,R_j}\mu$, with $x_j \in \Sigma$ and $R_j \to \infty$, has a subsequence converging to an m-uniform measure, with local convergence of supports in the Hausdorff metric.

Proof. The assumption $\mathcal{D}(\mu) < \infty$ implies that, for any $r > 0$ large enough (uniformly in $x \in \Sigma$), we have $G^\mu_r(x) \in (1/2, 2)$. Since $G^\mu_r(x) \geq (4\pi)^{-m/2} e^{-1/4\,\frac{\mu(B_r(x))}{r^m}}$, the upper bound in (2.1) follows. Also, if $B_r(x) \cap \Sigma \neq \emptyset$ then we can take x' in the intersection and bound $\mu(B_r(x)) \leq \mu(B_{2r}(x')) \leq C(n)(2r)^m$, giving (2.2) (up to enlarging C).

Given $a > 1$ and $x \in \Sigma$, expanding $\mathbb{R}^n \setminus B_{ar}(x) = \bigcup_{k=1}^\infty [B_{(k+1)ar}(x) - B_{kar}(x)]$, we deduce that for r large enough we have

$$\begin{aligned}\frac{1}{2} \leq G^\mu_r(x) &\leq C(n,a)\frac{\mu(B_{ar}(x))}{r^m} + \sum_{k=1}^\infty e^{-k^2a^2/4}\frac{\mu(B_{(k+1)ar}(x))}{r^m} \\ &\leq C(n,a)\frac{\mu(B_{ar}(x))}{r^m} + C(n)\sum_{k=1}^\infty e^{-k^2a^2/4}(k+1)^m a^m.\end{aligned}$$

Taking $a > 1$ so large that the last series is at most $1/(4C(n))$, we obtain $\mu(B_{ar}(x)) \geq \frac{r^m}{4C(n,a)}$, which gives the lower bound in (2.1).

Given rescalings $\mu_j := S_{x_j,R_j}\mu$ as in the statement, the coarse bounds (2.1) hold for all radii $r \geq R_0/R_j$, proving the existence of a subsequential limit ν and convergence of supports. Since

$$G^{\mu_j}_r(x) = G^\mu_{R_j r}(x_j + R_j x) = 1 + O_{\mu,r}(R_j^{-2})$$

for any $x \in \operatorname{spt}\mu_j$ (i.e., $x_j + R_j x \in \operatorname{spt}\mu$), thanks to the assumption $\mathcal{D}(\mu) < \infty$, in the limit we get $G^\nu_r(x) = 1$ for any $x \in \operatorname{spt}\nu$ (by a truncation argument as above). Rewriting

$$\begin{aligned}G^\nu_r(x) &= \pi^{-m/2}(4r^2)^{-m/2-1}\int_0^\infty 2se^{-s^2/(4r^2)}\nu(B_s(x))\,ds \\ &= \pi^{-m/2}(4r^2)^{-m/2-1}\int_0^\infty e^{-\sigma/(4r^2)}\nu(B_{\sqrt{\sigma}}(x))\,d\sigma,\end{aligned}$$

we see that $F(\sigma) := \nu(B_{\sqrt{\sigma}}(x))$ has Laplace transform $\int_0^\infty e^{-\tau\sigma}F(\sigma)\,d\sigma = \pi^{m/2}\tau^{-m/2-1}$, which is easily computed to agree with the transform of $\mathcal{H}^m(B^m_{\sqrt{\sigma}}(0)) = \omega_m\sigma^{m/2}$. By uniqueness, we get $\nu(B_s(x)) = \omega_m s^m$ for all $x \in \operatorname{spt}\nu$, proving that ν is m-uniform. □

Given $R > 0$, let us use the notation

$$\mathbb{E}_R[\psi] := (4\pi R^2)^{-m/2}\int_{\mathbb{R}^n}\psi(z)e^{-|z|^2/(4R^2)}\,d\mu(z), \quad Q_R := \frac{1}{2R^2}\mathbb{E}_R[z \otimes z] \tag{2.3}$$

(note that $\mathbb{E}_R[1] = G^\mu_R(0) = 1 + O_\mu(R^{-2})$ for $R \geq 1$). The matrix Q_R is normalized in such a way that it equals the orthogonal projection π_V when $\mu = \mathcal{H}^m \llcorner V$. Let us record a useful expansion.

Lemma 2.2. For every fixed $x \in \mathbb{R}^n$ and $R \geq R_0$, we have

$$4R^2(G^\mu_R(x) - G^\mu_R(0)) = 2\mathbb{E}_R[z]\cdot x + x\cdot Q_R x - |x|^2\mathbb{E}_R[1] + O_x(R^{-1}), \tag{2.4}$$

where O_x indicates an implicit constant depending on x and n, but not on μ.

Proof. Indeed, we have

$$G^\mu_R(x) = e^{-|x|^2/(4R^2)}\mathbb{E}_R[e^{z\cdot x/(2R^2)}]$$

and we can expand

$$e^{-|x|^2/(4R^2)} = 1 - \frac{|x|^2}{4R^2} + O_x(R^{-4}), \quad \mathbb{E}_R[e^{z\cdot x/(2R^2)}] = \mathbb{E}_R[1] + \frac{\mathbb{E}_R[z\cdot x]}{2R^2} + \frac{\mathbb{E}_R[(z\cdot x)^2]}{8R^4} + \varepsilon_{x,R},$$

where $|\varepsilon_{x,R}| \le O_x(R^{-6})\mathbb{E}_R[|z|^3 e^{|z||x|/(2R^2)}] \le O_x(R^{-6-m})\int_{\mathbb{R}^n}|z|^3 e^{-|z|^2/(8R^2)}\,d\mu(z) = O_x(R^{-3})$ (the last inequality can be obtained by an annular decomposition as in the previous proof). Note that, similarly, $\frac{\mathbb{E}_R[z\cdot x]}{2R^2} = O_x(R^{-1})$ and $\frac{\mathbb{E}_R[(z\cdot x)^2]}{8R^4} = O_x(R^{-2})$; hence, multiplying these two expansions, we get (2.4). □

We now show that, whenever one blow-down is a plane V, the blow-down is in fact unique and Σ, V are within a uniform neighborhood of each other.

Lemma 2.3. Assume that $\mu_j := S_{R_j}\mu \rightharpoonup \mathcal{H}^m \llcorner V$ (in duality with $C^0_c(\mathbb{R}^n)$) for some sequence $R_j \to \infty$. Then $\pi_{V^\perp}\Sigma$ is bounded, we have the full convergence

$$S_R\mu \rightharpoonup \mathcal{H}^m \llcorner V,\ Q_R \to \pi_V \quad \text{as } R\to\infty, \tag{2.5}$$

and, for some $C = C(\mu) > 0$,

$$\operatorname{dist}(p, \pi_V\Sigma) \le C \quad \text{for } p \in V. \tag{2.6}$$

Further, if we also have $\mathcal{D}(\mu) = 0$, then $b := \lim_{R\to\infty}\mathbb{E}_R[z]$ exists and $b \in V^\perp$, and moreover

$$|\pi_{V^\perp}x - b|^2 = |b|^2 \quad \text{for } x \in \Sigma. \tag{2.7}$$

The last identity can be seen as an analogue of [9, Theorem 3.6(1)] (in the special case $q = 1$).

Proof. Using (2.2) for the rescalings μ_j, it is straightforward to check that $Q^\mu_{R_j} = Q^{\mu_j}_1 \to Q^{\mathcal{H}^m\llcorner V}_1 = \pi_V$ (note that the analogue for $\mathbb{E}_{R_j}[z]$ does not hold, since this term is not normalized). Evaluating (2.4) at any $x \in \Sigma$, we then obtain a bound $|\mathbb{E}_{R_j}[z]\cdot x| \le C_x$ independent of j. Using finitely many such vectors x forming a basis of $L := \operatorname{span}\Sigma$, we obtain a uniform bound for $\pi_L\mathbb{E}_{R_j}[z]$, which equals $\mathbb{E}_{R_j}[z]$ since trivially $\mathbb{E}_{R_j}[z] \in L$. We can then extract a subsequential limit $b := \lim_{j\to\infty}\mathbb{E}_{R_j}[z] \in L$.

For each fixed $x \in \Sigma$, writing $x = p + q$ with $p \in V$ and $q \in V^\perp$, using (2.4) we deduce

$$|2b\cdot x - |q|^2| = \lim_{j\to\infty}|2\mathbb{E}_{R_j}[z]\cdot x + x\cdot Q_{R_j}x - |x|^2\mathbb{E}_{R_j}[1]| \le 8\mathcal{D}(\mu). \tag{2.8}$$

In particular, $2b\cdot x \ge -8\mathcal{D}(\mu)$. For every $v \in V$, the fact that $\operatorname{spt}\mu_j \to V$ provides $x_j \in \Sigma$ with $x_j/R_j \to v$. Since $b\cdot x_j/R_j \ge -4\mathcal{D}(\mu)/R_j$, in the limit we obtain $b\cdot v \ge 0$. Replacing v with $-v$ shows that $b \perp V$, and therefore (2.8) reads $||q-b|^2 - |b|^2| = |2b\cdot q - |q|^2| \le 8\mathcal{D}(\mu)$; in particular,

$$|\pi_{V^\perp}x - b|^2 \le |b|^2 + 8\mathcal{D}(\mu) \quad \text{for } x \in \Sigma, \tag{2.9}$$

which shows that $\pi_{V^\perp}\Sigma$ is bounded.

Any other blow-down ν, including those centered at varying support points, is thus supported in V. By Lemma 2.1 it is also m-uniform, which easily implies that $\nu = \mathcal{H}^m \llcorner V$ (indeed, by differentiation with respect to $\mathcal{H}^m\llcorner V$, we see that $\nu = \mathcal{H}^m\llcorner \operatorname{spt}\nu$ and, if we had $\operatorname{spt}\nu \ne V$, taking a relative ball $B \subset V\setminus\operatorname{spt}\nu$ with ∂B touching $\operatorname{spt}\nu$ at a point x_0, we would have $\nu(B_r(x_0)) \le \omega_m r^m/2 + o(r^m)$, a contradiction). This proves (2.5).

The set $\pi_V\Sigma$ is closed, since Σ is closed and the normal coordinates are bounded. If (2.6) fails, choose $p_j \in V$ with $d_j := \operatorname{dist}(p_j, \pi_V\Sigma) \to \infty$, nearest points $p'_j \in \pi_V\Sigma$, and $z'_j \in \Sigma$ with $\pi_V z'_j = p'_j$. As established earlier, the measures $S_{z'_j,d_j}\mu$ converge to $\mathcal{H}^m\llcorner V$, but their projected supports omit the open unit ball centered at the unit vector $(p_j - p'_j)/d_j$, giving a contradiction in the limit.

If $\mathcal{D}(\mu) = 0$, then (2.9) is an equality and gives (2.7). Also, evaluating (2.4) along any other sequence $R'_j \to \infty$, we get

$$0 = \lim_{j\to\infty} 4(R'_j)^2(G^\mu_{R'_j}(x) - G^\mu_{R'_j}(0)) = \lim_{j\to\infty} 2\mathbb{E}_{R'_j}[z]\cdot x - |\pi_{V^\perp}x|^2.$$

Thus, any subsequential limit of $\mathbb{E}_{R'_j}[z]$ has uniquely determined scalar products with all $x \in \Sigma$, i.e., with all vectors in L; since it belongs to L, it is uniquely determined, showing that the full limit $\lim_{R\to\infty} \mathbb{E}_R[z]$ exists. □

3. Rigid case and proof of Theorem 1.1

On $\mathbb{R}^m$, for any radius $r > 0$, let $k_r : \mathbb{R}^m \to (0,\infty)$ denote the usual Gaussian kernel and let $\mathcal{P}_r$ denote the corresponding convolution operator:

$$k_r(x) := (4\pi r^2)^{-m/2} e^{-|x|^2/(4r^2)}, \quad \mathcal{P}_r f := k_r * f. \tag{3.1}$$

This aligns with the convention, used throughout the paper, to use scale parameters (which have the dimension of length). With this notation, $(\mathcal{P}_{\sqrt{t}})_{t>0}$ is just the heat semigroup. In particular, we have the identities

$$\mathcal{P}_r\mathcal{P}_s f = \mathcal{P}_{\sqrt{r^2+s^2}} f, \quad \partial_r(\mathcal{P}_r f) = 2r\Delta(\mathcal{P}_r f), \quad \int_{\mathbb{R}^m} |x|^2 k_r(x)\,dx = 2mr^2. \tag{3.2}$$

Lemma 3.1. Let $f \in H^1_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^\ell)$ with $|f| = M$ a.e. for some constant $M \geq 0$. Assume, for all $r > 0$, that

$$\mathcal{P}_r f(0) = 0, \quad \mathcal{P}_r(|Df|^2)(0) \leq \frac{M^2}{r^2}, \tag{3.3}$$

$$\lim_{s\to\infty} s\mathcal{P}_r(f\cdot \mathcal{P}_s f)(0) = 0. \tag{3.4}$$

Then $f \equiv 0$.

Proof. Let $\Lambda_r := rD(\mathcal{P}_r f)(0)$. We first claim that

$$\lim_{r\to\infty} \Lambda_r = 0.$$

Differentiating the convolution and using $|f| \leq M$, we get

$$D(\mathcal{P}_r f)(0) = \frac{1}{2r^2}\int_{\mathbb{R}^m} k_r(x) f(x)\otimes x\,dx, \quad \|D^2\mathcal{P}_r f\|_{L^\infty} \leq C(m)M/r^2. \tag{3.5}$$

A simple Taylor expansion then gives $\mathcal{P}_s f(x) = D(\mathcal{P}_s f)(0)[x] + O(M|x|^2/s^2)$. Inserting this into (3.4), we get

$$o_r(s^{-1}) = \int_{\mathbb{R}^m} k_r(x) f(x)\cdot \mathcal{P}_s f(x)\,dx = \frac{2r}{s}\langle \Lambda_r, \Lambda_s\rangle + O(M^2 r^2/s^2),$$

where we used (3.5) to rewrite $D(\mathcal{P}_s f)(0)\cdot\int_{\mathbb{R}^m} k_r(x) f(x)\otimes x\,dx$ as $\frac{2r}{s}\langle\Lambda_r,\Lambda_s\rangle$. For each fixed $r > 0$, this implies $\langle\Lambda_r,\Lambda_s\rangle \to 0$ as $s\to\infty$. Since the span of all matrices Λ_r is finite-dimensional, this proves that indeed $\Lambda_s \to 0$ as $s\to\infty$.

Setting $f_r(x) := f(rx)$, since $|f_r| \leq M$ and

$$\mathcal{P}_1(|Df_r|^2)(0) = r^2\mathcal{P}_r(|Df|^2)(0) \leq M^2, \tag{3.6}$$

and moreover $k_1 > 0$ is bounded below by a positive constant on every compact set, by Rellich compactness there exist radii $r_j \to \infty$ such that

$$f_{r_j} \to g \quad \text{in } L^2_{loc}(\mathbb{R}^m;\mathbb{R}^\ell)$$

and a.e., for some $g \in H^1_{loc}(\mathbb{R}^m;\mathbb{R}^\ell)$. In particular, we have $|g| = M$ a.e. The convergence is also strong in $L^2(k_1\,dx)$, since

$$\int_{|x|>R} |f_{r_j} - g|^2 k_1(x)\,dx \leq 4M^2\int_{|x|>R} k_1(x)\,dx \to 0 \quad \text{as } R\to\infty$$

uniformly in j. Since the assignments $v \mapsto \mathcal{P}_1 v(0) = \int_{\mathbb{R}^m} k_1 v\,dx$ and $v \mapsto D(\mathcal{P}_1 v)(0) = \frac{1}{2}\int_{\mathbb{R}^m} k_1 v\otimes x\,dx$ are continuous on this weighted space, we have

$$|g| = M \text{ a.e.}, \quad \mathcal{P}_1 g(0) = \lim_{j\to\infty} \mathcal{P}_1 f_{r_j}(0) = \lim_{j\to\infty} \mathcal{P}_{r_j} f(0) = 0, \tag{3.7}$$

and also, recalling the definition of Λ_r, the previous bound (3.6), and lower semicontinuity of the Dirichlet energy,

$$D(\mathcal{P}_1 g)(0) = \lim_{j\to\infty} D(\mathcal{P}_1 f_{r_j})(0) = \lim_{j\to\infty} \Lambda_{r_j} = 0, \quad \mathcal{P}_1(|Dg|^2)(0) \le M^2. \tag{3.8}$$

To conclude, we will show that $M = 0$. For $\sigma \in (0,1)$, let

$$U_\sigma := \mathcal{P}_{\sqrt{1-\sigma}}(g), \quad J(\sigma) := \mathcal{P}_{\sqrt{\sigma}}(|U_\sigma|^2)(0), \quad K(\sigma) := \mathcal{P}_{\sqrt{\sigma}}(|DU_\sigma|^2)(0).$$

Since $\partial_\sigma \mathcal{P}_{\sqrt{\sigma}} = \Delta \mathcal{P}_{\sqrt{\sigma}}$ and $\partial_\sigma U_\sigma = -\Delta U_\sigma$, the identity $\Delta|v|^2 = 2v\cdot\Delta v + 2|Dv|^2$ gives

$$J'(\sigma) = \mathcal{P}_{\sqrt{\sigma}}(\Delta|U_\sigma|^2 + \partial_\sigma |U_\sigma|^2)(0) = \mathcal{P}_{\sqrt{\sigma}}(2|DU_\sigma|^2)(0) = 2K(\sigma), \tag{3.9}$$

and similarly

$$K'(\sigma) = 2\mathcal{P}_{\sqrt{\sigma}}(|D^2 U_\sigma|^2)(0), \quad K''(\sigma) = 4\mathcal{P}_{\sqrt{\sigma}}(|D^3 U_\sigma|^2)(0) \ge 0. \tag{3.10}$$

While these hold for $\sigma \in (0,1)$, at the endpoints we can use (3.7)–(3.8) and the fact that $|U_\sigma|^2 \to M^2$ in $L^1_{loc}(\mathbb{R}^m)$ as $\sigma \to 1$ (with $|U_\sigma|^2 \le M^2$), obtaining

$$J(0^+) = K(0^+) = 0, \quad J(1^-) = M^2.$$

By Jensen's inequality, we can also bound

$$K(\sigma) \le \mathcal{P}_{\sqrt{\sigma}}\mathcal{P}_{\sqrt{1-\sigma}}(|Dg|^2)(0) = \mathcal{P}_1(|Dg|^2)(0) \le M^2.$$

Since K is convex with $K(0^+) = 0$ and $K(1^-) \le M^2$, we must have $K(\sigma) \le \sigma M^2$, and therefore

$$M^2 = \int_0^1 J'(\sigma)\,d\sigma = \int_0^1 2K(\sigma)\,d\sigma \le \int_0^1 2\sigma M^2\,d\sigma = M^2.$$

The equality forces $K(\sigma) = \sigma M^2$, hence $K''(\sigma) = 0$ and thus $D^3 U_\sigma = 0$. We deduce that, for all $\sigma \in (0,1)$, U_σ is a bounded polynomial of degree at most two, hence constant. Consequently, by definition of K, we have $K(\sigma) = 0$. Since we obtained $K(\sigma) = \sigma M^2$ earlier, we arrive at $M = 0$. □

We next record the form in which this lemma will be applied.

Corollary 3.2. Let $(\kappa_x)_{x\in\mathbb{R}^m}$ be a (weakly measurable) family of probability measures on the sphere $\mathbb{S} := \{v \in \mathbb{R}^\ell : |v| = M\}$ and let $h(x) := \int_{\mathbb{S}} v\,d\kappa_x(v)$. Suppose that for each $r > 0$ we have

$$2r^2 A_r(x) = M^2 - v\cdot \mathcal{P}_r h(x) \quad \text{for } dx\otimes\kappa_x(v)\text{-a.e. } (x,v), \tag{3.11}$$

for a suitable bounded smooth function $A_r : \mathbb{R}^m \to [0,\infty)$, and assume moreover the identities

$$\mathcal{P}_r A_s = A_{\sqrt{r^2+s^2}}, \quad \mathcal{P}_r h(0) = 0, \quad A_r(0) = \frac{M^2}{2r^2} \tag{3.12}$$

for every $r, s > 0$. Then $\kappa_x = \delta_0$ for a.e. $x \in \mathbb{R}^m$.

Proof. Averaging (3.11) in v gives

$$2r^2 A_r = M^2 - h\cdot \mathcal{P}_r h. \tag{3.13}$$

Note that the left-hand side goes to zero in L^1_{loc} as $r \to 0$, since by (3.12) we have

$$\int k_{\sqrt{4-s^2}}(x) A_s(x)\,dx = \mathcal{P}_{\sqrt{4-s^2}} A_s(0) = A_2(0) \quad \text{for } s \in (0,2)$$

and $k_{\sqrt{4-s^2}}$ is bounded below by a positive constant on any given compact set when $s \in (0,1)$, which implies that A_s is locally uniformly bounded in L^1 for $s \in (0,1)$. Also, since $\mathcal{P}_r h \to h$ in L^2_{loc} (as $|h| \le M$), we have $h\cdot\mathcal{P}_r h \to |h|^2$ in L^1_{loc}. Hence, (3.13) yields $|h(x)| = M$ at a.e. x, at which points it follows that

$$\int_{\mathbb{S}} |v - h(x)|^2\,d\kappa_x(v) = M^2 - |h(x)|^2 = 0$$

and thus $\kappa_x = \delta_{h(x)}$ a.e. In particular, using also (3.13), for a.e. x we get

$$\int_{\mathbb{R}^m} |h(x+z) - h(x)|^2 k_r(z)\,dz = 2M^2 - 2h(x)\cdot\int_{\mathbb{R}^m} h(x+z)k_r(z)\,dz = 4r^2 A_r(x). \tag{3.14}$$

To obtain a gradient bound, we use the fact that the functions $z_i/(\sqrt{2}\,r)$ are orthonormal in $L^2(k_r(z)\,dz)$, and note that

$$\int_{\mathbb{R}^m} \frac{z_i}{\sqrt{2}\,r}(h(x+z)-h(x))k_r(z)\,dz = \int_{\mathbb{R}^m} \frac{z_i}{\sqrt{2}\,r}h(x+z)k_r(z)\,dz = \sqrt{2}\,r\,\partial_{x_i}\mathcal{P}_r h(x).$$

Bessel's inequality (applied to each component of the target $\mathbb{R}^\ell$) therefore gives

$$|D(\mathcal{P}_r h)|^2 \le \frac{1}{2r^2}\int_{\mathbb{R}^m} |h(x+z)-h(x)|^2 k_r(z)\,dz = 2A_r. \tag{3.15}$$

Using (3.12) again, for all $r,s>0$ we deduce that

$$\mathcal{P}_r(|D(\mathcal{P}_s h)|^2)(0) \le \mathcal{P}_r(2A_s)(0) = 2A_{\sqrt{r^2+s^2}}(0) = \frac{M^2}{r^2+s^2}.$$

Letting $s\to 0$, since $\mathcal{P}_s h \to h$ in L^2_{loc} we deduce that

$$h\in H^1_{loc}, \quad \mathcal{P}_r(|Dh|^2)(0) \le M^2/r^2. \tag{3.16}$$

Finally, from (3.13) we get

$$\mathcal{P}_r(h\cdot \mathcal{P}_s h)(0) = \mathcal{P}_r(M^2 - 2s^2 A_s)(0) = M^2 - 2s^2 A_{\sqrt{r^2+s^2}}(0) = M^2\frac{r^2}{r^2+s^2}.$$

We can then apply Lemma 3.1 with $f:=h$. □

Proof of Theorem 1.1. Without loss of generality, we can assume $0\in\Sigma := \operatorname{spt}\mu$. Since μ is fixed, in the sequel we will not indicate explicitly the dependence of error terms on μ and n. Applying (2.7) we find $b\in V^\perp$ such that, setting $M:=|b|$, we have

$$|\pi_{V^\perp}x - b|^2 = M^2 \quad \text{for } x\in\Sigma.$$

Let $\phi(x):=\pi_{V^\perp}x - b$. The previous identity says that $|\phi(x)|=M$ on all of Σ, while Lemma 2.3 gives $b=\lim_{R\to\infty}\mathbb{E}_R[z]$ and thus

$$\mathbb{E}_R[\phi(z)]\to 0 \tag{3.17}$$

(recall that $\mathbb{E}_R[\cdot]$ was introduced in (2.3)).

Let us identify V with $\mathbb{R}^m = \mathbb{R}^m\times\{0\}\subset\mathbb{R}^n$. For a scale $L\ge 1$, we blow down the first m components: let $\zeta_L:\mathbb{R}^n\to V\times V^\perp=\mathbb{R}^n$ be given by $\zeta_L(x):=(\pi_V x/L,\phi(x))$ and

$$\eta_L := L^{-m}(\zeta_L)_*\mu, \quad \rho_L := L^{-m}(\pi_V/L)_*\mu = (\pi_V)_*\eta_L. \tag{3.18}$$

By (2.5), we have $\rho_L \rightharpoonup dx'$, where we use x' to denote points in $\mathbb{R}^m$. More quantitatively, the coarse upper bound (2.2) and the fact that $|\pi_{V^\perp}x-b|=M$ for $x\in\Sigma$ give

$$\rho_L(B_r(x')) \le C(n)(Lr+M)^m/L^m = C(n)(r+M/L)^m \quad \text{for } x'\in\mathbb{R}^m,\ r\ge R_0/L. \tag{3.19}$$

After a subsequence, we can therefore extract a limit of η_L and disintegrate it with respect to π_V:

$$\eta_L \rightharpoonup dx'\otimes\kappa_{x'}, \tag{3.20}$$

where each $\kappa_{x'}$ is a probability measure on $\{u\in V^\perp : |u|=M\}$. We will show that, in fact, $\kappa_{x'}=\delta_0$ for a.e. x'.

For $(x_0',u_0)\in\operatorname{spt}\eta_L$, the assumption $\mathcal{D}(\mu)=0$ at the original support point $(Lx_0',u_0+b)\in\Sigma$ and scale Lr reads

$$G_{L,r}(x_0',u_0) := \int_{\mathbb{R}^n} k_r(x'-x_0')e^{-|u-u_0|^2/(4L^2r^2)}\,d\eta_L(x',u) = 1+o_r(L^{-2}). \tag{3.21}$$

For r in a compact subset of $(0,\infty)$, the error is uniform in r and in the support point. Letting

$$g_{L,r}:\mathbb{R}^m\to(0,\infty), \quad g_{L,r}:=\mathcal{P}_r\rho_L,$$

since on $\operatorname{spt}\eta_L$ it holds $|u-u_0|\le 2M$, on $L^{-1}\pi_V\Sigma$ it follows that

$$1+o_r(L^{-2}) \le g_{L,r} \le e^{M^2/(L^2r^2)}(1+o_r(L^{-2})); \tag{3.22}$$

recall that, by (2.6), $L^{-1}\pi_V\Sigma$ is a $(C(\mu)L^{-1})$-net in $\mathbb{R}^m$. Also, (3.19) bounds all spatial derivatives of $g_{L,r}$ uniformly in L, locally uniformly in $r>0$. We claim that, as a consequence,

$$\|g_{L,r}-1\|_{L^\infty}\le C(\mu,r)L^{-2}. \tag{3.23}$$

Indeed, we can assume $m>0$, since when $m=0$ the claim follows from (3.22). Let $f:=g_{L,r}-1$ and $\delta:=C(\mu)L^{-1}$, so that $L^{-1}\pi_V\Sigma$ is a δ-net. Fix $x'\in\mathbb{R}^m$ and choose a unit vector e parallel to $\nabla f(x')$ (we can take any unit vector if $\nabla f(x')=0$). By the net property, there exist $p_\pm\in L^{-1}\pi_V\Sigma$ such that $|p_\pm-(x'\pm\delta e)|\le\delta$. Consequently,

$$|p_\pm-x'|\le 2\delta,\quad \nabla f(x')\cdot(p_+-x')\ge 0,\quad \nabla f(x')\cdot(p_--x')\le 0.$$

Since by Taylor's formula we have

$$f(p_\pm)=f(x')+\nabla f(x')\cdot(p_\pm-x')+\varepsilon_\pm,\quad |\varepsilon_\pm|\le\frac{H|p_\pm-x'|^2}{2}\le 2H\delta^2,$$

where $H:=\|D^2f\|_{L^\infty}$, using (3.22) and the signs of the linear terms we obtain

$$o_r(L^{-2})-2H\delta^2\le f(p_-)-2H\delta^2\le f(x')\le f(p_+)+2H\delta^2\le O_r(L^{-2})+2H\delta^2,$$

which proves the claim (3.23) since the Hessian of $f=g_{L,r}-1$ is bounded uniformly in L,r (for r in a compact set).

We now reconstruct the ingredients needed to apply Corollary 3.2. Let

$$h(x'):=\int_{V^\perp}u\,d\kappa_{x'}(u),$$

so that $|h|\le M$. We also define

$$A_{L,r}:=L^2(g_{L,r}-1),\quad h_{L,r}(x_0'):=\int_{\mathbb{R}^n}k_r(x'-x_0')u\,d\eta_L(x',u). \tag{3.24}$$

The identities $\mathcal{P}_sA_{L,r}=A_{L,\sqrt{r^2+s^2}}$, $\partial_rA_{L,r}=2r\Delta A_{L,r}$, and (3.23) give uniform bounds (in L) for any derivative of $A_{L,r}$ with respect to x',r, on compact subsets of $\{r>0\}$. Hence, after a subsequence, we have $A_{L,r}\to A_r$ locally smoothly in $(x',r)\in\mathbb{R}^m\times(0,\infty)$. The L^∞ bound on $A_{L,r}$ for fixed $r>0$ allows us to interchange the limit with Gaussian convolutions, giving

$$\mathcal{P}_sA_r=\lim_{L\to\infty}\mathcal{P}_s(L^2(\mathcal{P}_r\rho_L-1))=\lim_{L\to\infty}L^2(\mathcal{P}_{\sqrt{r^2+s^2}}\rho_L-1)=A_{\sqrt{r^2+s^2}}. \tag{3.25}$$

Likewise, we have $h_{L,r}\to\mathcal{P}_rh$ locally smoothly, by definition of $h_{L,r}$ and (3.19).

Finally, expanding the term $e^{-|u-u_0|^2/(4L^2r^2)}=1-\frac{|u-u_0|^2}{4L^2r^2}+o_r(L^{-2})$ in (3.21), we get

$$1+o_r(L^{-2})=G_{L,r}(x_0',u_0)=g_{L,r}(x_0')-\int_{\mathbb{R}^n}k_r(x'-x_0')\frac{|u-u_0|^2}{4L^2r^2}\,d\eta_L(x',u)+o_r(L^{-2})$$

for any $(x_0',u_0)\in\operatorname{spt}\eta_L$, which rearranges to

$$2r^2A_{L,r}(x_0')=M^2g_{L,r}(x_0')-u_0\cdot h_{L,r}(x_0')+o_r(1), \tag{3.26}$$

where $o_r(1)$ denotes an error which goes to zero as $L\to\infty$, uniformly on compact subsets of $\{r>0\}$. As a consequence, for any fixed $r>0$, in the limit $L\to\infty$ we obtain

$$2r^2A_r(x')=M^2-u\cdot\mathcal{P}_rh(x')\quad\text{for }(x',u)\in\operatorname{spt}(dx'\otimes\kappa_{x'}). \tag{3.27}$$

In particular, averaging in u we get $2r^2A_r(x')=M^2-h(x')\cdot\mathcal{P}_rh(x')$ for a.e. $x'\in\mathbb{R}^m$ and, since $|h|,|\mathcal{P}_rh|\le M$, by continuity we have $A_r\ge 0$ everywhere.

To conclude, we have

$$h_{L,r}(0)=\int_{\mathbb{R}^n}k_{Lr}(x')u\,d\mu(x',u+b)=\mathbb{E}_{Lr}[e^{|\pi_{V^\perp}z|^2/(4L^2r^2)}\phi(z)];$$

since $\pi_{V^\perp}\Sigma$ is bounded, this equals $\mathbb{E}_{Lr}[\phi(z)]+O_r(L^{-2})$, which tends to zero as $L\to\infty$ by (3.17). Therefore,

$$\mathcal{P}_rh(0)=\lim_{L\to\infty}h_{L,r}(0)=0. \tag{3.28}$$

The original support point $0 \in \Sigma$ corresponds to $(0, -b) \in \operatorname{spt} \eta_L$ for every $L > 0$. Evaluating (3.27) at this point (note that (3.27) holds also at limits of points in $\operatorname{spt} \eta_L$, as $L \to \infty$) and using the fact that $\mathcal{P}_r h(0) = 0$, we also get

$$A_r(0) = \frac{M^2}{2r^2}.$$

Hence, we are under the assumptions of Corollary 3.2, which gives $M = 0$, as required. □

The following immediate consequence was first obtained in [9, Corollary 3.16], as one of the key rigidity steps towards Theorem 1.8.

Corollary 3.3. An m-uniform measure μ with one flat blow-down is flat.

Proof. By m-uniformity, we compute that $G_R^\mu = 1$ on $\operatorname{spt} \mu$. By Theorem 1.1, $\operatorname{spt} \mu$ is included in an affine m-plane. Using m-uniformity again, we see that μ must agree with the Hausdorff measure on this plane (see, e.g., the proof of Lemma 2.3 for the standard argument). □

4. Application to Preiss's theorem

We now show how to apply Corollary 3.3 in order to obtain Theorems 1.8 and 1.9. All the facts presented in this section are well known, but they are collected here in order to make the paper more self-contained. We will follow closely the presentations of [1, 7], which provide streamlined and insightful expositions of these topics. We will also give references to earlier works where such tools originated.

Given $\alpha \geq 0$, let $\mathcal{U}^\alpha$ denote the class of centered α-uniform measures ν, i.e., positive Radon measures on $\mathbb{R}^n$ such that $0 \in \operatorname{spt} \nu$ and

$$\nu(B_r(y)) = \omega_\alpha r^\alpha \quad \text{for } y \in \operatorname{spt} \nu,\ r > 0,$$

where $\omega_\alpha := \pi^{\alpha/2}/\Gamma(1 + \alpha/2)$, which is the volume of the α-dimensional unit ball when $\alpha \in \mathbb{N}$.

Consider a measure μ as in the statement of Theorem 1.8. For μ-a.e. x, using the notation $S_{x,r}\mu := r^{-\alpha}(z \mapsto (z - x)/r)_*\mu$, the following standard facts hold (see, e.g., [1, Proposition 3.4] or [7, Lemma 14.7] for the inclusion $\mathcal{T}_x\mu \subseteq \mathcal{U}^\alpha$ in (ii), and [1, Lemma 6.6] or [7, Theorem 14.16] for (iii), respectively):

(i) $\theta(x) := \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}$ exists in $(0, \infty)$;

(ii) the family $S_{x,r}\mu/\theta(x)$ is sequentially precompact as $r \to 0$, namely for any sequence $r_j \to 0$ there exists a subsequence (not relabeled) such that $\lim_{j \to \infty} S_{x,r_j}\mu/\theta(x)$ exists in the weak-* topology, and the set $\mathcal{T}_x\mu$ of all such limits satisfies

$$\mathcal{T}_x\mu \subseteq \mathcal{U}^\alpha;$$

(iii) if $\nu \in \mathcal{T}_x\mu$ then, for any $y \in \operatorname{spt} \nu$ and $s > 0$, also $S_{y,s}\nu \in \mathcal{T}_x\mu$.

Elements in $\mathcal{T}_x\mu$ are called *tangent measures* at x (note that normalization conventions are slightly inconsistent throughout the literature).

Remark 4.1. Note that, if we assume without loss of generality that $\mu(\mathbb{R}^n) < \infty$, then for μ-a.e. x (namely, for x where (i) holds) there exists $C_x > 0$ such that $\mu(B_R(x)) \leq C_x \theta(x) R^\alpha$ for all $R > 0$. In particular, the rescalings $S_{x,r}\mu/\theta(x)$ (and thus also the elements of $\mathcal{T}_x\mu$) belong to the set

$$\mathcal{M}^\alpha_{C_x} := \{\nu \in \mathcal{M}(\mathbb{R}^n) : \nu(B_R(0)) \leq C_x R^\alpha \text{ for all } R > 0\}, \tag{4.1}$$

where $\mathcal{M}(\mathbb{R}^n)$ denotes the set of all positive Radon measures on $\mathbb{R}^n$. By standard functional analysis, the weak-* topology of $\mathcal{M}(\mathbb{R}^n)$, restricted to $\mathcal{M}^\alpha_{C_x}$, can be metrized, yielding a compact metric space.

The following fact was first obtained in [5, Theorem 1].

Proposition 4.2. The exponent α is an integer in $\{0, \dots, n\}$ and, at every point x where (i)–(iii) above hold, $\mathcal{T}_x\mu$ contains a flat measure.

Proof. Fix x where (i)–(iii) hold. Let m be the minimal dimension of $\operatorname{span}(\operatorname{spt}\mu')$, as $\mu' \in \mathcal{T}_x\mu$ varies, and let $\nu \in \mathcal{T}_x\mu$ be a minimizer, so that $m = \dim \operatorname{span}(\operatorname{spt}\nu)$. Let us identify $\operatorname{span}(\operatorname{spt}\nu)$ with $\mathbb{R}^m$.

We must have $\alpha \le m$, since otherwise we could cover any compact $K \subset \mathbb{R}^m$ with a family of balls $\{B_r(x_i)\}_{i\in I}$ with $|I| = O_K(r^{-m})$ and deduce that $\nu(K) \le \omega_\alpha(2r)^\alpha \cdot |I| = O_K(r^{\alpha-m})$ (note that any ball $B_r(x)$ has $\nu(B_r(x)) \le \omega_\alpha(2r)^\alpha$, since if it intersects $\operatorname{spt}\nu$ at x' then $B_r(x) \subseteq B_{2r}(x')$), which would give $\nu(K) = 0$ once we let $r \to 0$, a contradiction.

Assume now that $\alpha < m$; we will reach a contradiction also in this case, proving that $\alpha = m$ is an integer. We cannot have $\operatorname{spt}\nu = \mathbb{R}^m$, since otherwise we could place at least $c(m)r^{-m}$ disjoint balls of any given radius $r \in (0,1)$ inside $B_1(0)$, giving $\nu(B_1(0)) \ge c(m)\omega_\alpha r^{\alpha-m}$ by uniformity, a contradiction since $r^{\alpha-m}$ blows up as $r \to 0$. We can then choose $p \notin \operatorname{spt}\nu$ and a nearest point $p' \in \operatorname{spt}\nu$, so that the ball $\overline{B}_{|p'-p|}(p)$ touches $\operatorname{spt}\nu$ at p'. Letting $v := p' - p$, it follows that any tangent $\eta \in \mathcal{T}_{p'}\nu \subseteq \mathcal{T}_x\mu$ is supported in the half-space $\{z : \langle z, v\rangle \ge 0\}$.

By uniformity, the smooth function

$$H(y) := \int_{\mathbb{R}^m} e^{-|z-y|^2}\, d\eta(z)$$

is constant on $\operatorname{spt}\eta$. Since $H(y) - H(0) = 2\langle w, y\rangle + O(|y|^2)$, with $w := \int_{\mathbb{R}^m} z e^{-|z|^2}\, d\eta(z)$, if $w \neq 0$ then any tangent at zero is supported in $w^\perp$ (as $H(y) - H(0)$ vanishes on $\operatorname{spt}\eta$). On the other hand, if $w = 0$ then

$$\int_{\mathbb{R}^m} \langle z, v\rangle e^{-|z|^2}\, d\eta(z) = 0,$$

giving $\langle z, v\rangle = 0$ for $z \in \operatorname{spt}\eta$ (since we had $\langle z, v\rangle \ge 0$), and hence $\operatorname{spt}\eta \subseteq v^\perp$. In both cases, we contradict the minimality of m.

Thus, $\alpha = m$ is an integer and, by the same argument, $\operatorname{spt}\nu$ equals its span, an m-dimensional plane, proving that ν is flat (see, e.g., the proof of Lemma 2.3). □

The cases $m = 0$ and $m = n$ in Theorem 1.8 are very classical: when $m = 0$, it is easy to check that the measure is atomic, i.e., a finite or countable sum of (multiples of) Dirac masses, while if $m = n$ then (by differentiation with respect to the Lebesgue measure $\mathcal{L}^n$) we have $\mu \ll \mathcal{L}^n$, matching the definition of rectifiable measure in these borderline cases. We therefore assume $1 \le m \le n-1$ for the rest of this section.

Given $\nu \in \mathcal{U}^m$, in the sequel we use again the notation $\mathbb{E}_R[\cdot] = \mathbb{E}^\nu_R[\cdot]$ from (2.3) and set

$$Q(\nu) := \frac{1}{2}\mathbb{E}_1[z \otimes z] \ge 0.$$

Using uniformity, we compute that $\mathbb{E}_R[1] = 1$ and $\mathbb{E}_R[|z|^2] = 2mR^2$, so that $\operatorname{tr} Q(\nu) = m$. Let us record how facts proved earlier simplify in this case, starting from the following special case of [9, Theorem 3.6(1)].

Lemma 4.3. If $\nu \in \mathcal{U}^m$ and λ is a blow-down, i.e., $S_{R_j}\nu \rightharpoonup \lambda$ for a sequence of scales $R_j \to \infty$, then there exists $b \in \operatorname{span}(\operatorname{spt}\nu)$ such that

$$|x|^2 = 2b \cdot x + x \cdot Q(\lambda)x \quad \text{for } x \in \operatorname{spt}\nu, \tag{4.2}$$

$$|y|^2 = y \cdot Q(\lambda)y \quad \text{for } y \in \operatorname{spt}\lambda. \tag{4.3}$$

Proof. Since $G^\nu_R \equiv 1$ on the support, (2.4) becomes

$$2\mathbb{E}_R[z] \cdot x + x \cdot Q(S_R\nu)x = |x|^2 + O_x(R^{-1}) \quad \text{for } x \in \operatorname{spt}\nu.$$

By uniformity of $S_{R_j}\nu$ and fast decay of the Gaussian weight, we have $Q(S_{R_j}\nu) \to Q(\lambda)$. As in the proof of Lemma 2.3, we get $\mathbb{E}_{R_j}[z] \to b$, so that (4.2) holds at every $x \in \operatorname{spt}\nu$. For $y \in \operatorname{spt}\lambda$, choose $x_j \in \operatorname{spt}\nu$ such that $x_j/R_j \to y$. Using these points in (4.2) and dividing by R_j^2, we deduce (4.3). □

On the set of positive Radon measures $\mathcal{M}(\mathbb{R}^n)$, consider the functional $\mathcal{F} : \mathcal{M}(\mathbb{R}^n) \to [0, \infty]$ given by

$$\mathcal{F}(\eta) := \min_{V \in \mathrm{Gr}_m(\mathbb{R}^n)} (4\pi)^{-m/2} \int_{\mathbb{R}^n} e^{-|z|^2/4} \operatorname{dist}(z, V)^2 \, d\eta(z), \tag{4.4}$$

where $\mathrm{Gr}_m(\mathbb{R}^n)$ is the set of all m-planes passing through the origin. It is easy to check that $\mathcal{F}|_{\mathcal{U}^m}$ is finite, continuous, and vanishes precisely at the flat measures. Let $\mathcal{Q}$ be the class of $\lambda \in \mathcal{U}^m$ satisfying (4.3) with their own matrix $Q(\lambda)$. By Lemma 4.3, every blow-down of a uniform measure belongs to $\mathcal{Q}$. The following is essentially the fundamental rigidity result [9, Theorem 3.14(1)] observed by Preiss for blow-downs.

Lemma 4.4. There exists $\delta_0(n) > 0$ such that any $\lambda \in \mathcal{Q}$ with $\mathcal{F}(\lambda) \leq \delta_0$ must be flat.

Proof. Assuming that the statement is false, there exist non-flat measures $\lambda_j \in \mathcal{Q}$ with $\mathcal{F}(\lambda_j) \to 0$. After a subsequence, they converge to a limit $\overline{\lambda}$ with $\mathcal{F}(\overline{\lambda}) = 0$, which is then flat. In other words, there exists an m-plane V such that

$$\lambda_j \rightharpoonup \mathcal{H}^m \llcorner V, \quad Q(\lambda_j) \to \pi_V.$$

Let us list the eigenvalues of $Q(\lambda_j)$ in decreasing order, as $\alpha_{j,1} \geq \cdots \geq \alpha_{j,n} \geq 0$, and choose an orthonormal eigenbasis $\{e_{j,1}, \dots, e_{j,n}\}$ of $Q(\lambda_j)$.

If $\alpha_{j,m} \geq 1$, then the fact that $\sum_k \alpha_{j,k} = \operatorname{tr} Q(\lambda_j) = m$ (with $\alpha_{j,k} \geq 0$) forces $\alpha_{j,1} = \cdots = \alpha_{j,m} = 1$ and $\alpha_{j,m+1} = \cdots = \alpha_{j,n} = 0$. Thus, in this case $Q(\lambda_j)$ is the orthogonal projection to an m-plane V_j and, by (4.3), we have $\operatorname{spt} \lambda_j \subseteq V_j$, contradicting our assumption that λ_j is not flat. We can then assume $\alpha_{j,m} = 1 - \delta_j$ for some $\delta_j > 0$.

Using again the fact that $\operatorname{tr} Q(\lambda_j) = m$, we get

$$\alpha_{j,k} \leq m - (m-1)(1-\delta_j) = 1 + (m-1)\delta_j \quad \text{for } k < m$$

(since all other summands in $\sum_{k=1}^m \alpha_{j,k} \leq m$ are at least $1 - \delta_j$), as well as $\alpha_{j,k} < 1$ for $k > m$. Thus, given $x \in \operatorname{spt} \lambda_j$, writing $x_k := \langle x, e_{j,k} \rangle$ for simplicity, using the assumption $\lambda_j \in \mathcal{Q}$ we find

$$0 = \sum_{k=1}^n (\alpha_{j,k} - 1) x_k^2 \leq \delta_j \bigg((m-1) \sum_{k<m} x_k^2 - x_m^2 \bigg).$$

We deduce that $\operatorname{spt} \lambda_j \subseteq \{x \in \mathbb{R}^n : x_m^2 \leq (m-1) \sum_{k<m} x_k^2\}$, a set which excludes $B_\rho(e_{j,m})$ for some small enough $\rho(m) > 0$. However, this contradicts the fact that $\operatorname{spt} \lambda_j$ converges to an m-plane V containing $\lim_{j\to\infty} e_{j,m}$ (after a subsequence). □

We are now ready to deduce Theorem 1.8 from Corollary 3.3.

Proof of Theorem 1.8. By restricting μ to a ball, we can assume without loss of generality that $\mu(\mathbb{R}^n) < \infty$. Fix a point $x \in \operatorname{spt} \mu$ where the properties (i)–(iii) listed above hold. Besides integrality of the dimension, Proposition 4.2 provides a flat measure $\zeta \in \mathcal{T}_x\mu$. Now assume by contradiction that there exists a non-flat $\xi \in \mathcal{T}_x\mu$. Thus, letting

$$\mu_r := S_{x,r}\mu / \theta(x),$$

there exist sequences $r_j \to 0$ and $s_j \to 0$, with $r_{j+1} < s_j < r_j$, such that

$$\mu_{r_j} \rightharpoonup \zeta, \quad \mu_{s_j} \rightharpoonup \xi.$$

The proof of the theorem is based on a continuity argument, exploiting the fact that $\mathcal{F}(\mu_r)$ is a continuous function of $r > 0$ (note that $\mathcal{F}$ is finite and continuous on the class $\mathcal{M}^m_{C_x}$ from (4.1)).

Combining Lemmas 4.3 and 4.4 with Corollary 3.3 proves that, whenever $\nu \in \mathcal{U}^m$ and $\liminf_{R\to\infty} \mathcal{F}(S_R\nu) < \delta_0$, in fact ν must be flat. Fixing any $0 < \delta < \min\{\delta_0, \mathcal{F}(\xi)\}$, since $\mathcal{F}(\mu_{s_j}) \to \mathcal{F}(\xi)$ we must have $\mathcal{F}(\mu_{s_j}) > \delta$ eventually, and similarly $\mathcal{F}(\mu_{r_j}) < \delta$. Thus, for j large enough we can let $\sigma_j := \max\{t \in [s_j, r_j] : \mathcal{F}(\mu_t) \geq \delta\}$. After a subsequence, $\mu_{\sigma_j} \rightharpoonup \upsilon \in \mathcal{U}^m$ with $\mathcal{F}(\upsilon) = \delta$. We claim that $\mathcal{F}(S_R\upsilon) \leq \delta$ for any $R \geq 1$ as well, which implies that υ is flat, and thus a contradiction.

To check this, note that $r_j/\sigma_j \to \infty$: indeed, if we had $r_j/\sigma_j \to \gamma \in [1,\infty)$ along a subsequence, then we would have $\zeta = S_\gamma \upsilon$ (as $\mu_{r_j} = S_{r_j/\sigma_j}\mu_{\sigma_j}$), which is impossible since ζ is flat while υ is not. As a consequence, we indeed have

$$\mathcal{F}(S_R\upsilon) = \lim_{j\to\infty} \mathcal{F}(S_R\mu_{\sigma_j}) = \lim_{j\to\infty} \mathcal{F}(\mu_{R\sigma_j}) \le \delta$$

for any $R \ge 1$, since eventually $\sigma_j \le R\sigma_j < r_j$ and, by definition of σ_j, $\mathcal{F}(\mu_t) \le \delta$ for all $t \in [\sigma_j, r_j]$.

We thus proved that all tangents at x are flat. By the rectifiability criterion of Marstrand–Mattila from [4, 6] (see also [1, Chapter 5] or [7, Theorem 16.7], or the argument at the end of the next subsection together with the appendix), we conclude that μ is rectifiable. □

4.1. **Proof of Theorem 1.9.** In this last part of the section, we just mention how the previous arguments need to be modified in order to obtain Theorem 1.9. First of all, we introduce a class of *near-uniform* measures: given any $\varepsilon > 0$, we set

$$\mathcal{U}^\alpha_\varepsilon := \{\nu \in \mathcal{M}(\mathbb{R}^n) : 0 \in \operatorname{spt}\nu,\ \omega_\alpha r^\alpha \le \nu(B_r(x)) \le (1+\varepsilon)\omega_\alpha r^\alpha \text{ for } x \in \operatorname{spt}\nu,\ r > 0\}.$$

Let $\mathcal{G}$ be the set of good points where the analogues of properties (i)–(iii) above hold, namely:

(i') $0 < \theta(x) := \Theta^\alpha_*(\mu, x) \le \Theta^{\alpha,*}(\mu, x) < \infty$ with $\Theta^{\alpha,*}(\mu, x) \le (1+\varepsilon_0)\Theta^\alpha_*(\mu, x)$;

(ii') the family $S_{x,r}\mu/\theta(x)$ is sequentially precompact as $r \to 0$ and the set $\mathcal{T}_x\mu$ of all possible limits $\lim_{j\to\infty} S_{x,r_j}\mu/\theta(x)$ satisfies

$$\mathcal{T}_x\mu \subseteq \mathcal{U}^\alpha_{\varepsilon_0};$$

(iii') if $\nu \in \mathcal{T}_x\mu$ then, for any $y \in \operatorname{spt}\nu$ and $s > 0$, also $S_{y,s}\nu \in \mathcal{T}_x\mu$ (same as (iii)).

Here, ε_0 is the same constant appearing in (1.6). Again we have $\mu(\mathbb{R}^n \setminus \mathcal{G}) = 0$ and, without loss of generality, we can assume $\mu(\mathbb{R}^n) < \infty$.

The fact that $\alpha \in \mathbb{N}$ for uniform measures, together with a simple compactness argument, yields $\varepsilon(\alpha, n) > 0$ small enough such that, whenever $\nu \in \mathcal{U}^\alpha_\varepsilon$, we must have $\alpha \in \mathbb{N}$. Thus, up to taking $\varepsilon_0(\alpha, n) > 0$ small enough, we can assume that $\alpha = m \in \mathbb{N}$, and actually $m \in \{1, \dots, n-1\}$. The following is an analogue of [9, Corollary 4.9].

Lemma 4.5. Given any $\delta \in (0, \delta_0)$ (where $\delta_0(n)$ is given by Lemma 4.4), up to taking ε_0 small enough, we have $\mathcal{F}(\nu) \le \delta$ for all $x \in \mathcal{G}$ and $\nu \in \mathcal{T}_x\mu$.

Proof. Recall that, given $\delta \in (0, \delta_0)$, for an m-uniform measure $\nu \in \mathcal{U}^m$ we cannot have

$$\mathcal{F}(\nu) = \delta, \quad \mathcal{F}(S_R\nu) \le \delta \text{ for } R \ge 1.$$

Another compactness argument yields $L(\delta, n) > 1$ and $\varepsilon(\delta, n) > 0$ such that, whenever $\nu \in \mathcal{U}^m_\varepsilon$ and $\mathcal{F}(\nu) = \delta$, then

$$\mathcal{F}(S_R\nu) > \delta \quad \text{for some } R \in [1, L]; \tag{4.5}$$

for this L, by compactness again, there exists also $\tau(\delta, n) > 0$ such that

$$\mathcal{F}(S_R\nu) > \tau \quad \text{for all } R \in [1, L], \tag{4.6}$$

up to shrinking ε further.

Next, there exists a tangent measure $\zeta \in \mathcal{T}_x\mu$ such that $\mathcal{F}(\zeta) < \tau$ (again, up to shrinking ε). Indeed, if we had a sequence of measures μ_j satisfying (1.6) with parameters $\varepsilon_j \to 0$, but violating this conclusion at generic points x_j, we could consider the set $\mathcal{T}$ given by the Kuratowski upper limit of the sets $\mathcal{T}_{x_j}\mu_j$ (i.e., the set of all possible subsequential limits of sequences $\nu_j \in \mathcal{T}_{x_j}\mu_j$). Note that $\mathcal{T} \subseteq \mathcal{U}^m$ is compact and, whenever $\nu \in \mathcal{T}$, $y \in \operatorname{spt}\nu$, and $r > 0$, we have $S_{y,r}\nu \in \mathcal{T}$. Hence, we could repeat the proof of Proposition 4.2 and find a flat $\nu \in \mathcal{T}$. This would mean that, along a subsequence, there exist $\nu_j \in \mathcal{T}_{x_j}\mu_j$ such that $\nu_j \rightharpoonup \nu$, and in particular $\mathcal{F}(\nu_j) \to \mathcal{F}(\nu) = 0$, a contradiction, proving the claim.

Finally, we claim that, as a consequence of the previous observations, all tangent measures $\nu \in \mathcal{T}_x\mu$ have $\mathcal{F}(\nu) \le \delta$. If this failed then, using the notation from the previous proof, we could find sequences $r_j, s_j \to 0$ with $r_{j+1} < s_j < r_j$, $\mathcal{F}(\mu_{r_j}) < \tau$, and $\mathcal{F}(\mu_{s_j}) > \delta$, since there would exist two tangent measures ζ, ξ where $\mathcal{F}(\zeta) < \tau$ and $\mathcal{F}(\xi) > \delta$, respectively. Taking $\sigma_j \in (s_j, r_j)$

as in the previous proof, for j large enough we should have $\mathcal{F}(\mu_{R\sigma_j}) > \tau$ for all $R \in [1, L]$, since otherwise we would contradict (4.6) in the limit $j \to \infty$; in particular, $r_j \geq L\sigma_j$.

Thus, by (4.5), for j large enough there would exist $R \in [1, L]$ (depending on j) such that $\mathcal{F}(\mu_{R\sigma_j}) > \delta$, contradicting the definition of σ_j since $R\sigma_j \in (\sigma_j, r_j)$. □

Consider now any compact, purely unrectifiable set $F \subseteq \mathcal{G}$. It suffices to show that $\mu(F) = 0$, which we will do by invoking a known rectifiability criterion, namely Proposition 6.1, proved in the appendix for the sake of completeness.

By standard measure theory, up to shrinking F to another compact set (with positive measure if $\mu(F) > 0$), we can assume without loss of generality that $\theta(x) \in [t, 3t/2]$ as $x \in F$ varies, for some uniform $t > 0$, so that $\Theta^m_*(\mu, x) \geq t$ and $\Theta^{m,*}(\mu, x) \leq 2t$, and actually we can further assume that

$$t/2 \leq \frac{\mu(B_r(x))}{\omega_m r^m} \leq 4t \quad \text{for } x \in F,\ r \in (0, r_0), \tag{4.7}$$

as well as $\mu(B_r(x)) \leq C_0 r^m$ for all $x \in F$ and $r > 0$ (for uniform $r_0, C_0 > 0$). In particular, $\mu_{x,r} := S_{x,r}\mu/\theta(x)$ belongs to the class $\mathcal{M}^m_{C_0/t}$ introduced in Remark 4.1, so that we can fix a distance $\mathfrak{d}$ metrizing the weak-* topology here. Up to shrinking F (and r_0) further, we can also assume that convergence to tangent measures is uniform in x, namely that

$$\mathfrak{d}(\mu_{x,r}, \mathcal{T}_x\mu) \leq \delta \quad \text{for } x \in F,\ r \in (0, r_0). \tag{4.8}$$

In the rest of the proof, we will tacitly assume that δ (and thus ε_0 in (1.6) and r_0 in (4.8)) is small enough, so that the corresponding conclusions of each subsequent step hold true.

Fix $\eta < c(m)/8$ in the statement of Proposition 6.1. Note that its assumption (6.1) follows immediately from (4.7). Moreover, since $\mathcal{F}(\nu) \leq \delta$ for all $\nu \in \mathcal{T}_x\mu$, using (4.8) and a compactness argument (in which $\delta \to 0$) we see that $\mu_{x,r}$ is close to a suitable flat measure $\mathcal{H}^m \llcorner V$. In particular, (6.2) holds for any $G \subseteq F$, since any point y in $F \cap B_{2r}(x)$ outside of $B_{\eta r}(V + x)$ would have $\mu(B_{\eta r/2}(y)) \geq t \cdot \frac{\omega_m \eta^m}{2^{m+1}} r^m$, which is impossible in the limit $\delta \to 0$ (since $(\mathcal{H}^m \llcorner V)(B_{\eta/2}(z)) = 0$ whenever $z \notin B_\eta(V)$).

By standard differentiation of measures, we can now find a compact $G \subseteq F$, with $\mu(G) \geq \mu(F)/2$, such that

$$\mu(B_{2r}(x) \setminus F) < t \cdot \omega_m \eta^m r^m/2 \quad \text{for } x \in G,\ r \in (0, r_1),$$

for a suitable $r_1 \in (0, r_0)$. It follows that (6.3) holds, up to replacing r_0 with r_1: indeed, given $x \in G$ and $r \in (0, r_1)$, for any $v \in V \cap B_1(0)$ we have $\mu(B_{\eta r}(x + rv)) > t \cdot \omega_m \eta^m r^m/2$ (since in the limit $(\mathcal{H}^m \llcorner V)(B_\eta(v)) = \omega_m \eta^m$) and $B_{\eta r}(x + rv) \subseteq B_{2r}(x)$. By our choice of G, we must have $B_{\eta r}(x + rv) \cap F \neq \emptyset$, i.e., $x + rv \in B_{\eta r}(F)$, as claimed. Proposition 6.1 then gives $\mu(G) = 0$, and hence $\mu(F) = 0$, as desired.

5. The quantitative width estimate

We now prove Theorem 1.4 for finite $\mathcal{D}(\mu)$. As usual, we let $\Sigma := \operatorname{spt}\mu$ and assume without loss of generality that $\mu \neq 0$. Recalling the assumption that a blow-down is flat, namely $\mathcal{H}^m \llcorner V$, Lemma 2.3 then shows that $\pi_{V^\perp}\Sigma$ is bounded. Letting

$$K := \overline{\pi_{V^\perp}\Sigma},$$

let $w \geq 0$ be the smallest radius of a ball $\overline{B}_w(q) \supseteq K$ in $V^\perp$. Up to a translation in $V^\perp$ and in V, we can assume that $q = 0$ and $0 \in \pi_V\Sigma$; let $\hat{q}_0 \in V^\perp$ be such that $\hat{x}_0 := (0, \hat{q}_0) \in \operatorname{spt}\mu$, where we identify $\mathbb{R}^n = V \times V^\perp$.

Since the claimed bound trivially holds for $w = 0$ and is easily seen to be scale-invariant, we can assume from now on that $w = 1$. Given $R > 0$, let β_R be the probability measure on K given by

$$\int_K \psi \, d\beta_R := \frac{(4\pi R^2)^{-m/2}}{G^\mu_R(\hat{x}_0)} \int_{\mathbb{R}^n} \psi(q) e^{-(|p|^2 + |q - \hat{q}_0|^2)/(4R^2)} \, d\mu(p, q),$$

where we write points $x \in \mathbb{R}^n$ as $(p, q) \in V \times V^\perp$. Let β be any subsequential limit of β_R as $R \to \infty$.

As shown in the proof of Lemma 2.3, there exists a vector $b \in V^\perp$, actually given by the center of mass of β, such that a variant of (2.8) holds: namely, translating μ by $-\hat{x}_0$, we obtain a measure ν with $0 \in \operatorname{spt}\nu$, so that (2.4) reads

$$4R_j^2(G^\nu_{R_j}(y) - G^\nu_{R_j}(0)) = 2\mathbb{E}^\nu_{R_j}[z]\cdot y + y\cdot Q^\nu_{R_j} y - |y|^2\mathbb{E}^\nu_{R_j}[1] + o_y(1).$$

Noting that $\mathbb{E}^\nu_{R_j}[z] \to b - \hat{q}_0$, $Q^\nu_{R_j} \to \pi_V$, and $\mathbb{E}^\nu_{R_j}[1] \to 1$, this is the same as

$$4R_j^2(G^\mu_{R_j}(x) - G^\mu_{R_j}(\hat{x}_0)) = 2(b-\hat{q}_0)\cdot(x-\hat{x}_0) + (x-\hat{x}_0)\cdot\pi_V(x-\hat{x}_0) - |x-\hat{x}_0|^2 + o_x(1),$$

and the right-hand side is simply $2(b-\hat{q}_0)\cdot(q-\hat{q}_0) - |q-\hat{q}_0|^2 + o_x(1)$. Taking the difference at two points $x', x'' \in \Sigma$ and passing to the limit, we get

$$\operatorname{osc}_{q\in K}(|q|^2 - 2b\cdot q) \le 8\mathcal{D}(\mu). \tag{5.1}$$

Recalling that $\overline{B}_1(0)$ is the smallest ball including K, we see that

$$0 \in \operatorname{co}(K \cap \partial B_1(0));$$

indeed, if this were not the case, we could find a unit vector $v \in V^\perp$ such that $\langle q, v\rangle > 0$ for all $q \in K \cap \partial B_1(0)$, which would easily imply that K is included in a ball of smaller radius, centered at εv, for $\varepsilon > 0$ small enough: otherwise, there would exist $q_\varepsilon \in K$ with $|q_\varepsilon - \varepsilon v|^2 \ge 1$, i.e., $-2\varepsilon\langle q_\varepsilon, v\rangle + \varepsilon^2 \ge 1 - |q_\varepsilon|^2 \ge 0$, giving a limiting $q_0 \in K$ with $|q_0| = 1$ and $\langle q_0, v\rangle \le 0$, impossible.

Consequently, there exist finitely many $q_i \in K \cap \partial B_1(0)$ and $\alpha_i \ge 0$ such that $\sum_i \alpha_i = 1$ and $\sum_i \alpha_i q_i = 0$. This gives $\sum_i \alpha_i(|q_i|^2 - 2b\cdot q_i) = 1$, which, combined with (5.1), implies

$$||q|^2 - 2b\cdot q - 1| \le \sum_i \alpha_i \cdot |(|q|^2 - 2b\cdot q) - (|q_i|^2 - 2b\cdot q_i)| \le 8\mathcal{D}(\mu) \quad \text{for } q \in K. \tag{5.2}$$

Integrating against β, we deduce that

$$1 - 8\mathcal{D}(\mu) \le \int_K (|q|^2 - 2b\cdot q)\,d\beta = \int_K |q|^2\,d\beta - 2|b|^2 \le 1 - 2|b|^2.$$

Since the subsequential limit β was arbitrary, we have shown that

$$\limsup_{R\to\infty}\left|\int_K q\,d\beta_R\right| \le 2\sqrt{\mathcal{D}(\mu)}. \tag{5.3}$$

Recalling (5.2), we then get

$$1 - 4\sqrt{\mathcal{D}(\mu)} - 8\mathcal{D}(\mu) \le |q|^2 \le 1 \quad \text{for } q \in K. \tag{5.4}$$

We now repeat the horizontal rescaling from the proof of Theorem 1.1: as in (3.18), identifying V with $\mathbb{R}^m = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$, we now let $\zeta_L(x) := (\pi_V x/L, \pi_{V^\perp} x)$ (in other words, we no longer translate by $-b$ vertically) and

$$\eta_L := L^{-m}(\zeta_L)_*\mu, \quad \rho_L := L^{-m}(\pi_V/L)_*\mu = (\pi_V)_*\eta_L.$$

Letting $L \to \infty$, after a subsequence, by (2.5) we have

$$\rho_L \rightharpoonup dx', \quad \eta_L \rightharpoonup dx' \otimes \kappa_{x'}, \tag{5.5}$$

where $\kappa_{x'}$ is a probability measure supported in $K \subseteq \overline{B}_1(0)$. We also set

$$h(x') := \int_K q\,d\kappa_{x'}(q), \quad \sigma(x') := \int_K |q|^2\,d\kappa_{x'}(q).$$

By (5.4), we also have

$$|h| \le 1, \quad 1 - 4\sqrt{\mathcal{D}(\mu)} - 8\mathcal{D}(\mu) \le \sigma \le 1. \tag{5.6}$$

As before, we define

$$g_{L,r} := \mathcal{P}_r\rho_L, \quad A_{L,r} := L^2(g_{L,r} - 1), \quad h_{L,r}(x'_0) = \int_{\mathbb{R}^n} k_r(x' - x'_0)q\,d\eta_L(x', q).$$

For $(x'_0, q_0) \in \operatorname{spt}\eta_L$, setting again

$$G_{L,r}(x'_0, q_0) := \int_{\mathbb{R}^n} k_r(x' - x'_0)e^{-|q-q_0|^2/(4L^2r^2)}\,d\eta_L(x', q) = G^\mu_{Lr}(Lx'_0, q_0),$$

we have $|L^2r^2(G_{L,r}-1)| \leq \mathcal{D}(\mu) + o_r(1)$ by definition of $\mathcal{D}(\mu)$ (with an error which is uniform over the support center and compact subsets of $\{r>0\}$). Since $|q-q_0| \leq 2$ on $\operatorname{spt}\eta_L$, while we can assume $\mathcal{D}(\mu) \leq 1$ without loss of generality, we see that (3.23) is still valid, with the same proof. In turn, this allows us to extract limits $A_r = \lim_{L\to\infty} A_{L,r}$ (after a subsequence), locally smoothly in (x',r), with

$$A_r \geq -\mathcal{D}(\mu)/r^2, \quad \mathcal{P}_s A_r = A_{\sqrt{s^2+r^2}}. \tag{5.7}$$

Again, we also have $h_{L,r} \to \mathcal{P}_r h$.

The same computation used to get (3.26) now gives

$$\begin{aligned} 4r^2 A_{L,r}(x_0') = \int_{\mathbb{R}^n} k_r(x'-x_0')|q|^2\, d\eta_L(x',q) - 2q_0 \cdot h_{L,r}(x_0') + |q_0|^2 g_{L,r}(x_0') \\ + 4L^2r^2(G_{L,r}(x_0',q_0)-1) + o_r(1). \end{aligned}$$

Passing to the limit as $L\to\infty$, we obtain

$$|4r^2 A_r(x') - \mathcal{P}_r\sigma(x') + 2q\cdot \mathcal{P}_r h(x') - |q|^2| \leq 4\mathcal{D}(\mu) \tag{5.8}$$

for (x',q) in the support of $dx'\otimes\kappa_{x'}$. By (5.4) and (5.6), we deduce that

$$|2r^2 A_r(x') - 1 + q\cdot \mathcal{P}_r h(x')| \leq 4\sqrt{\mathcal{D}(\mu)} + 10\mathcal{D}(\mu) \quad \text{for } (x',q) \in \operatorname{spt}(dx'\otimes\kappa_{x'}). \tag{5.9}$$

Moreover, (3.17) is now replaced by

$$\limsup_{L\to\infty} |h_{L,r}(0)| = \limsup_{R\to\infty} \left| \int_K q\, d\beta_R \right| \leq 2\sqrt{\mathcal{D}(\mu)},$$

thanks to (5.3) (as before, factors such as $e^{-|q-q_0|^2/(4R^2)}$ disappear in the limit). Thus, we obtain

$$|\mathcal{P}_r h(0)| \leq 2\sqrt{\mathcal{D}(\mu)}. \tag{5.10}$$

Since $\hat{x}_0 = (0,\hat{q}_0) \in \operatorname{spt}\eta_L$ for every L (and (5.9) holds also at limits of points in $\operatorname{spt}\eta_L$), using (5.9) and (5.10) at this point yields

$$\left| A_r(0) - \frac{1}{2r^2} \right| \leq \frac{3\sqrt{\mathcal{D}(\mu)} + 5\mathcal{D}(\mu)}{r^2}. \tag{5.11}$$

Now assume by contradiction that the bound stated in Theorem 1.4 is false. We can then find measures μ_j, normalized as above (in particular, such that $\overline{B}_1(0)$ is the smallest ball including $\pi_{V^\perp}\operatorname{spt}\mu_j$), with $\mathcal{D}(\mu_j)\to 0$. For each of them, we construct functions $A_r^{(j)}, h^{(j)}$ and measures $\kappa_{x'}^{(j)}$ as above (using a sequence of scales $L\to\infty$ which may depend on j).

Since the measures $dx'\otimes\kappa_{x'}^{(j)}$ are locally uniformly bounded, after a subsequence they converge to a limit $dx'\otimes\kappa_{x'}$. Also, since $|h^{(j)}|\leq 1$, we can extract a limit h (with respect to weak-* convergence in L^∞), so that $\mathcal{P}_r h^{(j)} \to \mathcal{P}_r h$ smoothly, and similarly we can take limits $A_r^{(j)} \to A_r$ locally smoothly in (x',r), thanks to the bound (5.11) and (5.7). By (5.7) again, in the limit $j\to\infty$ we have

$$A_r \geq 0, \quad \mathcal{P}_s A_r = A_{\sqrt{s^2+r^2}},$$

while (5.10) and (5.11) give

$$\mathcal{P}_r h(0) = 0, \quad A_r(0) = \frac{1}{2r^2},$$

and (5.9) becomes

$$2r^2 A_r(x') = 1 - q\cdot \mathcal{P}_r h(x') \quad \text{for } (x',q)\in\operatorname{spt}(dx'\otimes\kappa_{x'}).$$

Finally, since $dx'\otimes d\kappa_{x'}^{(j)}$ is supported in $\mathbb{R}^m \times [\overline{B}_1^{n-m}(0) \setminus B_{1-\delta_j}^{n-m}(0)]$ by (5.4), where $\delta_j := 4\sqrt{\mathcal{D}(\mu_j)} + 8\mathcal{D}(\mu_j) \to 0$, we see that the limit measure $dx'\otimes\kappa_{x'}$ is supported in $\mathbb{R}^m \times \partial B_1^{n-m}(0)$, so that $\kappa_{x'}$ is supported on the unit sphere $\mathbb{S} := \partial B_1^{n-m}(0)$ (up to negligible modification). Also, given any $\psi \in C_c^0(\mathbb{R}^m)$, we have

$$\int_{\mathbb{R}^m} h\psi\, dx' = \lim_{j\to\infty} \int_{\mathbb{R}^m} h^{(j)}\psi\, dx' = \lim_{j\to\infty} \int_{\mathbb{R}^n} q\psi(x')\, dx'\otimes d\kappa_{x'}^{(j)}(q) = \int_{\mathbb{R}^n} q\psi(x')\, dx'\otimes d\kappa_{x'}(q),$$

proving that $h(x') = \int_{\mathbb{S}} q\, d\kappa_{x'}(q)$, i.e., $h(x')$ is the center of mass of $\kappa_{x'}$, for a.e. x'.

We are thus under the assumptions of Corollary 3.2 with $M = 1$; since the latter forces $M = 0$, we obtain a contradiction.

6. Appendix: a rectifiability criterion

The following is a simpler, weaker version of [9, Lemma 5.2] and [9, Theorem 5.3]. Its proof adapts arguments from the rectifiability criterion of Marstrand–Mattila, as presented in [1, Chapter 5] and [7, Chapter 16].

Proposition 6.1. Given integers $1 \le m \le n-1$, there exists a constant $c(m) > 0$ such that the following holds. Let μ be a positive Radon measure on $\mathbb{R}^n$, let $G \subseteq F$ be compact sets, with G purely unrectifiable, and suppose that

$$ar^m \le \mu(B_r(y)) \le br^m \quad \text{for } y \in F,\ r \in (0, r_0), \tag{6.1}$$

for uniform parameters $0 < a \le b$ and $r_0 > 0$. Suppose also that, for every $x \in G$ and $r \in (0, r_0)$, there exists an m-plane $V = V(x, r)$ such that

$$F \cap B_{2r}(x) \subseteq B_{\eta r}(V + x), \tag{6.2}$$

$$(V + x) \cap B_r(x) \subseteq B_{\eta r}(F), \tag{6.3}$$

with $\eta < c \cdot (a/b)$. Then we have $\mu(G) = 0$.

Let us assume without loss of generality that $a = 1$.

Lemma 6.2. Given any m-plane V, for every $x \in G$ and $\sigma \in (0, r_0/3)$ we have

$$\mathcal{H}^m(\pi_V[G \cap \overline{B}_\sigma(x)]) \le C(m)\eta \cdot \mu(B_{2\sigma}(x)). \tag{6.4}$$

Proof. Assuming without loss of generality that $\eta < 1/100$, note that for $\mathcal{H}^m$-a.e. $y \in G \cap \overline{B}_\sigma(x)$ there exists $z \in G \cap B_\sigma(y) \setminus \{y\}$ such that

$$|\pi_V(z - y)| < \eta |z - y|. \tag{6.5}$$

Indeed, the Borel set $S \subseteq G \cap \overline{B}_\sigma(x)$ of points y where this fails cannot have $\mathcal{H}^m(S) > 0$, since otherwise we would have $|\pi_V(z - y)| \ge \eta |z - y|$ whenever $y, z \in S \cap B$, for any ball $B = B_{\sigma/2}(p) \subset \mathbb{R}^n$, implying that S is included in a finite union of Lipschitz graphs over V; this contradicts the assumption that G is purely unrectifiable.

Given any such $y \in G \cap \overline{B}_\sigma(x)$, taking $z \in B_\sigma(y)$ as in (6.5) and setting $r_y := |z - y| < \sigma$, we now apply (6.2) at y, with radius $r := 3r_y < r_0$. We obtain an m-plane W such that $F \cap B_{6r_y}(y) \subseteq B_{3\eta r_y}(W + y)$ and thus, since $z \in G \subseteq F$, there exists a unit vector $w \in W$ (approximating $(z - y)/r_y$) with $|\pi_V w| \le C\eta$, for an absolute constant C.

In particular, the least singular value of $\pi_V|_W$ is at most $C\eta$. Thus, considering the disk $D_y := (W + y) \cap B_{5r_y}(y)$, the projection $\pi_V[B_{3\eta r_y}(D_y)]$ is included in a rectangle of diameter at most $C(m)r_y$, with one side of length at most $C\eta r_y$. Since

$$G \cap B_{5r_y}(y) \subseteq B_{3\eta r_y}(W + y) \cap B_{5r_y}(y) \subseteq B_{3\eta r_y}(D_y),$$

recalling that $r_y^m \le \mu(B_{r_y}(y))$ (by (6.1) and $a = 1$), we then get

$$\mathcal{H}^m(\pi_V[G \cap B_{5r_y}(y)]) \le C(m)\eta r_y^m \le C(m)\eta\mu(B_{r_y}(y)). \tag{6.6}$$

The claim follows by applying Vitali's covering lemma to the balls $\{B_{r_y}(y)\}$, as y varies among the points such that (6.5) holds for some z (its complement projects to a negligible set). □

Proof of Proposition 6.1. Assume by contradiction that $\mu(G) > 0$. Let $x \in G$ be a μ-density point for G; up to a translation, we will assume that $x = 0$ and write balls $B_r(0)$ simply as B_r. The upper bound in (6.1) gives

$$\mu(B_\sigma \setminus G) = o(\sigma^m). \tag{6.7}$$

We can then take $\sigma < r_0/3$ such that $\mu(B_\sigma \setminus G) < \min\{\eta^m, 1/(4 \cdot 80^m)\}\sigma^m$.

Next, let $V := V(0,\sigma)$ and, for any $v \in V$, let $D_r(v) := V \cap B_r(v)$ be the disk of center v and radius r in V. By Lemma 6.2, the sets

$$K := \pi_V(G \cap \overline{B}_\sigma), \quad D := D_{\sigma/4}(0)$$

contain 0 and satisfy

$$\mathcal{H}^m(D \setminus K) \geq \frac{\omega_m \sigma^m}{4^m} - C(m)\eta b \cdot (2\sigma)^m \geq \frac{\omega_m \sigma^m}{2 \cdot 4^m}, \tag{6.8}$$

once we take $c(m) \leq \omega_m/(C(m)2^{3m+1})$ in the statement. Moreover, letting

$$\rho : D \setminus K \to (0, \sigma/4), \quad \rho(v) := \mathrm{dist}(v, K),$$

we claim that

$$\rho < 2\eta\sigma. \tag{6.9}$$

Indeed, for each $v \in D \setminus K$, (6.3) (with $x := 0$, $r := \sigma$, and the plane V) yields some $p \in F$ with $|p - v| < \eta\sigma$. Thus, $B_{\eta\sigma}(p) \subset B_\sigma$ projects into $D_{2\eta\sigma}(v)$, which cannot be disjoint from K, as this would imply $B_{\eta\sigma}(p) \cap G = \emptyset$ and thus

$$(\eta\sigma)^m \leq \mu(B_{\eta\sigma}(p)) \leq \mu(B_\sigma \setminus G),$$

contradicting our choice of σ. Since we just proved that $D_{2\eta\sigma}(v)$ must intersect K, (6.9) follows.

Applying Vitali's covering lemma to $\{D_{2\rho(v)}(v) \mid v \in D \setminus K\}$, we get a disjoint family $\{D_{2\rho_i}(v_i)\}_{i\in I}$ such that $D \setminus K \subseteq \bigcup_{i\in I} D_{10\rho_i}(v_i)$. Thus, by (6.8), we have

$$\sum_{i\in I} \rho_i^m \geq \frac{\sigma^m}{2 \cdot 40^m}. \tag{6.10}$$

Denoting by $\mathcal{C}_s(v) := \pi_V^{-1}(D_s(v))$ the cylinder above $D_s(v)$, we now let

$$I' := \{i \in I : F \cap B_{\sigma/2} \cap \mathcal{C}_{\rho_i/2}(v_i) = \emptyset\}.$$

The indices $i \notin I'$ have a small contribution to the sum in (6.10): indeed, given $i \notin I'$, taking $p_i \in F \cap B_{\sigma/2} \cap \mathcal{C}_{\rho_i/2}(v_i)$, we have $B_{\rho_i/2}(p_i) \subseteq B_\sigma$. Since this ball projects into $D_{\rho_i}(v_i)$, it does not intersect G. Also, as $i \in I \setminus I'$ varies, these balls are disjoint, since the disks $D_{\rho_i}(v_i)$ are disjoint. We deduce that

$$\sum_{i\in I\setminus I'} \rho_i^m \leq 2^m \sum_{i\in I\setminus I'} \mu(B_{\rho_i/2}(p_i)) \leq 2^m \mu(B_\sigma \setminus G) < \frac{\sigma^m}{4 \cdot 40^m},$$

again thanks to our choice of σ. Consequently, recalling (6.10), we have

$$\sum_{i\in I'} \rho_i^m \geq \frac{\sigma^m}{4 \cdot 40^m}. \tag{6.11}$$

Fix now an index $i \in I'$ and choose $z_i \in G \cap \overline{B}_\sigma$ such that $|\pi_V z_i - v_i| = \rho_i$. By (6.2) we have

$$|z_i| \leq |\pi_V z_i| + \eta\sigma \leq |v_i| + \rho_i + \eta\sigma \leq (\sigma/4) + 3\eta\sigma < \sigma/3.$$

Let $W_i := V(z_i, r_i)$, where $r_i := \rho_i/(8\eta) < \sigma/4$. The least singular value of $\pi_V|_{W_i}$ is at most 16η: if not, then $w_i := \pi_V|_{W_i}^{-1}(v_i - \pi_V z_i)$ would have

$$\pi_V(z_i + w_i) = v_i, \quad |w_i| \leq \frac{\rho_i}{16\eta} < \frac{\sigma}{8}.$$

By (6.3) (at center z_i and scale r_i), we could also find a point $p_i \in F \cap B_{\eta r_i}(z_i + w_i)$, thus in $F \cap B_{\sigma/2} \cap \mathcal{C}_{\rho_i/2}(v_i)$ (as it projects into $D_{\eta r_i}(v_i)$ and $\eta r_i = \rho_i/8 < \eta\sigma/4$, so that $|p_i| \leq \sigma/3 + \sigma/8 + \eta\sigma/4 < \sigma/2$), contrary to the assumption $i \in I'$. Hence, we can find a unit vector $\hat{w}_i \in W_i$ such that $|\pi_V \hat{w}_i| \leq 16\eta$.

Let us take $N := \lceil 1/(32\eta) \rceil$ values $t_{i,1}, \dots, t_{i,N} \in [0, r_i/4]$, with $|t_{i,k} - t_{i,k'}| \geq \rho_i$. For each $t_{i,k}$, (6.3) gives $q_{i,k} \in F$ with

$$|q_{i,k} - (z_i + t_{i,k}\hat{w}_i)| < \eta r_i = \rho_i/8.$$

In particular, the balls $B_{\rho_i/4}(q_{i,k})$ are disjoint. Recalling that $|z_i| < \sigma/3$, we have

$$|q_{i,k}| + \frac{\rho_i}{4} < |z_i| + |t_{i,k}| + \frac{\rho_i}{8} + \frac{\rho_i}{4} < \frac{\sigma}{3} + \frac{r_i}{4} + \rho_i < \frac{\sigma}{2},$$

so that $B_{\rho_i/4}(q_{i,k}) \subset B_{\sigma/2}$. Also, recalling that $|\pi_V z_i - v_i| = \rho_i$, we have

$$|\pi_V q_{i,k} - v_i| + \frac{\rho_i}{4} < |\pi_V z_i - v_i| + 16\eta|t_{i,k}| + \frac{\rho_i}{8} + \frac{\rho_i}{4} < \frac{3\rho_i}{2} + 16\eta \cdot \frac{r_i}{4} = 2\rho_i,$$

so that $B_{\rho_i/4}(q_{i,k})$ projects into $D_{2\rho_i}(v_i)$. Thus, we have

$$B_{\rho_i/4}(q_{i,k}) \subseteq B_{\sigma/2} \cap \mathcal{C}_{2\rho_i}(v_i),$$

which means that also the family $\{B_{\rho_i/4}(q_{i,k})\}_{i\in I',k}$ is disjoint, giving

$$\frac{\sigma^m}{256 \cdot 160^m \eta} \leq \sum_{i\in I'} \frac{\rho_i^m}{64\eta \cdot 4^m} \leq \sum_{i\in I'} N(\rho_i/4)^m \leq \sum_{i\in I'} \sum_{k=1}^{N} \mu(B_{\rho_i/4}(q_{i,k})) \leq \mu(B_\sigma) \leq b\sigma^m$$

by (6.11), a contradiction once we choose $c(m)$ small enough in the statement. □

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