

A rank-constrained support inequality

Modifications of the Gennaioli–Rindler argument

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1 The theorem

We use the heat-quotient calculation of Gennaioli–Rindler [1, §3, Lemma 3.2 and (3.9)]. The purpose of this note is to isolate the changes needed when only m directions are elliptic.

Theorem 1.1. *Let $1 \leq m < N$ and let $\mathcal{K} \subset \mathbb{R}^{N \times N}$ be compact and convex, with*

$$c := \inf_{A \in \mathcal{K}} s_m(A) > 0, \quad L := \sup_{A \in \mathcal{K}} |A| < \infty.$$

Let $\mu = \theta \mathcal{H}^m \llcorner M$ be a finite positive m -rectifiable measure, with $\mathcal{H}^m(M) < \infty$. If

$$\sigma = A\mu, \quad A(x) \in \mathcal{K} \quad \mu\text{-a.e.}, \quad D = \text{Div } \sigma$$

is a finite vector-valued measure, then

$$\mu(\mathbb{R}^N) \leq C(N, m, c, L) \mathcal{H}^m(M)^{1/m} |D|(\mathbb{R}^N). \quad (1.1)$$

Singular values are decreasing, matrix norms are Frobenius norms, and Div is row-wise divergence: $(\text{Div } \sigma)_i = \sum_j \partial_j \sigma_{ij}$. Write $q = N - m$.

2 The modified heat calculation

Initially assume that μ has compact support and mass one, and set $d = |D|(\mathbb{R}^N)$. All spatial integrals below are over $\mathbb{R}^N$. Use elapsed heat time, rather than GR's reversed time:

$$P_t \eta = (4\pi t)^{-N/2} e^{-|\cdot|^2/(4t)} * \eta.$$

Work first on $[\varepsilon, T]$, with $0 < \varepsilon < T$. Compact support gives Gaussian tails, so the smooth heat quotients and all integrations by parts used below are justified by spatial cutoffs.

1. Let the divergence envelope follow the heat scale. Put

$$u = P_t \mu, \quad r = P_t |D|, \quad v = P_t \sigma, \quad S = u + \sqrt{t} r, \quad B = \frac{v}{S}, \quad h = \frac{r}{2\sqrt{t}}. \quad (2.1)$$

Thus

$$(\partial_t - \Delta)S = h \geq 0, \quad (\partial_t - \Delta)v = 0, \quad \int S = 1 + \sqrt{t} d.$$

Convexity gives

$$\frac{v}{u} \in \mathcal{K}, \quad B = \frac{u}{S} \frac{v}{u}, \quad |B| \leq L. \quad (2.2)$$

The denominator in (2.1) is the analogue of GR's (3.3). Their Lemma 3.2 calculation now has one extra source term:

$$\frac{d}{dt} \int \frac{|v|^2}{S} = -2 \int S |\nabla B|^2 - \int h |B|^2. \quad (2.3)$$

It has the favorable sign. Since $|v| \leq Lu$ and $u \leq S$, the energy on the left is at most L^2 , and therefore

$$\int_{\varepsilon}^T \int S |\nabla B|^2 \leq L^2/2.$$

GR's first-order identity (3.9) is unchanged:

$$B \nabla \log S = \eta - \operatorname{Div} B, \quad \eta := \frac{P_t D}{S}. \quad (2.4)$$

Their defect estimate becomes

$$\int S |\eta|^2 \leq \int \frac{r^2}{S} \leq \frac{d}{\sqrt{t}}.$$

Consequently

$$\int_{\varepsilon}^T \int S (|\nabla B|^2 + |B \nabla \log S|^2) \leq C_{N,L} (1 + \sqrt{T} d). \quad (2.5)$$

The choice $\sqrt{t}$ will also make the loss of rank integrable at zero.

2. Select scalar mass without restricting the stress. For now let $\mu_0 \leq \mu$ be any nonzero submeasure, and put

$$a = P_t \mu_0, \quad g = \mu_0(\mathbb{R}^N), \quad J(t) = \int a \log \frac{a}{S}.$$

The inequalities $a \leq u \leq S$ and Jensen give

$$g \log \frac{g}{1 + \sqrt{t} d} \leq J(t) \leq 0.$$

The same heat chain rule as in GR's (3.7) gives

$$J'(t) = - \int a \left| \nabla \log \frac{a}{S} \right|^2 - \int \frac{a h}{S}. \quad (2.6)$$

In particular,

$$\int_{\varepsilon}^T \int a \left| \nabla \log \frac{a}{S} \right|^2 \leq g \log \frac{1 + \sqrt{T} d}{g}. \quad (2.7)$$

Use $a \leq S$, $|B| \leq L$, and $B \nabla \log a = B \nabla \log S + B \nabla \log(a/S)$ to conclude that

$$\int_{\varepsilon}^T \int a (|\nabla B|^2 + |B \nabla \log a|^2) \leq C_{N,L} \Phi_T, \quad \Phi_T := 1 + \sqrt{T} d + g \log \frac{1 + \sqrt{T} d}{g}. \quad (2.8)$$

Thus the energy of the full stress controls the logarithmic gradient of the selected scalar heat flow. No derivative of the selection is taken.

3. Replace full ellipticity by a dimensional estimate. This is the replacement for GR's Lemma 3.3. Choose $\chi \in C^\infty([0, \infty); [0, 1])$, zero on $[0, c^2/16]$ and one on $[c^2/4, \infty)$, and set

$$Q = \chi(B^T B), \quad \Pi = Q^2.$$

Smooth spectral calculus on $|B| \leq L$ gives

$$0 \leq \Pi \leq I, \quad |Qz|^2 \leq C_c |Bz|^2, \quad |\nabla Q| \leq C_{N,c,L} |\nabla B|. \quad (2.9)$$

On $\{u \geq \sqrt{t}r\}$, (2.2) gives $s_m(B) \geq c/2$, hence $\text{tr } \Pi \geq m$. Its complement has a -mass at most $\sqrt{t}d$, because $a \leq u < \sqrt{t}r$ there. It follows that

$$\int a \text{tr}(I - \Pi) \leq qg + m\sqrt{t}d. \quad (2.10)$$

For a Gaussian convolution,

$$\nabla^2 \log a \geq -\frac{I}{2t}. \quad (2.11)$$

Indeed differentiating the Gaussian mixture gives $-I/(2t)$ plus a positive covariance matrix divided by $4t^2$. For a smooth vector field e , two integrations by parts give

$$-\int a e^T \nabla^2 \log a e = \int a \left[(\text{div } e + e \cdot \nabla \log a)^2 - \text{tr}((De)^2) \right]. \quad (2.12)$$

To check the identity, write $b = a^{-1} \text{div}(ae)$ and integrate $\int ab^2 = -\int ae \cdot \nabla b$ once more; the score terms cancel.

Let e_α be the columns of the symmetric matrix Q , so that $\Pi = \sum_\alpha e_\alpha \otimes e_\alpha$. With $I(a) := \int a |\nabla \log a|^2$, equations (2.11) and (2.10) yield

$$I(a) = -\int a \text{tr}(\nabla^2 \log a) \leq \frac{qg}{2t} + \frac{md}{2\sqrt{t}} - \sum_\alpha \int a e_\alpha^T \nabla^2 \log a e_\alpha.$$

Use (2.12), $|\text{tr}((De)^2)| \leq |De|^2$, and (2.9). This proves

$$I(a) \leq \frac{qg}{2t} + \frac{md}{2\sqrt{t}} + C_{N,c,L} \int a (|\nabla B|^2 + |B \nabla \log a|^2). \quad (2.13)$$

Combining with (2.8),

$$\boxed{\int_\varepsilon^T I(a_t) dt \leq \frac{qg}{2} \log(T/\varepsilon) + C_{N,m,c,L} \Phi_T.} \quad (2.14)$$

The normal Gaussian coefficient is exact: no multiple of $I(a)$ was absorbed.

3 The geometric input and the conclusion

Choose the submeasure. Rectifiability and inner regularity provide finitely many disjoint compact sets $E_i \subset M$, each in a graph of slope at most one, such that

$$E = \bigcup_i E_i, \quad \mu_0 = \mu \llcorner E, \quad g = \mu(E) \geq \frac{1}{2}.$$

Their union satisfies the uniform tube bound

$$\limsup_{\rho \downarrow 0} \rho^{-q} \mathcal{L}^N(E_\rho) \leq C_{N,m} \mathcal{H}^m(E), \quad E_\rho := \{x : \text{dist}(x, E) < \rho\}. \quad (3.1)$$

Here is the short verification. Extend each graph as a 1-Lipschitz map; a vertical fiber of $(E_i)_\rho$ lies in a q -ball of radius 2ρ , and its base lies in $(\pi_i E_i)_\rho$. Therefore

$$\mathcal{L}^N((E_i)_\rho) \leq \omega_q (2\rho)^q \mathcal{L}^m((\pi_i E_i)_\rho).$$

Compactness makes the last factor converge to $\mathcal{L}^m(\pi_i E_i) \leq \mathcal{H}^m(E_i)$. Summation proves (3.1), with a constant independent of the number of pieces.

Use the small support assumption. Set

$$Z_t = \int \exp\left(-\frac{\text{dist}(x, E)^2}{4t}\right) dx.$$

Layer cake and (3.1) give

$$Z_t = \int_0^\infty \frac{\rho}{2t} e^{-\rho^2/(4t)} \mathcal{L}^N(E_\rho) d\rho, \quad \limsup_{t \downarrow 0} t^{-q/2} Z_t \leq C_{N,m} \mathcal{H}^m(E). \quad (3.2)$$

The part of this integral above any fixed positive radius is exponentially small, since E is compact.

Compare the probability density a_t/g with $Z_t^{-1} \exp[-\text{dist}(x, E)^2/(4t)]$. By convexity of $z \log z$ we have

$$H(a_t) = \int a_i \log a_i \geq g \log g - g \log Z_t - \frac{1}{4t} \int \text{dist}(x, E)^2 a_t(x) dx.$$

Since μ_0 is supported on E , the last integral is at most $2Ntg$. Consequently

$$\liminf_{t \downarrow 0} \left[H(a_t) + \frac{qg}{2} \log t \right] \geq g \log \frac{g}{\mathcal{H}^m(M)} - C_{N,m}g. \quad (3.3)$$

This finite-graph argument is the only use of rectifiability.

Integrate and choose the scale. Since a_t is caloric,

$$H(a_\varepsilon) - H(a_T) = \int_\varepsilon^T I(a_t) dt, \quad H(a_T) \leq g \log g - \frac{Ng}{2} \log(4\pi T).$$

Combine these with (2.14), let $\varepsilon \downarrow 0$, and use (3.3). After canceling $g \log g$,

$$-g \log \mathcal{H}^m(M) \leq -\frac{mg}{2} \log T + C_{N,m,c,L} \Phi_T. \quad (3.4)$$

If $d = 0$, letting $T \rightarrow \infty$ is a contradiction. Otherwise set $T = d^{-2}$. Since $g \in [1/2, 1]$, Φ_T is bounded, and (3.4) yields

$$1 \leq C_{N,m,c,L} \mathcal{H}^m(M)^{1/m} d.$$

Normalization proves (1.1) for arbitrary compactly supported mass. Finally apply this case to $\mu_R = \xi_R \mu$ and $\sigma_R = \xi_R \sigma$, where ξ_R is a smooth cutoff equal to one on B_R , supported in B_{2R} , and $|\nabla \xi_R| \leq C/R$. The matrix density remains in $\mathcal{K}$, the carrying-set measure does not increase, and

$$|\text{Div } \sigma_R|(\mathbb{R}^N) \leq |D|(\mathbb{R}^N) + \frac{CL}{R} \mu(\mathbb{R}^N).$$

Let $R \rightarrow \infty$. This completes the proof of Theorem 1.1. $\square$

References

- [1] L. Gennaioli and F. Rindler, *Concentration phenomena and the Vanishing Mass Conjecture*, arXiv:2608.20899v1 (2026), §3. <https://arxiv.org/abs/2608.20899v1>.