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The construction of Real Numbers in Homotopy Type Theory

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Anno Accademico 2019/2020

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Introduction

This thesis presents an account of the construction of the real numbers in the context of Homotopy Type Theory (HoTT) [16], a foundational theory which extends Per Martin-Löf's intuitionistic type theory and constitutes the formal realization of Univalent Foundations. Univalent Foundations are a program of foundations of mathematics alternative to Zermelo-Fraenkel set theory, which can be used for both constructive and non-constructive mathematics. In particular, HoTT and more in general intuitionistic type theory is formulated in a constructive context, where the classical Axiom of Choice and the law of excluded middle are not available, although they can consistently be assumed [16, Ch.3].

The main idea underlying the Univalent Foundations project is about the convenience, rather than the theoretical power, of a foundational system with respect to another. Indeed, in the univalent perspective, instead of developing mathematics in the setting of set theory, where mathematical structures have to be built up from the empty set and the "natural operations" of set theory (that is, the ones that straightly come along with ZF axioms, such as subset, union, intersection), it is more convenient to work in a formal system (type theory) where we are given at the outset the world of spaces, i.e. types, and were operations that do make sense for spaces (namely, that preserve or enrich the structures) find a correspondence with natural operations on the basic objects. Moreover, this setting seems to enrich, in the above sense of convenience, ZF set theory, since it is still possible to recover sets in HoTT, and it turns out to be significantly easier to extract sets from the universe of (homotopy) types than it is to construct homotopy types from sets.

Whereas in Zermelo-Fraenkel set theory the basic objects are sets, in type theory the basic objects are types. Moreover, standard extensional set-theoretic constructions such as union, intersection or subsets (that is, the axiom of separation) do not make sense for types; instead, types are constructed by means of inductive definitions. The prototypical example of inductive definition is given by the Peano axioms of arithmetic. Interestingly, the category-theoretic descriptions of product or coproduct by means of their universal properties constitute other examples of inductive definitions. When formalized in type theory, the Peano axioms and the universal properties of the product and coproduct obey a common formal pattern, since they give rise to initial objects in suitable classes of types; these kind of constructions are abstracted in the general notion of *inductive types*. Indeed, the real field itself will be constructed as an inductive object.

The set-theoretic foundations are usually formulated inside first order logic. By contrast, type theory is not formulated inside any superstructure, and provides its own deductive system. As a deductive system, type theory differs from classical first-order logic in that propositions (the objects of derivations) are replaced by judgments. The judgment $a : A$, where a is a term and A is a type, asserts that the term a is of type A, hence in particular that the type A is inhabited. In this sense, inhabitedness of a type replaces the classical concept of provability of a proposition, a point of view generally referred to as propositions-as-types. In this context, the term a can be regarded as a witness, or a proof, that proposition A holds. It is important to observe that a judgment includes its proof: in other words, two valid judgments $a : A$ and $b : A$ cannot be regarded as interchangeable, whereas classically two proofs of the same theorem are logically equivalent. For example, if A and B are types representing propositions, then we associate to the logical formula $A \vee B$ the type $A + B$, namely the coproduct of A and B, or their disjoint union; the type $A + B$ is inductively defined by the two inclusions $\mathrm{inl} : A \to A + B$ and $\mathrm{inr} : B \to A + B$. In particular, one can prove that any term $u : A + B$ is either equal to $\mathrm{inl}(a)$ for an $a : A$ or to $\mathrm{inr}(b)$ for a certain $b : B$. Hence, under the propositions-as-types paradigm, an element $u : A + B$ not only tells that the proposition $A \vee B$ is true, but it also specifies which one between A or B is true.

Besides types as sets and types as propositions, there is a third interpretation of types, which is specific to Homotopy Type Theory, namely types as topological spaces: under this point of view, one can read the judgment $a : A$ as asserting that a is a point of the space A. Thanks to this interpretation, we can clarify the nature of identity in HoTT. In fact, like in intuitionistic type theory, in HoTT we have two notions of equality between terms, namely judgmental equality $a \equiv b : A$ and propositional equality $a =_A b$. While the former is a judgment, the latter is a type. On the one hand, judgmental equality behaves similarly to the equality of the deductive system of first order logic; on the other, the type $a =_A b$ can be understood as the space of paths in the space A connecting the point a to the point b. Another essential feature specific to Homotopy Type Theory is the Univalence Axiom, introduced by Voevodsky [6]. Roughly speaking, it asserts that if two types A and B are homotopically equivalent, i.e. $A \simeq B$, then they are equal as types $A =_{\mathcal{U}} B$, namely they are connected by a path in the universe $\mathcal{U}$ of all types.

In this setting, we discuss the construction in HoTT of the real numbers, intended as a complete archimedean ordered field. The property that real numbers have to embody is the geometrical *continuity* of the line, with respect to the discontintuity of rationals; however, more than one formalization is possible. Traditionally, real numbers are constructed as a completion of the rationals, either under Dedekind cuts or under the limits of Cauchy sequences. However, these formalizations present technical difficulties in a constructive setting, namely the need of the Axiom of Choice or the law of excluded middle to prove both Cauchy completeness of Cauchy reals and isomorphism between Dedekind and Cauchy reals. We illustrate how HoTT's inductive constructions allow to define Cauchy reals as the *free complete metric space generated by the rationals*, a construction that might have independent interest, and we also investigate the conditions needed to make Cauchy and Dedekind reals coincide in HoTT.

The set-theoretical construction of Cauchy reals is carried out by quotienting the subset of $\mathbb{Q}^{\mathbb{N}}$ of rational Cauchy sequences by the equivalence relation of being arbitrarily close. Nevertheless, when proving Cauchy completeness of the quotient, we have to rely on the Axiom of countable Choice, for there is provably no canonical nor procedural way to construct sections [9]. We overcome this problem by looking up to a construction of the real interval suggested by Martìn H. Escardò and Alex K. Simpson in [3]. They identify completeness of the real line with convexity of midpoint algebras, that is, types equipped with a binary operation m which respects the axioms of bisection; then, they define the interval as the *free convex body over two generators*. Indeed, consider an arbitrary sequence of points $x_0, x_1, \dots$ in an ordinary Euclidean convex space A. Let z be any point of A and consider the derived sequence

$$m(x_0, z), m(x_0, m(x_1, z)), m(x_0, m(x_1, m(x_2, z))), \dots$$

If A is bounded, then this is a Cauchy sequence whose unique limit point lies in A and is independent from z. Thus, any sequence $x_0, x_1, \dots$ determines a unique point $m(x_0, m(x_1, m(x_2, \dots)))$ obtained by infinitely iterating the binary operation m over the sequence. Escardò and Simpson's notion of completeness for a midpoint algebra A is to ask that the limit of such infinite iterations always exists.

Following this idea of free structure, we give an inductive definition of Cauchy reals $\mathbb{R}_c$, which amounts to see $\mathbb{R}_c$ as the Cauchy complete free metric space generated by $\mathbb{Q}$. This construction is an example of higher inductive-inductive type, a generalization of inductive types which furnishes a way to define possibly more than one free structure at the same time, together with equational laws between their elements.

Successively, we investigate the construction of the Dedekind reals. Classically, Dedekind cuts are defined as subsets of rationals, a notion which is not available in type theory. To work around this impasse we need to postulate the existence of a type Ω of mere propositions closed under countable conjunctions and disjunctions and existential quantifiers. Under this simplifying assumption, the definition mimics the classical one, in that reals are defined by the property of being located. However, there is more than one possible formalization of the locatedness property inside HoTT; we investigate these different formalizations of locatedness and we prove that, under either the Axiom of Choice or the law of excluded middle, they coincide and give rise to a type of Dedekind reals equivalent to Cauchy reals.

HoTT is currently a field of active research, and even the construction of the reals is not unproblematic. In fact, both Cauchy and Dedekind constructions of real numbers lead to a type with the algebraic structure of a complete archimedean field, but not with the desired homotopical properties: for instance, in HoTT the reals are not contractible. More generally, it is object of current study how to relate synthetic and analytic definitions of structures in HoTT. The prototypical example is given by the circle: we can define $\mathbb{S}^1$ either as an Higher Inductive Type (see Chapter 2), which turns out to have non trivial foundamental group and the expected homotopical properties, or as the classical subset of $\mathbb{R}^2$ of points of norm equal to 1, which turns out to be discrete.
Another open problem is to clarify the consistency strength of HoTT with respect to ZFC. On the one hand, we know that ZFC plus two strongly inaccessible cardinals models HoTT; on the other, if we add to HoTT the Axiom of Choice and classical logic, then HoTT interprets ZFC. However, researchers are studying how to relate constructive versions of set theory with HoTT, and in general we still lack an exact classification.

In conclusion, in this work we illustrate the formalism of type theory and we investigate the different constructions proposed for a type-theoretic version of the reals. We see how HoTT's characteristic feature of inductive types allows a definition of the reals of independent interest, and investigate conditions to recover equivalence of the resulting types of reals.

CHAPTER 1

Homotopy Type Theory

The Univalent Foundations replace set theory with a different formal system, Homotopy Type Theory, whose basic objects are spaces: Martin-Löf's dependent type theory, originally conceived as a constructive foundation for mathematics where everything has computational content, turns out to admit topological interpretations. There are correspondences between types and propositions, and between types and spaces, and they are reflected in type-theoretic operations, which from time to time assume a logical or homotopical meaning and have a related construction in set theory.

First of all, we give an overview of dependent (or intuitionistic or Martin Löf's) type theory, together with the homotopical interpretation provided by Homotopy Type Theory. Secondly, we present Voevodsky's Univalence Axiom and some of the consequences it brings to the theory, such as Function Extensionality.

1. Type theory and set theory: a comparison

The idea of a theory of types was originally suggested by Bertrand Russell in the first years of 1900 [14] to avoid the paradoxes in the logical foundations of mathematics that were under investigation at the time, and was developed further over the following years by other mathematicians; in the 1970s, Per Martin-Löf developed his version of type theory (called dependent, intuitionistic or simply Martin-Löf's type theory), intended as a rigorous framework for the formalization of constructive mathematics and as a foundational language for mathematics, alternative to Zermelo-Frenkel set-theoretic foundations. Type theory differs from set theory in several ways and on several different levels; we will give just a brief overview of the differences, which will be pointed out in the next sections as they emerge.

The set-theoretic foundations are usually formulated inside first order logic. By contrast, type theory is not formulated inside any superstructure, and provides its own deductive system – that is, rules for deriving judgments. In the deductive system of first-order logic there is only one kind of judgment, the relation of formal provability between (finite) collections of formulas, whereas type theory is based on two forms of judgments:

- $a : A$ which can be read as " *a is an element of type A* ";
- $a \equiv b : A$, read as " *a and b are judgmentally equal elements of type A* ".

It is possible to identify propositions with (possibly a subclass of) types and the proofs of propositions with elements of the corresponding type: under this interpretation, we read the judgment $a : A$ as "a is a proof of the proposition represented by the type A, or in other words that a is a witness of the truth of A". This identification is known as the *Curry-Howard correspondence*, and it is what makes type theory's internalization of logic possible. It is important to observe that in classical logic all propositions are either true or false; in type theory a term of a type A is regarded as an *evidence* of the validity of the proposition it represents, and there may be more then one proof of the latter.

Types play also the role of mathematical objects, and one could also treat a type more like a set than like a proposition, in which case the judgment $a : A$ could be interpreted as $a \in A$. In this regard, it is important to point out the structural difference between the two. On the one hand, $a \in A$ is an atomic proposition that can be combined in more complex propositions; for instance, we may use $a \in A$ to write $a \in A \iff b \in B$. On the other hand, $a : A$ is a judgment, and as such is a metatheoretical feature, which cannot be further combined.

The innovation of Homotopy Type Theory is precisely the detachment from the interpretation of types as (strange) sets, which paves the way for a homotopical understanding of type theory. In HoTT a type can also be seen as a topological space, and the judgment $a : A$ expresses the fact that the term a is a point of the space A; the logical constructions on types are regarded as homotopy-invariant constructions on the spaces they represent. Suppose also to have already defined a type, called the *identity type*, that implements, internally to type theory, the logical notion of identity $a =_A b$ of two objects $a, b : A$ of the same type, so that under the types-as-propositions correspondence an element $p : a =_A b$ represents a proof that a and b are equal.
In Homotopy Type Theory $p : a =_A b$ can be understood as a path from the point a to the point b in the space A; the *identity family* Id_A, namely the collection of the types of the identifications between elements of A, becomes in this way the space $A^{[0,1]}$ of all (continuous 6) paths in A from the unit interval. We remark that in HoTT spaces are treated purely homotopically, not topologically: there is no notion of open subset, but there are only homotopical notions such as paths between paths and homotopies between paths.
We summarize the different points of view of the type-theoretic operations in 1.

Types	Logic	Sets	Homotopy
$B : A \to \mathcal{U}$	predicate	family of sets	fibration
$b(x) : B(x)$	conditional proof	family of elements	section
$\mathbf{0}, \mathbf{1}$	$\bot, \top$, $\emptyset, \{\emptyset\}$	$\emptyset$, $*$	
$A + B$	$A \vee B$	disjoint union	coproduct
$A \times B$	$A \wedge B$	set of pairs	product space
$A \to B$	$A \implies B$	set of functions	function space
$\sum_{x:A} B(x)$	$\exists (x : A).B(x)$	disjoint sum	total space
$\prod_{x:A} B(x)$	$\forall (x : A).B(x)$	product	space of sections
Id_A	equality $- =_A -$	$\{(x, x) \mid x \in A\}$	path space $A^{[0,1]}$

2. Formal type theory

There is more than one formulation of Homotopy Type Theory. Here we have chosen to define terms, judgments and rules of inference inductively in the style of natural deduction; for brevity, we omit the description of the syntax of terms (the objects that the judgments of type theory relate). The following presentation of type theory as a formal system will not be extremely rigorous and we will lighten some of the formalism: our main goal is to emphasize the points which highlight the main differences with the classical foundations of mathematics; a rigorous formalization can be found in the Appendix of [16], or also in [13].

In formal type theory there are three kind of judgments:

$$\Gamma\,\mathrm{ctx} \qquad\qquad \Gamma \vdash a : A \qquad\qquad \Gamma \vdash a \equiv a' : A$$

which are specified by providing inference rules for deriving them. The first one states that Γ is a well-formed context, the second that in the context Γ a is an object of type A and the third a and a' are two judgmentally equal terms of type A in the context Γ.

Throughout the following section, we may adopt the following convention:

$x, y, z, \ldots$ and $x_i, \ldots$ represent variables;
$a, b, c \ldots$ stand for terms;
$A, B, C, \ldots$ denote types.

The judgments are explicitly formulated in an ambient **context**, or a list of assumptions, of the form

$$x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$$

We may abbreviate contexts with letters like Γ, Δ. An element $x_i : A_i$ of the context expresses the assumption that the variable x_i has type A_i. If the type A_i in the context represents a proposition, then the assumption is a type-theoretic version of hypothesis, which however still holds externally to type theory, in the metatheory. The judgment $a : A$ in the context Γ is written as

$$\Gamma \vdash a : A$$

and it means that $a : A$ under the assumptions listed in Γ. We write $B[a/x]$ for the sostitution of a term a for a free occurences of the variable x in the term B. The judgment ctx is the one left implicit when presenting type theory informally.

An inference rule has the form

$$\frac{J_1 \ldots J_k}{J}$$

and, as usual with systems inspired by Gentzen's natural deduction, it says that we may derive the conclusion J provided that we have already derived the hypotheses $J_1, \ldots, J_k$. It is worth noting that, being judgments rather than types, the hypotheses $J_1, ..., J_k$ are not *internal* to type theory, in the sense that they cannot be formulated inside the theory via the propositions as type correspondence etc; they are instead hypotheses in the deductive system, namely in the metatheory. Lastly, a *derivation* of a judgment is a tree constructed from such inference rules.

We postulate an infinite hierarchy of type universes

$$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, ...$$

The universes form a cumulative hierarchy of types; indeed, universes are governed by the following rules:

- $\mathcal{U}_m : \mathcal{U}_n$ for $m < n$;
- if $A : \mathcal{U}_m$ and $m \leq n$, then $A : \mathcal{U}_n$;
- if $\Gamma \vdash A : \mathcal{U}_n$ and x is a variable that doesn't appear in Γ then $\vdash (\Gamma, x : A)\,\mathrm{ctx}$.

The third rule expresses the idea that an object of a universe can serve as a type and stand to the right colon of a judgment. A structural rule of dependent type theory says that every type A must inhabit some universe $\mathcal{U}_i$, as well as any term comes equipped with the type it inhabits. Usually, we avoid to explicitly mention the level i and we assume that levels can be assigned in a consistent way; we may write $A : \mathcal{U}$ omitting the level. This style of writing universes is reffered to as **typical ambiguity**.

The first judgment, $\Gamma\,\mathrm{ctx}$, formally expresses the fact that Γ is a **well-formed context**; its inference rules are the following:

$$\frac{}{.\,\mathrm{ctx}} \qquad \frac{x_1 : A_1, ..., x_{n-1} : A_{n-1} \vdash A_n : \mathcal{U}_i}{(x_1 : A_1, ..., x_n : A_n)\,\mathrm{ctx}}$$

with the side condition to the second rule that the variable x_n must be distinct from the variables $x_1, ..., x_{n-1}$.

The context holds assumption: there is a rule of weakening which says that it is possible to derive the judgments listed in the context.

The main judgment of type theory is that of the form $a : A$: it asserts that the **term a is an inhabitant of the type A**. We already postulated that for every type A there exists an universe $\mathcal{U}_i$ such that $A : \mathcal{U}_i$. We have also the principles of *substitution* and of *weakining*, usually not needed to be explicitly assumed, that assert that substitution of terms and weakining by a type in context preserve well-formedness of types and terms.

$$\mathrm{Sbst}_1 \frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] : B[a/x]} \qquad \mathrm{Wkg}_1 \frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma, \Delta \vdash b : B}{\Gamma, x : A, \Delta \vdash b : B}$$

There are a number of ways that $a : A$ has traditionally been motived:

(1) A is a set and a is an element of A;
(2) A is a proposition and a is a proof of A.

The starting point of Homotopy Type Theory is to consider a third alternative to these motivations:

(3) A is a topological space and a is a point of A.

It is important to remark the following substantial difference between types and sets: in set theory there is no formal distinction between sets and elements, while in type theory the fact that an element is of a given type is part of its very nature as a static information. For this reason, while in set theory when we say "if X is a natural number then ... " we are actually quantifyng over all sets and then assuming the proposition $X \in \mathbb{N}$ true, i.e.

$$\forall X(X \in \mathbb{N} \implies \phi(X))$$

in type theory we cannot do assertions like "if the term x is of type $\mathbb{N}$ then $\cdots$", since if we are given x then we are also given the type it belongs to.
Moreover, we mention that type-checking, that is, checking that a term is of a certain type, is decidable, and can be done algoritmically [12].

The last judgment $a \equiv a' : A$ is the one of **judgmental equality** between terms (notice that the rule of universes implies that every type can be seen as a term). Judgmental equality behaves similarly to the equality of the deductive system of first order logic. Indeed, there are rules of substitution and weakening, meaning that substitution of terms and weakining by a type in context preserve judgmental equality of types and terms.

$$\text{Sbst}_2 \frac{\Gamma \vdash a : A \qquad \Gamma, x : A\Delta \vdash b \equiv c : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \equiv c[a/x] : B[a/x]} \quad \text{Wkg}_2 \frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma, \Delta \vdash b \equiv c : B}{\Gamma, x : A, \Delta \vdash b \equiv c : B}$$

In addition to the judgmental equality rules that will be given for each type former through the *computation rules*, judgmental equality is also required to be an equivalence relation on terms of a given type. Note that the judgment $A \equiv B : \mathcal{U}$ is well-formed, so judgmental equality makes sense also for types; indeed, it induces an equivalence relation on terms and types.

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}$$

Moreover, the *conversion rules* demand this equivalence relation to be respected by typing:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash A \equiv B : \mathcal{U}_i}{\Gamma \vdash a : B} \qquad \frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash A \equiv B : \mathcal{U}_i}{\Gamma \vdash a \equiv b : B}$$

Additionally, for all the type formers that we will introduce, there will be assumed (but omitted for brevity) rules stating that each constructor preserves definitional equality in each of its arguments.

It is fundamental to observe that the notion of equality given by the judgmental equality is external to the theory. We will introduce in 5 an internal implementation of equality, namely a type $a =_A b$ such that $p : a =_A b$ represents a witness that $a : A$ and $b : A$ are equal.

3. Intuitionistic type theory: type constructors

Having stated the judgments of type theory and their inference rules, it is now possible to group the remaining rules of type theory into **type formers**, or **type constructors**: they are sets of rules expressing ways to construct types, possibly making use of previously constructed types, together with rules for the construction and the behavior of elements of these type. It is worth noting that, in set theory, the only rules are the ones of first-order logic, namely the inference rules for logical connectives and quantifiers, and all the information about the behavior of sets is contained in the axioms of Zermelo-Frenkel theory. By contrast, intuitionistic type theory is procedural, in the sense that it is constituted only (mostly) by rules; it is indeed this constructivity property that enables one to see it as a programming language. As we will see, however, Homotopy Type Theory adds to it the Univalence Axiom and Higher Inductive Types.

There is a general pattern for the systematic introduction of a type former:

- a **formation rule**, which states when the type former can be applied;
- **introduction rules** (or **constructors**), that prescribes how to inhabit that type;
- **elimination rules** (or **eliminators**), stating how to use elements of that type (typically, how to construct functions *into* or *out of* the type);
- **computation rules**, which are judgmental equalities expressing how an eliminator acts on a costructor, i.e. what hapens when elimination rules are applied to results of introduction rules;
- optional **uniqueness principles**, which are judgmental equalities expressing uniqueness of maps into or out of that type. When the uniqueness principle is not taken as a rule of judgmental equality, it is often anyway provable as a propositional equality from the other rules for the type, and in this case it is called as **propositional uniqueness principle**

As an example, we present the *cartesian product types.* Its formation rule states that given types $A, B\colon \mathcal{U}$ we may construct the type $A \times B : \mathcal{U}$. Formally, it consists in the following inference rule:

$$\frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma \vdash B : \mathcal{U}_i}{\Gamma \vdash A \times B : \mathcal{U}_i}$$

To construct inhabitants of $A \times B$ it suffices to have elements $a\colon A$ and $b\colon B$; in other words, its constructor is the map

$$\mathrm{pair} : A \to B \to (A \times B)$$

We may denote $\mathrm{pair}(a)(b) \equiv (a, b)$. Observe that while in set theory ordered pairs are defined as particular sets and then collected together to form the cartesian product, in type theory they are a primitive notion, which comes together with that of cartesian product itself. The elimination rule, or *induction principle for cartesian product*, prescribes how to define dependent functions out of the product type. If one considers any type $C : \mathcal{U}$, then the eliminator states that to produce a function $f : A \times B \to C$ it is enough to construct $g\colon A \to B \to C$, which will tell how f computes on the canonical inhabitants (i.e. the constructor) of $A \times B$, in the sense that $f((a, b)) :\equiv g(a)(b)$. By generalizing from non-dependent types $C : \mathcal{U}$ to dependent type families $C : A \times B \to \mathcal{U}$, namely an object which associates to any $u : A \times B$ a type $C(u) : \mathcal{U}$, we get the full induction principle:

$$\frac{\Gamma \vdash C : A \times B \to \mathcal{U} \qquad \Gamma \vdash g\colon \prod_{x\colon A} \prod_{y\colon B} C(x, y)}{\Gamma \vdash \mathrm{ind}(C, g) : \prod_{u\colon A \times B} C(u)}$$

whose computation rule is $f((a, b)) :\equiv g(a)(b)$. From a mathematical perspective, computation rules constitute the defining equations of the primitive constructions: applying a computation rule corresponds to replacing something by its definition.

Observation 1.1. The definition of cartesian product is an example of the more general construction of **definition by induction**, where a type is characterized by the data we have to provide to construct maps out of it.

4. Proposition as types

In type theory, the paradigm of **propositions as types** says that every logical proposition can be identified with a type, whose terms will be the possibly different proofs of the proposition it represents, and that in turn every type corresponds to a proposition. In type theory, showing that a proposition is true is identified with exhibiting an element of the correspondent type, and we regard elements of this types as *evidences* or *witnesses* of the proposition. The identification of propositions with the class of *all* types yields a strongly constructive conception of logic, leading, for example, to the provability of the Axiom of Choice [16, p. 3.8]. For this reason Homotopy Type Theory uses what is called the "(-1)-truncated" logic, in which only the (homotopy) (-1)-types represent propositions, namely the types whose inhabitedness does not carry any extra information apart from the fact that the proposition they represent is true. HoTT's (-1)-truncated logic will be debated in more details in 3 after the introduction of Higher Inductive Types and of *propositional truncation types* among them.

For now, we just observe that what makes the identification of propositions with (a subclass of) types achievable is the possibility to translate logical connectives and quantifiers (which are *logic* operations on propositions) into type formers (namely *type-theoretic* operations). The following table synthesizes this correspondence for propositional logic:

Propositional Logic	Type Theory
True	1
False	0
$A \wedge B$	$A \times B$
$A \vee B$	$A + B$
$A \to B$	$A \to B$
$A \iff B$	$A \to B \times B \to A$
$\neg A$	$A \to 0$

In each case the rules for constructing and using elements of the type on the right correspond to the inference rules of the logical connectives on the left. For what concerns predicate logic, types act in a dual way, since they serve as propositions and also as domains to quantify over. In fact, a predicate over a type A is represented as a dependent family $P : A \to \mathcal{U}$. The translations of quantifiers is expressed by the following correspondence:

Predicate Logic	Type Theory
$\forall(x : A).P(x)$	$\prod\limits_{x:A} P(x)$
$\exists(x : A).P(x)$	$\sum_{x:A} P(x)$

In type theory it is also possible to represent higher order logic, since it permits to quantify over all propositions or over all predicates. For example, taking advantage of typical ambiguity, one may write the type

$$\prod_{P:A\to\mathcal{U}}\prod_{a,b:A} P(a)\to P(b)$$

which corresponds to the proposition *for all properties P of A and every $a,b:A$, if P(a) then P(b).* Of course, this proposition lives in a different universe from P itself:

$$\frac{A:\mathcal{U}_i \qquad P:A\to\mathcal{U}_i}{\prod_{P:A\to\mathcal{U}}\prod_{a,b:A} P(a)\to P(b):\mathcal{U}_{i+1}}$$

5. Identity type

The **identity type** is the implementation of the notion of equality *inside* type theory, and it can be considered as the actual starting point of divergence of this theory from set theory. In the view of types as propositions, the proposition that two elements of the same type $a,b:A$ are equal must correspond to some type, and since this type will depend on the terms a and b, the identity type over A has to be a type family which depends on two copies of A. We may write the family of identity types of A as $\mathrm{Id}_A : A\to A\to\mathcal{U}$, so that $\mathrm{Id}_A(a,b)$ is the type representing the proposition of equality between a and b, for which we may also use the notation $a=_A b$. Here we give a formal presentation of the identity type of A.

- Formation rule:
$$\frac{\Gamma\vdash A:\mathcal{U}_i}{\Gamma\vdash \mathrm{Id}_A : A\to A\to\mathcal{U}_i}$$
- Introduction rule:
$$\frac{\Gamma\vdash a:A}{\Gamma\vdash \mathrm{refl}_a : a=_A a}$$
- Induction principle (*Path Induction*) and computation rule:
  - Path Induction:
$$\frac{\Gamma\vdash C:\prod_{x,y:A}(x=_A y)\to\mathcal{U} \qquad \Gamma\vdash c:\prod_{z:A} C(z,z,\mathrm{refl}_z)}{\Gamma\vdash \mathrm{ind}(C,c):\prod_{x,y:A}\prod_{p:x=_A y} C(x,y,p)}$$

    The computation rule for path induction tells that such an f computes on the constructor so that for every $x:A$
$$\mathrm{ind}(C,c)(x,x,\mathrm{refl}_x) :\equiv c(x)$$

The principle asserts that to produce a function from the identity type family over A to another typ $C:\mathcal{U}$ it is enough to specify its values on the elements of the form refl_x, as $x:A$ varies.

Treatment of equality had been one of the main issues of intuitionistic type theory, and mostly because the type resulting from this definition is far from being as trivial as expected. In particular, the *principle of uniqueness of identity proofs* , according to which there is at most one proof that two objects are equal, i.e.

$$\prod_{A:\mathcal{U}}\prod_{x,y:A}\prod_{p,q:x=_A y} p=_{a=_A y} q$$

is not provable from the identity eliminator, as shown by Hofmann and Streicher in [11]. However, reformulating axioms of identity so that the principle of uniqueness of identity proofs is provable makes type checking is no longer decidable [5], contrarily to intuitionistic type theory [12], with a consequent loss of the computational properties of type theory as a programming language.

In Homotopy Type Theory, the type $a =_A b$ may be regarded as the space of the paths in A between the two points a and b. Under this homotopical interpretation, the principle of uniqueness of identity proof isn't a desirable property anymore: as a space can be not simply connected, there can be more than one witness of the equality of two objects, more than one *identification.*

	Type Theory	Homotopy Type Theory
$A : \mathcal{U}$	A is a type	A is a space
$a : A$	a is a term of type A	a is a point in the space A
$a =_A b$	a and b are equal terms of type A	There exists a path from a to b in the space A

Identity types may be iterated: given any type A and any $x, y : A$ such that $p, q : x =_A y$, it is possible to form the type $p =_{x=_A y} q$ of paths(/identifications) between paths (/identifications), and then the type $r =_{p=_{x=_A y} q} s$ and so on. The resulting structure is a tower of identity types which corresponds to that of the continuous paths and (higher) homotopies between them in a space. This additional structure of the internal notion of equality of type theory is absent in classical first-order logic and arises from the induction principle of the identity type. Furthermore, those that in set theory are *properties* of equality, namely the ones that make it a congruence relation on sets (reflexivity, symmetry, transitivity and being respected by operations) become *operations* on paths in the identity type under the homotopical interpretation of HoTT: for every type $A : \mathcal{U}$ and every $x, y, z : A$ we have the correspondences of the table below.

Type Theory	Equality	Homotopy
$\mathsf{refl}_x : x =_A x$	reflexivity	constant path
$-^{-1} : x = y \to y = x$	symmetry	inversion of paths
$\cdot : x = y \to y = z \to x = z$	transitivity	concatenation of paths

Concatenation and inversion of paths, together with the constant loop, make the identity type $x =_A x$ a group, where the group laws hold only up to higher-order paths; fortunately, identity up to homotopy is precisely the equality of our theory.

6. Functions are functors and type families are fibrations

The induction principle of the identity type implies that any function preserves identifications, in the sense that it sends terms connected by a path (i.e. equal) to terms still connected by a path. This is a form of continuity for functions in type theory, and more precisely it is a form of functoriality, where a map is intended to be **functiorial** with respect to the groupoid structure of a type (the tower of identity types that arises from the $x =_A y$).

LEMMA 1.2. *Let $A, B : \mathcal{U}$ and $f : A \to B$. For any two elements $x, y : A$ there exists*

$$\mathrm{ap}_f : (x =_A y) \to (f(x) =_B f(y))$$

An analogous property of continuity can be recovered also for dependent functions $f : \prod_{x:A} B(x)$, albeit a direct generalization is not possible: if we have a term $p : x =_A y$, then $f(x) : B(x)$ and $f(y) : B(y)$ are elements of distinct types, hence a priori it doesn't make sense to ask whether $f(x)$ is equal to $f(y)$ or not. The key point is the interpretation of dependent type families as fibrations, for which we need the following lemma.

LEMMA 1.3 (Transportation). *Let $A : \mathcal{U}$ and $P : A \to \mathcal{U}$ a type family over A. For any two elements $x, y : A$, if there is $p : x =_A y$ then there exists a function $transport^P(p, -) : P(x) \to P(y)$.*

PROOF. By induction, it suffices to assume that $p \equiv \mathrm{refl}_x$, in which case we can define $\mathrm{transport}^P(\mathrm{refl}_x, -) :\equiv id_{P(x)} : P(x) \to P(x)$. □

When the dependent family where we transport the path p is clear from the context, we simply write $p_* :\equiv \mathrm{transport}^P(p, -)$.
Logically speaking, the transport lemma correspond to the principle of *indiscernibility of identicals*, in the sense that any predicate P respects equality: if $x = y$ then $P(x)$ holds if and only if $P(y)$ holds. Homotopically, the transportation lemma gives a way to view type families over A as fibrations, since it provides a transport operation on dependent families that can have the role of a path-lifting operation.

Now we can specify in what way every type family $P : A \to \mathcal{U}$ can be regarded as a fibration.

DEFINITION 1.4 (Fibration, ZF). In classical homotopy theory, a *fibration* $\phi : E \to A$ is a continuous map between two topological spaces E (said to be the *total space*) and A (the *base space*) that has what is know as the *homotopy-lifting property*. This last one specifies to the *path-lifting property*, which we will take as the defining one. It states that for any point $e : E$, any continuous path $\gamma : [0, 1] \to A$ such that $\gamma(0) = \phi(e)$ can be uniquely lifted (up to homotopy) to a path $\tilde{\gamma} : [0, 1] \to E$ starting at e, in the sense that $\forall t : [0, 1]\ \phi(\tilde{\gamma}(t)) = \gamma(t)$.

Type-theoretically, we may think of a type family $P : A \to \mathcal{U}$ as (inducing) a fibration where A is the base space and $E :\equiv \sum_{x:A} P(x)$ the total space. The path-lifting property can be stated as follows and its proof relies indeed on the transportation lemma.

LEMMA 1.5 (Path lifting property). *Let $P : A \to \mathcal{U}$ be a type family over A and assume there exists $u : P(x)$ for some $x : A$. Then for any path $p : x =_A y$ there exists*

$$\mathrm{lift}(u, p) : (x, u) = (y, p_*(u))$$

in $\sum_{x:A} P(x)$ such that $\mathrm{pr}_1(\mathrm{lift}(u, p)) = p$, where pr_1 is the projection on the first component.

At this point, it is finally possible to assert a form of continuity, or better functoriality, for dependent functions in type theory.

Proposition 1.6. *For any* $f : \prod_{x:A} P(x)$ *there exists a map*

$$\mathrm{apd}_f : \prod_{p:x=_A y} p_*(f(x)) =_{P(y)} f(y)$$

To define a path $\tilde{p}$ from $(x, u) : P(x)$ to $(y, v) : P(y)$ in the total space $\sum_{x:A} P(x)$ that lifts $p : x =_A y$, where lifting p means that $\mathrm{pr}_1(\tilde{p}) = p$, it suffices to factorize through $\mathrm{lift}(u, p)$ and then to use $\mathrm{apd}_f(p)$ to connect $p_*(f(x))$ with $f(y)$ in $P(y)$. Given the canonicity of the factorizatio, done by $\mathrm{lift}(-,-)$, we may identify the lifting path $\tilde{p}$ with $\mathrm{apd}_f(p) : p_*(f(x)) =_{P(y)} f(y)$.

We introduce a special notation for dependent paths, shown in Figure 1[1]. Given a type $X : \mathcal{U}$ and a dependent family over X $P : X \to \mathcal{U}$, if $p : x =_X y$, then for any two $u : P(x)$ and $v : P(y)$ we write:

$$u =^P_p v :\equiv (\mathrm{transport}^P(p, u) =_{P(y)} v)$$

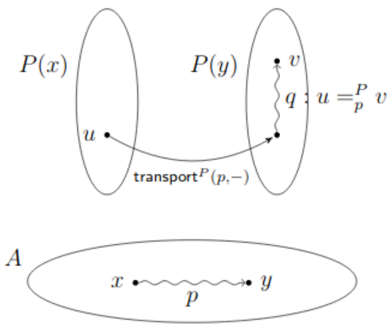


Figure 1: q is a dependent path lying over p

7. The Univalence Axiom

The Univalence Axiom relates two kinds of identifications between types $A, B : \mathcal{U}$, namely identity the identity type $A =_{\mathcal{U}} B$ and the type of equivalences $A \simeq B$, the latter yet to be introduced. The identity type $A =_{\mathcal{U}} B$ can be understood as the space of paths from type A to type B in the universe $\mathcal{U}$. However, a priori, it does not seem to convey the notion of isomorphism between mathematical structures, where an isomorphism is intended to be a bijective map that preserves the structure of the objects. For this reason, we define the type of equivalence maps between A and B.

In general, in HoTT, when two (dependent) maps are pointwise equal we say that they are *homotopically equivalent*.

Definition 1.7. For any type $A : \mathcal{U}$ and any family $P : A \to \mathcal{U}$, for any two dependent functions $\phi, \psi : \prod_{x:A} P(x)$, we define $\phi \sim \psi$ as the **type of homotopies**

[1]Figure from [7]

between ϕ and ψ, that is

$$\phi \sim \psi :\equiv \prod_{x:A} \phi(x) =_{P(x)} \psi(x)$$

Classically, a function $f : A \to B$ is an isomorphism (of sets) if there exists $g : B \to A$ such that $g(f(x)) = x$ for every $x : A$ and $f(g(y)) = y$ for every $y : B$. Using the above notation, in HoTT this condition corresponds to the requirement that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$.

DEFINITION 1.8. Given $A, B : \mathcal{U}$ and $f : A \to B$, the *type of the quasi-inverses of* f is by definition

$$\mathrm{qinv}\, f :\equiv \sum_{g:B\to A} (g \circ f \sim \mathrm{id}_A) \times (f \circ g \sim \mathrm{id}_B)$$

Unfortunately, the type $\mathrm{qinv}\, f$ carries more structure than the correspondent proposition, for that there exist types A and B, $f : A \to B$ and $e_1, e_2 : \mathrm{qinv}\, f$ such that $\neg(e_1 = e_2)$ (see Theorem 4.1.3 in [16]); technically, we say that $\mathrm{qinv}\, f$ is not a *mere proposition*. To get the desirable property that a function can be an isomorphism in at most one way we consider the following type:

DEFINITION 1.9 (Equivalence). Given a function $f : A \to B$ we say that f is an **equivalence** if $\mathrm{isequiv}\, f$ is inhabited, where

$$\mathrm{isequiv}\, f :\equiv (\sum_{g:B\to A} f \circ g \sim \mathrm{id}_B) \times (\sum_{h:B\to A} h \circ f \sim \mathrm{id}_A)$$

The theory is now endowed with a type whose inhabitedness establish that two types are equivalent.

DEFINITION 1.10 (Equivalence types). For every two types $A, B : \mathcal{U}$ we define the type of equivalences between A and B as

$$A \simeq B :\equiv \sum_{f:A\to B} \mathrm{isequiv}\, f$$

If $A \simeq B$ is inhabited, we may say t

It can be shown that the family of relations $\_ \simeq \_ : \mathcal{U} \to \mathcal{U} \to \mathcal{U}$ induces an equivalence relation on the universe $\mathcal{U}$.

Transportation gives a first relation between the identity type and the type of equivalences.

PROPOSITION 1.11. *There exists*

$$\mathrm{idtoeqv} : \prod_{A,B:U} A =_{\mathcal{U}} B \to A \simeq B$$

PROOF. Although the inhabitedness of the above type directly follows from an application of the Path Induction principle, it is possible to construct idtoeqv explicitly and that is precisely what we are going to do, specifying how idtoeqv acts on a generic path in $A =_{\mathcal{U}} B$. Let $A, B : \mathcal{U}$ be types and $p : A =_{\mathcal{U}} B$ be a path between them. The identity function $\mathrm{id}_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ may be regarded as a type family indexed by the universe $\mathcal{U}$, and thus we can transport p along it, getting a transport function $p_* : A \to B$. The claim is that p_* is an equivalence: if so, we can define $\text{isequiv}\, p \equiv (p_*, u)$, where $u : \text{isequiv}\, p_*$. We need to show that the following type is inhabited:

$$\prod_{A,B:\mathcal{U}} \prod_{p:A=_{\mathcal{U}}B} \text{isequiv}\, p_*$$

By Path Induction it suffices to assume that $p \equiv \text{refl}_A$, in which case $p_* \equiv \text{id}_A$, that clearly is an equivalence, being its own inverse. □

Observation 1.12. One could have defined another element $\text{idtoeqv}' : \prod_{A,B:U} A =_{\mathcal{U}} B \to A \simeq B$ via Path Induction, prescribing that $\text{idtoeqv}'\, A, A, \text{refl}_A \equiv \text{id}_A$. Since the two maps coincide on (A, A, refl_A) for every $A : \mathcal{U}$, using once more Path Induction we would have had that for every $A, B : \mathcal{U}$ and $p : A =_{\mathcal{U}} B$

$$\text{idtoeqv}\, A, B, p = \text{idtoeqv}'\, A, B, p$$

Nevertheless, in order to conclude that $\text{idtoeqv} = \text{idtoeqv}'$ it is necessary the function extensionality principle, which is an instance of the Univalence Axiom.

Nonetheless, intuitionistic type theory, and specifically Path Induction, the universal property of the identity type, do not provide a way to prove idtoeqv is an equivalence: Voevodsky formulated this property as an axiom – more precisely as an axiom schema, one statement about each universe.

Definition 1.13 (Univalence Axiom). For any $A, B : \mathcal{U}$ $\text{idtoeqv}\, A, B : A =_{\mathcal{U}} B \to A \simeq B$ is an equivalence. We call its inverse $\text{ua}_{A,B}$.

It should be noticed that adding the Univalence Axiom to the rules of dependent type theory makes the theory loose its procedural property, which was so advantageous from a computational point of view. Nevertheless, Voevodsky conjectured, albeit for a system not including Higher Inductive Types, that the interferences to the constructivity of HoTT arising from the Univalence Axiom can be bypassed, and indeed in the mathematical community Univalence Axiom and Higher Inductive Types are believed to be constructive in some way (unlike LEM, AC); and several researchers are attempting to give an alternative description of HoTT where Univalence and HIT compute, see for example the cubical type theory [2].

The Univalence Axiom brings several groundbreaking consequences.

First of all, that idtoeqv is an equivalence gives support in the language of logic for the usual mathematical practice of identifying isomorphic structures, a principle of informal mathematics called **structure identity principle (SIP)**. Indeed, for any property of types $P : \mathcal{U} \to \mathcal{U}$, a path $p : A =_{\mathcal{U}} B$ induces a function

$$p_* : P(A) \to P(B)$$

Hence, by promoting equivalences into identities, Univalence Axiom makes properties invariant under equivalence: one cannot express a property in the formal language of type theory that fails to be invariant under equivalence. In Homotopy Type Theory Book, [16, Sec.9.8], SIP is formulated inside the mathematical theory and is indeed proven, with a proof that strongly relies on the Univalence Axiom.
Observe that the structure identity principle is technically incompatible with set theory, where mathematical objects are often constructed by arbitrary encodings that give them an additional internal structure. For instance, it is not the case that $\mathbb{N} = \{x : \mathbb{Z} | x \geq 0\}$ if $\mathbb{Z}$ is defined as the quotient of $\mathbb{N} \times \mathbb{N}$ by the usual equivalence

relation; the strongest assertion that can be made is that $\mathbb{N} \simeq \{x : \mathbb{Z} | x \geq 0\}$. Similarly, if one defines the disjoint union of two sets X, Y as the set of pairs whose second components are 0 or 1 if the first component belongs to X or Y respectively, it is not true that for any three sets X, Y, Z $X \sqcup (Y \sqcup Z) = (X \sqcup Y) \sqcup Z$, although they clearly are isomorphic: $\{0,1\} \sqcup \{0,1\} = \{0,1\} \times \{0,1\}$, but $\{1,2\} \sqcup \{1,2\} \neq \{1,2\} \times \{1,2\}$, although evidently $\{0,1\} \simeq \{1,2\}$. The reason why these equalities do not hold is that the elements of the two sets considered are different. This behavior of set theory makes it intensional, since the theory allows to talk about encodings of objects; on the other hand, in type theory it is not possible to talk about these intentional aspects, and so the theory may be rearranged so that it is allowed to identify objects that have the same extensional behavior, and this is precisely what is done by the Univalence Axiom.

Secondly, univalent foundations definitely reject the interpretation for the formal system of type theory in classical mathematics, since Univalence Axiom is inconsistent with the principle of uniqueness of identity proofs; from an homotopical point of view this means that not every type can be regarded as (the homotopic version of) a set, where the notion of set is *internal* to the theory.
It is in fact possible to recover the concept of *set* in HoTT: topologically speaking, a set may be regarded as a space whose connected components are contractible, and so homotopically equivalent to a discrete space. In this sense, when we force the identity type to behave in a classical way, namely to be a proposition, we are prescribing that every type is a set.

Definition 1.14 (h-Set). A type A is said to be an *h-set* if the following type is inhabited:

$$\mathrm{isSet}(A) :\equiv \prod_{x,y:A} \prod_{p,q:x=_A y} p = q$$

Proposition 1.15. *In Univalent foundations it is not the case that every type is a set. In other words, it is inhabited the type*

$$(\prod_{X:\mathcal{U}} \mathrm{isSet}(X)) \to 0$$

Proof. Consider the type of booleans $\mathbf{2}$. Define $e : \mathbf{2} \simeq \mathbf{2}$ by case analysis as $e(1_\mathbf{2}) \equiv 0_\mathbf{2}$ and $e(0_\mathbf{2}) \equiv 1_\mathbf{2}$. Of course, the identity $\mathrm{id}_\mathbf{2}$ is an equivalence too, and it is evident that $\neg(e = \mathrm{id}_\mathbf{2})$. By the Univalence Axiom these two equivalences correspond respectively to two different paths of type $\mathbf{2} = \mathbf{2}$, respectively to $\mathrm{ua}(\mathrm{id}_\mathbf{2}) = \mathrm{refl}_\mathbf{2}$ and $\mathrm{ua}(e)$. Hence by $\neg(e = \mathrm{id}_\mathbf{2})$ we get $\neg(\mathrm{refl}_\mathbf{2} = \mathrm{ua}(e))$. □

Additionally, univalence endows Homotopy Type Theory with extensionality, in the sense that, set-theoretically speaking, every function may be identified with its graph.

Axiom (Function extensionality). *For every* $f, g : \prod_{x:A} P(x)$ *there exists*

$$\mathrm{funext}_{f,g} : f \sim g \to f =_{\prod_{x:A} P(x)} g$$

Observation 1.16. Path induction allows to construct a map

$$\mathrm{happly} : f =_{\prod_{x:A} P(x)} g \to f \sim g$$

Theorem 1.17. *Univalence Axiom implies function extensionality.*

A proof of the above theorem can be found in [16, Sec.4.9].

Eventually, it is essential to mention that the Univalence Axiom is consistent with HoTT: it was, in fact, formulated by Voevodsky after he recognized that the model of HoTT he constructed in Kan simplicial sets satisfied the invariance property illustrated above [6].

CHAPTER 2

Inductive Types

In this section, we will see how structural features of the definition of a natural numbers type $\mathbb{N}$ suggest a generalization of that style of axiomatization into a new kind of construction of types, called *inductive types.* The main idea behind an inductive definition is to characterize a type W by the data necessary to construct maps out of W, namely objects of type $W \to X$. In this regard, an inductive type should be intended as freely generated by its constructors, where the free structure has to be intended in an algebraic manner. More generally, Higher Inductive Types (HIT) permit the construction of new types determined by an inductive definition together with equational laws between the generators. While inductive types are part of the original Martin Löf's type theory, Higher Inductive Types (HIT) are a feature introduced by Homotopy Type Theory. Consequences of the introduction of HIT are of a transversal nature: they contribute in an essential way to synthetic homotopy type theory, for they furnish a way to represent homotopical structures and constructions (in a synthetic way); they permit the formalization of *n-truncations* and the definition of *quotient types*; they pave the way for introduction of even higher-dimensional higher inductive types such as *higher inductive-inductive types.* This latter construction, in particular, is of fundamental importance for a well-behaved definition of Cauchy real numbers and permits a new approach to some foundational problems.

1. Natural numbers and inductive types

The main idea associated to natural numbers is that every element of $\mathbb{N}$ is either $0 : \mathbb{N}$ or of the form $\mathrm{succ}(n)$ for some previously constructed element $n : \mathbb{N}$. This description, though, is basically extensional, for it describes a structure via the elements it should contain. However, Peano axiomatization of natural numbers does not really require such extensionality, for essentially it prescribes that a structure may be called $\mathbb{N}$ if it is equipped with an element $0 : \mathbb{N}$ and a function $\mathrm{succ} : \mathbb{N} \to \mathbb{N}$ such that the three of them satisfy the induction schema, which basically prescribes how to construct maps out of $\mathbb{N}$. This is precisely the idea at the basis of the more general definition of inductive types.

In defining the type of natural numbers as an inductive type, type theory postulates the existence of a type $\mathbb{N} : \mathcal{U}$ whose constructors are the following:

$$0 : \mathbb{N}$$

$$\mathrm{succ} : \mathbb{N} \to \mathbb{N}$$

The non-dependent eleminator prescribes that, in order to construct a non-dependent function $f : \mathbb{N} \to C$ out of the natural numbers, it is sufficient to equip C of a structure that mimics the one of $\mathbb{N}$, i.e. to provide a starting point $c_0 : C$ and a map

$c_s : \mathbb{N} \to C \to C$, giving rise to an f that computes on the constructors in the following way:

$$f(0) :\equiv c_0$$
$$f(\mathrm{succ}(n)) :\equiv c_s(n, f(n))$$

This principle is the **primitive recursion principle**, and it can be expressed type-theoretically through the recursor

$$\mathrm{rec}_{\mathbb{N}} : \prod_{C:\mathcal{U}} C \to (\mathbb{N} \to C \to C) \to \mathbb{N} \to C$$

with the associated computational rules

$$\mathrm{rec}_{\mathbb{N}}(C, c_0, c_s, 0) :\equiv c_0$$
$$\mathrm{rec}_{\mathbb{N}}(C, c_0, c_s, \mathrm{succ}(n)) :\equiv c_s(n, \mathrm{rec}_{\mathbb{N}}(C, c_0, c_s, n))$$

Let it be noted that the introduction of $\mathbb{N}$, as well as the introduction of any new inductive type, implies a modification of type theory until now introduced, due to the introduction of a new set of inference rules together with *axioms* that prescribe the inhabitedness of certain types. Intensional type theory is intended to be type theory where the definition of identity is intensional (i.e. as given in the previous chapter) together with the stipulation of the existence of inductive types.

In the formal definition of the type of natural numbers $\mathbb{N}$, the recursion principle is generalized to dependent type families $C : \mathbb{N} \to \mathcal{U}$, prescribing how to construct dependent functions from $\mathbb{N}$ into those types.

Definition 2.1 (Natural numbers). The type of natural numbers is a type $\mathbb{N} : \mathcal{U}$ whose constructors are the following:

$$0 : \mathbb{N}$$
$$\mathrm{succ} : \mathbb{N} \to \mathbb{N}$$

The induction principle (eliminator) for $\mathbb{N}$ prescribes how to construct a dependent function from $\mathbb{N}$ to a dependent type family $C : \mathbb{N} \to \mathcal{U}$.

$$\mathrm{ind}_{\mathbb{N}} : \prod_{C:\mathbb{N}\to\mathcal{U}} C(0) \to (\prod_{n:\mathbb{N}} C(n) \to C(\mathrm{succ}(n))) \to \prod_{n:\mathbb{N}} C(n)$$

The associated computational rules are The computational rule is expressed as:

$$\mathrm{ind}_{\mathbb{N}}(C, c_0, c_s, 0) :\equiv c_0$$
$$\mathrm{ind}_{\mathbb{N}}(C, c_0, c_s, \mathrm{succ}(n)) :\equiv c_s(n, \mathrm{ind}_{\mathbb{N}}(C, c_0, c_s, n))$$

.

Observation 2.2. The eliminator for natural numbers makes clear the terminology of *induction principle*. In fact, under the propositions-as-types correspondence (even if applied to a proper subclass of types, such as with (-1)-truncated logic of HoTT) proving a proposition corresponds to inhabiting the associated type, and a property of natural numbers is a (particular) dependent family $P : \mathbb{N} \to \mathcal{U}$. With these associations in mind, the above induction principle says that if one can prove $P(0)$ and $P(\mathrm{succ}(n))$ for every n, assuming $P(n)$, then we have $P(n)$ for all $n : \mathbb{N}$.

Multi-variable functions can be defined by primitive recursion as well, by currying and allowing C to be a function type; consequently, we can endow $\mathbb{N}$ with an addition $+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ and a multiplication $\cdot : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ which are defined as in PA axioms. At first sight, it may seem that only primitive recursive functions can be defined using the primitive recursion schema, namely the non-dependent eliminator. However, observe that we can take the codomain $C : \mathcal{U}$ to be a function type, for instance $C \equiv A \to B$. Hence, by means of higher function types, that is, functions with other functions as arguments, all computable functions can, in fact, be constructed.

EXAMPLE 1. The Ackermann function ack : $\mathbb{N} \to \mathbb{N} \to \mathbb{N}$ is a typical example of a computable function which is not primitive recursive. It is defined as

$$\operatorname{ack}(m,n) :\equiv \begin{cases} \operatorname{succ}(n) & \text{if } m = 0 \\ \operatorname{ack}(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ \operatorname{ack}(m-1,\operatorname{ack}(m,n-1)) & \text{id } m,n > 0 \end{cases}$$

In type theory, we can define ack only using $\operatorname{rec}_{\mathbb{N}}$. In fact, the following map ack satisfies the required equations, as it is easy to check.

$$\operatorname{ack} \equiv \operatorname{rec}_{\mathbb{N}}(\mathbb{N} \to \mathbb{N}, \operatorname{succ}, \lambda(m,r).\operatorname{rec}_{\mathbb{N}}(\mathbb{N}, r(1), lambda(n,s).r(s)))$$

In mathematics there are several examples of inductive structures: in set theory, ordinals generalize the way of reasoning about natural numbers, codifying the concept of mathematical induction; in first-order logic, formulas are usually defined inductively, namely by starting from some atomic formulas and then applying to them the constructions given by logical operations; in algebra, we see inductive definitions of substructures generated by a subset of elements (for instance the subgroup of G generated by elements $b_1, \ldots, b_n$ is the *least* subgroup H of G such that $b_1, \ldots, b_n$ are in H and if x, y are in H then $x \cdot y^{-1}$ is in H too). As we said, the main idea behind an inductive definition of a type W is that it prescribes the data necessary to construct maps out of W. Formally, an **inductive type** W is specified by:

- a finite collection of **constructors**, where a constructor is a function of some number of arguments with codomain W, allowed to take arguments from the inductive type being defined, but only strictly positively (that is, W is banned from appearing on the left of an arrow in the domain of its constructors);
- an **induction principle**, i.e. the elimination rule, which prescribes how to define dependent functions out of W. In particular, in order to define $f : \prod_{w:W} P(w)$, the type family $P : W \to \mathcal{U}$ is required to be equipped with constructor data lying over those of W. The induction principle specifies to the **recursion principle**, which establishes how to construct non-dependent maps out of W, and is usually strong enough to prove a **uniqueness principle**, which characterizes uniquely the functions that the eliminator yields.

Inductive types is that they should be intended as freely generated by their constructors: all the elements of an inductive type should be obtained by repeatedly applying its constructors. From this perspective, the induction principle expresses the fact that to prove a property of an inductively defined type W or to construct a function out of W it suffices to consider the case when the input $w : W$ arises from one of the constructors, allowing recursive call of the function being defined on the inputs of those constructors.

The uniqueness principle, which is evidently propositional (i.e. it is a type, not a judgment), applies even to functions that only satisfy the recurrences up to propositional equality of paths, and in fact for this reason it will be necessary to switch to *homotopy inductive types* to have their characterization as homotopy-initial algebras. In particular, the uniqueness principle for natural numbers is stated in the following way.

LEMMA 2.3 (Uniqueness principle). *Let $E : \mathbb{N} \to \mathcal{U}$ be a dependent family over $\mathbb{N}$ and $g, h : \prod_{n:\mathbb{N}} E(n)$ be two functions such that*

$$f(0) = e_z \quad \textit{and} \quad g(0) = e_z$$

$$\prod_{n:\mathbb{N}} f(\mathrm{succ}(n)) = e_s(n, f(n)) \quad \textit{and} \quad \prod_{n:\mathbb{N}} g(\mathrm{succ}(n)) = e_s(n, g(n))$$

Then $f = g$.

PROOF. By induction on the type family $\lambda(x : \mathbb{N}).D(x) :\equiv \lambda(x : \mathbb{N}).f(x) = g(x)$. The base case follows from the hypothesis, while for the inductive case, assume $n : \mathbb{N}$ such that $f(n) = g(n)$. Then $f(\mathrm{succ}(n)) = e_s(n, f(n)) = e_s(n, g(n)) = g(\mathrm{succ}(n))$. Then we have that $f \sim g$, i.e. a pointwise equality between f and g. Function extensionality implies $f = g$. □

For what concerns uniqueness of inductive constructions, we may ask whether or not two types that satisfy the same induction principle may result in equivalent types, and so, for the Univalence Axiom, if they may be identified. Indeed, if two types W and W' satisfy the same induction principle, then not only are they equivalent, i.e. $W \simeq W'$, but the equivalence between them is canonical: the recursion principle for W and W' gives rise to maps $W \to W'$ and $W' \to W$, and then the induction principle for each proves that both composites are equal to the identity; finally, thanks to the Univalence Axiom, they result equal as types. Path Induction guarantees that any construction or proof relating to one type can be transferred to the other *by substitution*, that is, by the transport operation on paths. In fact, if one considers the type of the function or theorem as a type-indexed family $P : \mathcal{U} \to \mathcal{U}$, the given object (proof-term or construction-term) will be an element of $P(W)$, that we can transport along the path of $W = W'$.

As a final remark, we want to point out that the identity family over a type A, i.e. $\mathrm{Id}_A : A \to A \to \mathcal{U}$, has been defined as an inductive type, with constructor

$$\mathrm{refl}_{-} : \prod_{x:A} \mathrm{Id}_A(x, x) \equiv \prod_{x:A} x =_A x$$

From this definition it may be extracted an induction principle stating that, given any $C : \prod_{x,y:A} (x =_A y) \to \mathcal{U}$ along with structure data lying over that of Id_A, namely together with $d : \prod_{x:A} C(x, x, \mathrm{refl}_x)$, there exists $f : \prod_{x,y:A} \prod_{p:x=_A y} C(x, y, p)$ such that it computes on the constructor as $f(x, x, \mathrm{refl}_x) \equiv d(x)$: this is exactly the Path Induction. Path Induction expresses the fact that Id_A is *freely generated* by elements of the form refl_x, as x varies in A. Logically, this means that to give an element of any other family $C(x, y, p)$ dependent on a generic element $(x, y, p) : \mathrm{Id}_A$ it suffices to consider the cases where that is of the form (x, x, refl_x), as x varies in A. Homotopically, Path Induction prescribes that the type $\sum_{x,y:A} (x =_A y)$, called the *free path space*, which corresponds to the space of paths in A with free endpoints, is freely (or inductively) generated by the constant loop refl_x at each point $x : A$, that is by terms of the form (x, refl_x). As it is always possible to retract one path to the constant loop if we don't require both endpoints to be fixed, it can be shown via Path Induction that for every $x, y : A$ and $p : x =_A y$ the type $(x, y, p) = \sum_{x,y:A} (x=_A y)$ (x, x, refl_x) is inhabited. Let it be noted that it is the identity *family* that is freely generated by elements of the form refl_x as $x : A$ varies, not the identity *type* $x =_A x \equiv \text{Id}_A(x, x)$, and in fact it is not provable that every element of $x =_A x$ is of the form refl_x. The crucial aspect is allowing one of the endpoints to vary while retracting the path: Path Induction does not provide a way to give an element of a family $C(p)$ where p has two fixed endpoints a and b, as it is not the case that for any space A any path in $x =_A x$ is equal to refl_x. Indeed, by the Univalence Axiom we can actually exhibit a counterexample, that is, an element of

$$\neg(\prod_{A:\mathcal{U}} \prod_{x:A} \prod_{p:x=_A x} p =_{x=_A y} \text{refl}_x)$$

as shown in Proposition 1.15

2. Higher Inductive Types

In the inductive definition of a type, say W, we only concern ourselves with objects of type W (namely points of the space W). Nevertheless, it might be desirable to also specify properties of such objects, in particular those expressible via equations between them. This style of definition of new types, that is, inductive definitions together with equational laws, is embodied by Higher Inductive Types, a class of type-forming rules of relatively recent introduction. As we have seen, an inductive definition provides some kind of free structure, without any additional condition, similarly to how in algebra one defines the group freely generated by some elements. However, one may want to cover also the construction of structures governed by certain laws between their elements, as one may want to define groups determined not only by their generators but also by relations between them. By the same token, we can use HIT to construct the truncation of a type P (that is, a new type $\|P\|$ whose elements are all equal) or even to address the construction of Cauchy real numbers. Actually, the original motivation behind Higher Inductive Types was the homotopical interpretation of type theory, which required a sort of translation of homotopy-theoretic constructions in a style proper to type theory, that is to say, in a *synthetic* style. For instance, one of the fundamental objects of homotopy theory is the circle $\mathbb{S}^1$, which can be constructed as a topological cell complex with one point (0-cell) and one path (1-cell) with both endpoints glued to the point. From the homotopy-type-theoretic perspective paths correspond to elements of identity types, and this suggests considering a type $\mathbb{S}_1$ that is "inductively generated" by:

- a point base : $\mathbb{S}^1$;
- a path loop : base $=_{\mathbb{S}^1}$ base.

In analogy to the inductive definitions studied above, we would like to characterize the circle with an universal property, captured by an induction principle. We may then think to the fact that a function out of the circle $f : \mathbb{S}^1 \to X$ is completely determined by the image of the base point and the image of the path loop : base $=_{\mathbb{S}^1}$ base. This is embodied by the induction principle, stating that for any given $P : \mathbb{S}^1 \to \mathcal{U}$, to construct $f : \prod_{x:\mathbb{S}^1} P(x)$ it suffices to provide

- an element $b : P(\text{base})$;
- a path $l : b =^P_{\text{loop}} b$.

The path $l : b =^P_{\text{loop}} b$ is intended to be a path in the total space $\sum_{x:\mathbb{S}^1} P(x)$ from b to itself that "lies over" loop. Let it be noted that this induction principle, in analogy to (lower) inductive type, expresses a sort of initiality of $\mathbb{S}^1$, in the sense that it can be seen as "the least object" with the given properties.

Other fundamental constructions can be performed following a similar pattern, such as the cylinder, the interval and spheres or various dimensions, but also the more complex suspensions and pushouts; finding a general syntactic description of valid higher inductive types, though, is an area of current research. What can be said is that, in general, like an ordinary inductive definition, an higher inductive definition is specified by a list of *constructors*, each of which is a (dependent) function where the type being defined may appear only strictly positively; unlike in an ordinary inductive definition, in a higher one the output type of a constructor may be not only the type being defined, say W, but also some identity type of it, such as $u =_W v$ or more generally an iterated identity type. What is more, the initiality property of inductive types can be recovered for Higher Inductive Types too as a characterizing property, as shown, for instance, in the work of Kristina Sojakova [15]. Among Higher Inductive Types there can be found *quotient types* and *truncations*, the formers permitting the definitions of $\mathbb{Z}$ and $\mathbb{Q}$.

3. Quotient types

A particularly important class of higher inductive types is the class of *quotients*. Let A be a set and $R : A \to A \to \text{Prop}$ a family of mere propositions. The **set-quotient** of A by R, A/R, is the higher inductive type generated by

- A function $q : A \to A/R$;
- For each $a, b : A$ such that $R(a, b)$, an equality $q(a) =_{A/R} q(b)$;
- The 0-truncation constructor: for all $x, y : A/R$ and $r, s : x = y$, we have $r = s$.

Note that A/R is a set by definition. Classically, the usual case to consider is when R is an equivalence relation, i.e. we have

- *reflexivity*: $\prod_{a:A} R(a, a)$;
- *symmetry*: $\prod_{a,b:A} R(a, b) \to R(b, a)$;
- *transitivity*: $\prod_{a,b,c:A} R(a, b) \to R(b, c) \to R(a, c)$.

In this case, the set-quotient A/R has additional good properties, such as $R(a, b) \simeq q(a) =_{A/R} q(b)$. We may write the equivalence relation $R(a, b)$ infix as $a \sim b$. Several expected properties apply to set quotients, among which there are the ones stated in the next lemmas.

LEMMA 2.4. *The function $q : A \to A/R$ is surjective. In addition, set-quotients are characterized by the following universal property: for any set B, precomposing with q yields an equivalence*

$$(A/R \to B) \simeq \sum_{f:A\to B} \prod_{a,b:A} R(a, b) \to f(a) = f(b)$$

Proof of the previous properties can be found in [16]; unsurprisingly, they strongly rely on the induction principle for set-quotients. It is worth mentioning the existence of an impredicative way to define quotients by equivalence relations, which mimics the set-theoretic approach to its construction, although we will not focus on it.

However, there are special cases, for instance when there are *canonical representatives* of the equivalence classes, where it is not necessary a general construction of quotients, thanks to the following lemma.

LEMMA 2.5. *Suppose* $\sim$ *is a relation on a set* $A : \mathcal{U}$ *and there exists an idempotent* $r : A \to A$ *such that* $r(x) = r(y) \simeq x \sim y$ *for all* $x, y : A$*. (This implies* $\sim$ *is an equivalence relation.) Then the type*

$$A/\sim :\equiv \sum_{x:A} r(x) = x$$

satisfies the universal property of the set-quotient of A *by* $\sim$ *and hence is equivalent to it. In other words, there is a map* $q : A \to A/\sim$ *such that for every set* B*, precomposition with* q *induces an equivalence*

$$(A/\sim \to B) \simeq \sum_{g:A\to B} \prod_{x,y:A} (x \sim y) \to g(x) = g(y) \tag{1}$$

PROOF. Let $i : \prod_{x:A} r(r(x)) = r(x)$ witness idempotence of r. We define $q : A \to A/\sim$ by $q(x) :\equiv (r(x), i(x)$. Since A is a set, we have $q(x) = q(y)$ if and only if $r(x) = r(y)$, hence (by assumption) if and only if $x \sim y$. We define a map e from left to right in (1) by

$$e(f) :\equiv (f \circ q, \_)$$

where the underscore denotes the following proof: if $x, y : A$ and $x \sim y$, then $q(x) = q(y)$ as observed above, hence $f(q(x)) = f(q(y))$. To see that e is an equivalence, consider the map e' in the opposite direction defined by

$$e'(g, s)(x, p) :\equiv g(x)$$

Given any $f : (A/\sim) \to B$,

$$e'(e(f))(x, p) \equiv f(q(x)) \equiv f(r(x), i(x)) = f(x, p)$$

where the last equality holds because$p : r(x) = x$ and so $(x, p) = (r(x), i(x))$ because A is a set. Similarly we compute

$$e(e'(g, s)) \equiv e(g \circ \mathrm{pr}_1) \equiv (g \circ \mathrm{pr}_1 \circ q, \_)$$

Because B is a set we need not worry about the $\_$ part, while for the first component we have

$$g(\mathrm{pr}_1(q(x))) \equiv g(r(x)) = g(x)$$

, where the last equation holds because $r(x) \sim x$, and g respects $\sim$ by the assumption s. $\square$

The previous lemma applies to $\mathbb{Z}$ and to $\mathbb{Q}$, with the idempotents $r : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ and $s : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ respectively defined by

$$r(a, b) = \begin{cases} (a - b, 0) & \text{if } b \le a \\ (0, b - a) & \text{otherwise} \end{cases} \quad , \quad s(k, h) = (k/gcd(h, k + 1), (h + 1)/gcd(h, k + 1))$$

The above definition by cases can be performed thanks to the decidability of the order on $\mathbb{N}$. More precisely, we define the partial order $\le$ on $\mathbb{N}$ as

$$\le : \mathbb{N} \to \mathbb{N} \to \mathcal{U}$$

$$(n \leq m) :\equiv \sum_{k:\mathbb{N}} (n + k = m)$$

Observe that $\text{isProp}(n \leq m)$ for any $n, m : \mathbb{N}$. In fact, for any two elements $p, q : (n \leq m)$ we have that $n + \text{pr}_1(p) = n + \text{pr}_1(q) = m$, and by $\mathbb{N}$-induction we can prove that $\prod_{n:\mathbb{N}}(n + \text{pr}_1(p) = n + \text{pr}_1(q) \to \text{pr}_1(p) = \text{pr}_1(q))$. Again, induction proves that $\leq$ is indeed a partial order. Finally, decidability of $\leq$ follows from the decidability of equality in $\mathbb{N}$, proven in Example 2.

Analogously to what we have done with $\mathbb{N}$, it can be shown that equality of $\mathbb{Z}$ and of $\mathbb{Q}$ is decidable. In addition, we can endow $\mathbb{Z}$ of the structure of an ordered ring and $\mathbb{Q}$ with that of an ordered field, both with decidable orders.

CHAPTER 3

Logic in Homotopy Type Theory

In contrast to classical foundations, type theory internalizes logic: under the propositions-as-types interpretation, mathematical statements and their proofs become first-class mathematical objects. However, this correspondence between types and propositions applied to the class of *all* types yields a logic which is constructive in an algoritmical sense. Indeed, types have in general more structure than propositions as intended classically, since they can carry more information by having different inhabitants. For example, having a term of type $A + B$ not only means that $A \vee B$ is true, but it gives also the information of *which* between A and B is true, thanks to the (propositional) uniqueness principle for Σ-types. A more significant example consists in the provability of the axiom of choice under this propositions-as-types correspondence.

In set theory, the Axiom of Choice can be stated by saying that for any two sets X, Y and any binary relation $P \subseteq X \times Y$, if for any $x \in X$ there exists $y \in Y$ such that $(x, y) \in P$ then there exists a function (called a *choice function*) $g : X \to Y$ such that for any $x \in X$ we have $(x, gx) \in P$. Formally:

$$\forall(X,Y)\forall(P \subseteq X\times Y)(\forall x \in X \exists y \in Y(x,y) \in P) \to (\exists g \in X \to Y \forall x \in X(x,gx) \in P)$$

A naïf translation of the Axiom of Choice in type theory following the standard propositions-as-types paradigm (and generalizing the cartesian product to a dependent product type) is:

$$\mathrm{AC}_\infty :\equiv \prod_{X:\mathcal{U}} \prod_{Y:X\to\mathcal{U}} \prod_{P:\prod\limits_{x:X} Y(x)\to\mathcal{U}} \left(\prod_{x:X} \sum_{a:Y(x)} P(x,a) \to \sum_{g:\prod\limits_{x:X} Y(x)} \prod_{x:X} P(x,gx) \right)$$

LEMMA 3.1. *The propositions-as-types translation of the Axiom of Choice AC_∞ is provable.*

PROOF. Fix $X : \mathcal{U}$ and type families $Y : X \to \mathcal{U}$, $P : \prod\limits_{x:X} A(x) \to \mathcal{U}$. We need to construct an inhabitant of

$$\prod_{x:X} \sum_{a:Y(x)} P(x,a) \to \sum_{g:\prod\limits_{x:X} Y(x)} \prod_{x:X} P(x,gx)$$

Suppose having an $f : \prod\limits_{x:X} \sum_{a:Y(x)} P(x,a)$, we define the required $g : \prod\limits_{x:X} Y(x)$ as $g(x) :\equiv \mathrm{pr}_1(f(x))$. $\square$

Observe that the latter does not amount to an actual proof of the Axiom of Choice, because the hypothesis $\prod\limits_{x:X} \sum_{a:Y(x)} P(x,a)$ already condenses in itself the choice function. Hence, the formula AC_∞ does not really convey the content of the Axiom of Choice, since the existential quantification in the hypothesis translated as a Σ-type makes the choosing of the choice function g explicit.

Univalence Axiom, for its part, turns out to be inconsistent with other classical principles such as the law of double negation and the law of excluded middle.

Proposition 3.2. *It is not the case that for all $A : \mathcal{U}$ we have $\neg\neg A \to A$.*

Proof. We read "it is not the case that.." as the operator $\neg$, thus in order to prove the statement, it suffices to assume given some $f : \prod_{A:\mathcal{U}} \neg\neg A \to A$ and construct an element of 0. The idea is that f, as any other function in type theory, is continuous: by univalence, this implies that f is natural with respect to equivalences of types. From this and a fixed-point-free autoequivalence we will be able to extract a contradiction. Let $e : \mathbf{2} \simeq \mathbf{2}$ be the equivalence defined by $e(1_{\mathbf{2}}) :\equiv 0_{\mathbf{2}}$ and $e(0_{\mathbf{2}}) :\equiv 1_{\mathbf{2}}$. Let $p : \mathbf{2} = \mathbf{2}$ be the path corresponding to e by the Univalence Axiom. Then we have $f(\mathbf{2}) : \neg\neg\mathbf{2} \to \mathbf{2}$ and

$$\mathrm{apd}_f(p) : p_*(f(\mathbf{2})) =^{A\to(\neg\neg A\to A)} f(\mathbf{2})$$

with the transport made in $A \to (\neg\neg A \to A)$. Hence, for any $u : \neg\neg\mathbf{2}$ we have

$$\mathrm{happly}(\mathrm{apd}_f(p), u) : p_*(f(\mathbf{2}))(u) =^{A\to(\neg\neg A\to A)} f(\mathbf{2})(u)$$

Transporting $f(\mathbf{2}) : \neg\neg\mathbf{2} \to \mathbf{2}$ along p in $A \to (\neg\neg A \to A)$ is equal to the function which transports its argument along p^{-1} in the type family $A \to \neg\neg A$, applies $f(\mathbf{2})$, then transports the result along p in the type family $A \to A$:

$$p_*(f(\mathbf{2}))(u) = p_*(f(\mathbf{2}))(p_*^{-1}(u))$$

However, any two points $u, v : \neg\neg\mathbf{2}$ are equal by function extensionality, since for any $x : \neg\mathbf{2}$ we have $u(x) : \mathbf{0}$ and thus we can derive any conclusion, in particular $u(x) = v(x)$. Thus, we have $p_*^{-1}(u) = u$, and so from $\mathrm{happly}(\mathrm{apd}_f(p), u)$ we obtain an equality $p_*(f(\mathbf{2}))(u) = f(\mathbf{2})(u)$. Finally, transporting in the type family $A \to A$ along the path p is equivalent to applying the equivalence e, thus we have $e(f(\mathbf{2})(u)) = f(\mathbf{2})(u)$. However, we can also prove by case analysis on x that $\prod_{x:\mathbf{2}} \neg(e(x) = x)$. Thus, applying an inhabitant of the latter type to $f(\mathbf{2})(u)$ and the previous one we obtain an element of $\mathbf{0}$. □

Corollary 3.3. *It is not the case that for all $A : \mathcal{U}$ we have $A + \neg A$.*

Proof. Suppose we had $g : \prod_{A:\mathcal{U}}(A+\neg A)$. We will show that then $\prod_{A:\mathcal{U}} \neg\neg A \to A$, so that we can apply the previous lemma. Thus, suppose $A : \mathcal{U}$ and $u : \neg\neg A$, we want to construct an element of A. Now $g(A) : A + \neg A$, so by case analysis we may assume either $g(A) \equiv \mathrm{inl}(a)$ for some $a : A$, or $g(A) \equiv \mathrm{inr}(w)$ for some $w : \neg A$. In the first case, we have $a : A$, while in the second case we have $u(w) : 0$ and so we can obtain anything we wish (such as A). Thus, in both cases we have an element of A, as desired. □

However, it is still possible for Homotopy Type Theory to maintain the invariance of the language given by Univalence and constructivity and computability proper to type theory, together with consistentcy with axioms of classical logic, by considering as propositions only *propositionally truncated types*, namely types whose inhabitedness does not carry any extra information.

DEFINITION 3.4 (Mere proposition). A type $P : \mathcal{U}$ is a mere proposition when the following type is inhabited:

$$\text{isProp}(P) :\equiv \prod_{x,y:P} x =_P y$$

Namely all elements of P are equal.

OBSERVATION 3.5. For any type $P : \mathcal{U}$, $\text{isProp}(P)$ is a mere proposition.

For mere propositions, equivalence amounts to simple logical equivalence.

LEMMA 3.6. *If P and Q are mere propositions such that $P \to Q$ and $Q \to P$, then $P \simeq Q$.*

PROOF. Suppose given $f : P \to Q$ and $g : Q \to P$. For any $x : P$, we have $g(f(x)) = x$ since P is a mere proposition. Similarly, for any $y : Q$ we have $f(g(y)) = y$ since Q is a mere proposition, thus f and g are quasi-inverses. □

DEFINITION 3.7. We define the type of mere propositions in an universe $\mathcal{U}_i$ as

$$\text{Prop}_{\mathcal{U}_i} :\equiv \sum_{X:\mathcal{U}} \text{isProp}(X)$$

By means of typical ambiguity, we may write simply Prop, or $\text{Prop}_{\mathcal{U}}$.

Under this new paradigm, logical principles on propositions have to be referred only to *mere propositions*. The law of excluded middle in Homotopy Type Theory becomes then:

$$\text{LEM} :\equiv \prod_{A:\mathcal{U}} \text{isProp}(A) \to A + \neg A$$

We note that the counterexamples given in 3.2 and 3.3 do not apply in these cases, since **2** is not a mere proposition.

Even if we do not assume LEM, there still may be mere propositions P, and more generally types P, for which the type $P + \neg P$ is inhabited.

DEFINITION 3.8 (Decidability). A type A is said to be decidable if it is inhabited the type $A + \neg A$.
Similarly, given a type $X : \mathcal{U}$, the dependent family $P : X \to \mathcal{U}$ is said to be decidable if $\prod_{x:X} P(x) + \neg P(x)$.

LEM states that all mere propositions are decidable.

EXAMPLE 2. Natural numbers have decidable equality, i.e.

$$\prod_{x,y:\mathbb{N}} (x = y) + \neg(x = y)$$

To see this, we proceed by induction on x and case analysis on y. Case analysis on y is justified by the fact that we can prove by induction that $\prod_{y:\mathbb{N}}(y = 0) + \sum_{n:\mathbb{N}} y = \text{succ}(n)$. In the base case we have $x \equiv 0$. If $y \equiv 0$, then $\text{inl}(\text{refl}_0) : x = y$, while if $y \equiv \text{succ}(n)$, by [16, p. 2.13.2](characterization of the identity type of $\mathbb{N}$) we have $\neg(0 = \text{succ}(n))$. For the inductive step, let $x \equiv \text{succ}(n)$. If $y \equiv 0$, we use [16, p. 2.13.2] again. Finally, if $y \equiv \text{succ}(m)$, the inductive hypothesis gives $(m = n) + \neg(m = n)$. In the first case, if $p : m = n$, then $\text{ap}_{\text{succ}}(p) : \text{succ}(m) = \text{succ}(n)$. In the second case, [16, p. 2.13.3] yields $\neg(\text{succ}(m) = \text{succ}(n))$.

By means of higher inductive types we can define *propositional truncation*, an additional type former which "truncates" a type down to a mere proposition.

DEFINITION 3.9 (Propositional truncation). For any type $A : \mathcal{U}$, its propositional truncation is the higher inductive type $\|A\| : \mathcal{U}$ generated by the following constructors:

- a map $| - | : A \to \|A\|$, which assigns to any $a : A$ an element $|a| : A$;
- an element of $\mathrm{isProp}(\|A\|)$.

The recursion principle for propositional truncation states that

if B is a mere proposition, then any $f : A \to B$ induces a $g : \|A\| \to B$ such that $g(|a|) \equiv f(a)$ for all $a : A$.

We can now formulate the proper version of the Axiom of Choice in Homotopy Type Theory.

DEFINITION 3.10 (Axiom of choice). Assume a type $X : \mathcal{U}$ and type families $A : X \to \mathcal{U}$ and $P : \prod_{x:X} A(x) \to \mathcal{U}$ and moreover that X is a set, $A(x)$ is a set for all $x : X$ and that $P(x, a)$ is a mere proposition for all $x : X$ and $a : A(x)$. The axiom of choice asserts that under these assumptions

$$\mathrm{AC} :\equiv \left(\prod_{x:A} \left\| \sum_{a:A(x)} P(x,a) \right\| \right) \to \left\| \sum_{g:\prod_{x:X} A(x)} \prod_{x:X} P(x, g(x)) \right\|$$

LEMMA 3.11. *The axiom of choice AC is equivalent to the following statement: for any set X and any $\Upsilon : X \to \mathcal{U}$ such that each $\Upsilon(x)$ is a set we have*

$$(\prod_{x:X} \|\Upsilon(x)\|) \to \left\| \prod_{x:X} \Upsilon(x) \right\|$$

This corresponds to the well-known version of the axiom of choice as "the cartesian product of a family of nonempty sets is nonempty".

PROOF. $\left\| \sum_{g:\prod_{x:X} A(x)} \prod_{x:X} P(x, g(x)) \right\|$ is equivalent to $\left\| \prod_{x:X} \sum_{a:A(x)} P(x,a) \right\|$, thus AC is equivalent to the second formulation with $\Upsilon(x) :\equiv \sum_{a:A(x)} P(x,a)$. Conversely, instantiating the variables in AC with $A(x) :\equiv \Upsilon(x)$ and $P(x,a) :\equiv \mathbf{1}$ we get the second formulation. Hence the two are logically equivalent, and since they both are mere propositions (easy to check), by lemma 3.6 they are equivalent types. □

From operations that can be performed on paths of identity types (namely concatenation and inversion), it follows that a mere proposition is also a set, i.e.

$$\prod_{P:\mathcal{U}} \mathrm{isProp}(P) \to \mathrm{isSet}(P)$$

In this fashion, the law of excluded middle and the Axiom of Choice may be consistently assumed as axioms of the theory, as validated by Voevodsky, LeFanu Lumsdaine and Kapulkin's model of Homotopy Type Theory [6]. This is interesting, since it shows how Homotopy Type Theory does not *force* a constructive manner of doing mathematics, but still it permits to leave out all classical assumptions until really needed.

CHAPTER 4

Cauchy Reals in Homotopy Type Theory

In classical logic, Cauchy Reals are, by definition, the completion of $\mathbb{Q}$ under limits of Cauchy sequences, and this construction can be generalized to the *Cauchy completion* $\mathcal{C}T$ of an arbitrary metric space T. The effective construction of the completion $\mathcal{C}T$ (necessary to show the existence of such a structure) is classically carried out by quotienting the space of Cauchy sequences in T by the equivalence relation of "being arbitrarily close". To prove the quotient $\mathcal{C}T$ is Cauchy-complete, then, any Cauchy-sequence in $\mathcal{C}T$ is lifted to a sequence of sequences in T, and a limit of the original sequence is constructed by means of the one lifted. However, for the lifting of the sequence (precisely, the choice of a representative), it is necessary the Axiom of (countable) Choice ($\mathrm{AC}_{\aleph_0}$), and this makes such an approach to the construction of Cauchy reals and general Cauchy completions unfeasible in constructive settings.

One may take a different approach, which avoids the issues linked to the nonconstructive principles of AC or LEM. The idea at the base of this new approach is that $\mathcal{C}T$ can be regarded as the *free complete (pre)metric space generated by* T, the operation of the structure (with respect to which we consider its *free* nature) being a "take the limit" map, i.e. a function $\lim : Approx(\mathcal{C}T) \to \mathcal{C}T$. In fact, the construction of the completion can be performed by means of higher inductive types, which can contain the limit operation in the very definition of the type itself. It is indeed this free structure, together with the need to equip it with a (pre)metric, that leads to the definition of the completion $\mathcal{C}T$ as an **higher inductive-inductive type** (HITT), that is, a type together with a dependent family over the type whose definitions are simultaneous and of an inductive nature, with constructors involving also identity types.

After having defined the Cauchy-completion, our main aim is to prove the following theorem, which justifies the construction carried out.

THEOREM 4.1. *Let* $(T, \approx^T)$ *be a premetric space, then its Cauchy completion, denoted as* $(\mathcal{C}T, \approx^{\mathcal{C}T})$*, is a Cauchy complete premetric space. In addition, if* T *is already Cauchy complete, then* $\mathcal{C}T = T$

In order to prove the above theorem we do not assume any nonconstructive principle. This puts Homotopy Type Theory in an in-between level with respect to constructive and classical logic: while the former, that is at the base of type theory, cannot prove Cauchy-completeness of reals, as shown in [9], HoTT can; moreover, HoTT can consistently be reduced to classical logic by adding the LEM to its axioms, and if we do so the traditional construction of Cauchy reals becomes equivalent to the one exposed in this chapter.

1. Cauchy completion of premetric spaces

Definition 4.2 (Premetric space). A premetric space is a type T togethert with a parametric mere relation $\_ \approx \_ : \mathbb{Q}^+ \to T \to T \to \mathrm{Prop}$, sometimes denoted as $\approx^T$, verifying the following properties:

- *reflexivity*: $\prod_{\epsilon:\mathbb{Q}_+} \prod_{x:T} x \approx_\epsilon x$;
- *symmetry*: $\prod_{\epsilon:\mathbb{Q}_+} \prod_{x,y:T} x \approx_\epsilon y \to y \approx_\epsilon x$;
- *separatedness*: $\prod_{x,y:T} \left(\prod_{\epsilon:\mathbb{Q}_+} x \approx_\epsilon y\right) \to x =_T y$;
- *triangularity*: $\prod_{x,y,z:T} \prod_{\epsilon,\delta:\mathbb{Q}_+} (x \approx_\epsilon y \to y \approx_\delta z \to x \approx_{\epsilon+\delta} z)$;
- *roundedness*: $\prod_{\epsilon:\mathbb{Q}_+} \prod_{x,y:T} x \approx_\epsilon y \leftrightarrow \left\| \sum_{\delta:\mathbb{Q}_+} \delta < \epsilon \times x \approx_\delta y \right\|$.

$\approx$ is called the closeness relation of T, with $x \approx_\epsilon y$ read as "x and y are ϵ-close", or "the distance between x and y is less than ϵ".

From now on, T will always stand for a premetric space, with the closeness relation denoted by $\approx$.

Observation 4.3. Separatedness implies that a premetric space is a set.

Observation 4.4. $\mathbb{Q}$ is a premetric space, with the closeness relation given by the absolute value:

$$\approx: \mathbb{Q}_+ \to \mathbb{Q} \to \mathbb{Q} \to \mathrm{Prop}, \qquad x \approx_\epsilon y :\equiv |x - y| < \epsilon$$

The typical Cauchy completion of a metric space uses the notion of *Cauchy sequence*; we define a Cauchy-sequence in T as a map $x : \mathbb{N} \to T$ such that

$$\prod_{\epsilon:\mathbb{Q}_+} \sum_{N:\mathbb{N}} \prod_{n,k \geq N} x_n \approx_\epsilon x_k$$

The inner existential is not truncated, and this allows to compute rates of convergence by extracting the *modulus of convergence*, a function $M : \mathbb{Q}_+ \to \mathbb{N}$ such that $\prod_{\epsilon:\mathbb{Q}_+} \prod_{m,k \geq M(\epsilon)} x_m \approx_\epsilon x_k$. It is convenient, instead, to work with *Cauchy approximations*, an equivalent notion that allows to formally delete the dependence on $\mathbb{N}$ and carries the same information about the mutual distance between the elements as the original Cauchy condition, that is, the same information about its limit.

Definition 4.5 (Cauchy Approximation). A map $x : \mathbb{Q}_+ \to T$ is a Cauchy approximation if it satisfies the condition

$$\mathrm{isCApp}(x) :\equiv \prod_{\delta,\epsilon:\mathbb{Q}_+} x_\delta \approx_{\epsilon+\delta} x_\delta$$

The type of Cauchy approximation in T is defined as

$$\mathit{Approx}(T) :\equiv \sum_{x:\mathbb{Q}_+ \to T} \mathrm{isCApp}(x)$$

Definition 4.6 (Limit). Let $x : \mathbb{Q}_+ \to T$ be a Cauchy approximation. A term $l : T$ is said to be a limit of x if

$$\prod_{\epsilon,\delta:\mathbb{Q}_+} x_\epsilon \approx_{\epsilon+\delta} l$$

By separatedness of $\approx$, if the limit exists then it is unique. Let it be observed that isCApp : $(\mathbb{Q}_+ \to T) \to \mathcal{U}$ is a family of mere propositions, so, for the characterization of the identity type of Σ-types, an element $u : \mathit{Approx}(T)$ is completely determined by its first projection $\mathrm{pr}_1(u) : \mathbb{Q}_+ \to T$; by an abuse of notation, they may be identified (when there is no risk of ambiguity). With these definitions in hand we can formally state what it means for a premetric space to be *Cauchy-complete.*

Definition 4.7 (Cauchy-completeness). A premetric space T is Cauchy-complete if every Cauchy approximation in T admits a limit. Equivalently, T is Cauchy-complete if there exists a function

$$\lim : \mathit{Approx}(T) \to T$$

that maps every Cauchy-approximation into its limit.

Definition 4.8 (Cauchy-completion). Let $T : \mathcal{U}$ be a premetric space with $\approx : T \to T \to \mathbb{Q}_+ \to \mathrm{Prop}$ its closeness relation. The Cauchy-completion of T is the higher inductive-inductive type $(\mathcal{C}T, \approx)$, where $\mathcal{C}T : \mathcal{U}$ is a type and $\approx : \mathcal{C}T \to \mathcal{C}T \to \mathbb{Q}_+ \to \mathrm{Prop}$ a family of mere relations generated by the following constructors:

$$\mathcal{C}T - \text{constructors} : \begin{cases} \eta : T \to \mathcal{C}T \\ \\ \lim : \mathit{Approx}(\mathcal{C}T) \to \mathcal{C}T \\ \\ \dfrac{u,v{:}\mathcal{C}T \qquad \prod_{\epsilon:\mathbb{Q}_+} u \approx_\epsilon v}{\mathrm{eq}_{\mathcal{C}\mathrm{T}}(u,v) : u =_{\mathcal{C}T} v} \ \textit{(separatedness)} \end{cases}$$

$$\approx -\text{constructors} : \begin{cases} \dfrac{q,r{:}T \qquad \epsilon{:}\mathbb{Q}_+ \qquad q \approx_\epsilon r}{\eta q \approx_\epsilon \eta r} \\ \\ \dfrac{q{:}T \qquad x{:}\mathit{Approx}(\mathcal{C}T) \qquad \epsilon,\delta{:}\mathbb{Q}_+ \qquad \eta q \approx_{\epsilon-\delta} x_\delta}{\eta q \approx_\epsilon \lim(x)} \\ \\ \dfrac{r{:}T \qquad y{:}\mathit{Approx}(\mathcal{C}T) \qquad \epsilon,\delta{:}\mathbb{Q}_+ \qquad y_\delta \approx_{\epsilon-\delta} \eta r}{\lim(y) \approx_\epsilon \eta r} \\ \\ \dfrac{u,v{:}\mathcal{C}T \qquad \epsilon{:}\mathbb{Q}_+ \qquad h,k : u \approx_\epsilon v}{h = k} \end{cases}$$

We observe that the action of quotienting is embodied in the third $\mathcal{C}T$-constructor for $\mathcal{C}T$, namely the one expressing the property of *separetedness*, for it prescribes the identification of terms that are arbitrarily ϵ-close.

Let it be noted the proof-irrelevance of the previous definition: if $x : \mathit{Approx}(\mathcal{C}T)$, the element $\lim(x) : \mathcal{C}T$ actually depends not only on the Cauchy approximation x, but also on the proof $p : \mathrm{isCApp}(x)$, as well as the path $\mathrm{eq}_{\mathcal{C}\mathrm{T}}(u,v) : u =_{\mathcal{C}T} v$ depends from the witness $r : \prod_{\epsilon:\mathbb{Q}_+} u \approx_\epsilon v$, i.e. we have $\lim(x,p)$ and $\mathrm{eq}_{\mathcal{C}\mathrm{T}}(u,v,r)$; the omission is justified from the fact that we have isProp(isCApp(x)) and isProp(: $\prod_{\epsilon:\mathbb{Q}_+} u \approx_\epsilon v$), i.e. the elements left out are terms of mere propositions. By the same token, the last constructor for $\approx$ justifies not having given names to the constructors of $\approx$.

2. Induction principle

Analogously to lower and higher inductive types, the induction principle for an higher inductive-inductive type as $(\mathcal{C}T, \approx)$ should mirror its initiality in an appropriate category; in particular, we expect that in order to construct (dependent) functions out of the Cauchy completion of a premetric space into arbitrary types it is necessary to equip the latters of initial data that "lay over" those that define $(\mathcal{C}T, \approx)$. Dealing with an higher inductive-inductive type instead of the simpler higher inductive types, namely with a construction that defines at the same time a type and a family of mere relations over that type, the form of the induction principle will be at least as complex as the definition. In particular, the requirement of initiality justifies the type families $(\mathcal{C}T, \approx)$-induction principle must apply to and the mutual dependencies that they have to present.

$\mathcal{C}$T-induction principle applies to any pair of type families:

$$A : \mathcal{C}T \to \mathcal{U}$$
$$B : \prod_{x,y:\mathcal{C}T} A(x) \to A(y) \to \prod_{\epsilon:\mathbb{Q}_+} (x \approx_\epsilon y) \to \mathcal{U}$$

It may be written $(x, a) \frown^\zeta_\epsilon (y, b)$ for $B(x, y, a, b, \epsilon, \zeta)$; when such a $\zeta : x \approx_\epsilon y$ exists it is unique, so sometimes ζ will be omitted and we may write just $(x, a) \frown_\epsilon (y, b)$, remembering that the relation is defined only when $x \approx_\epsilon y$. In addition, given $x : \mathit{Approx}(\mathcal{C}T)$, we call an element $a : \prod_{\epsilon:\mathbb{Q}_+} A(x_\epsilon)$ such that

$$\prod_{\epsilon,\delta:\mathbb{Q}_+} (x_\delta, a_\delta) \frown (x_{\epsilon,a_\epsilon})$$

a **dependent Cauchy approximation** over x (with respect to $\frown$) . Hence, given families A and $\frown$ as above, the hypotheses of the induction principle for $(\mathcal{C}T, \approx)$ consist of the following data, one for each constructor of the Cauchy-completion:

- for any $q : T$, an element $f_q : A(\eta q)$;
- for any $x : \mathit{Approx}(\mathcal{C}T)$ and for any dependent Cauchy approximation a over x, an element $f_{x,a} : A(\lim(x))$;
- for any $u, v : \mathcal{C}T$ such that $\prod_{\epsilon:\mathbb{Q}_+} u \approx_\epsilon v$, and for any $a : A(u)$, $b : A(v)$ such that $\prod_{\epsilon:\mathbb{Q}_+} (u, a) \frown_\epsilon (v, b)$, an element of $a =^A_{\mathrm{eq}_{\mathcal{C}T}(u,v)} b$;
- for any $q, r : T$, $\epsilon : \mathbb{Q}_+$, if $q \approx_\epsilon r$ then $(\eta q, f_q) \frown_\epsilon (\eta r, f_r)$;
- for any $q : T$, $y : \mathit{Approx}(\mathcal{C}T)$, $\epsilon, \delta : Q_+$, b dependent Cauchy approximation over y, if $\eta q \approx_{\epsilon-\delta} y_\delta$ then
$$(\eta q, f_q) \frown_{\epsilon-\delta} (y_\delta, b_\delta) \implies (\eta q, f_q) \frown_\epsilon (\lim(y), f_{y,b})$$
- for any $r : T$, $x : \mathit{Approx}(\mathcal{C}T)$, $\epsilon, \delta : Q_+$, a dependent Cauchy approximation over x, if $x_\delta \approx_{\epsilon-\delta} \eta r$, then
$$(x_\delta, a_\delta) \frown_{\epsilon-\delta} (\eta r, f_r) \implies (\lim(x), f_{x,a}) \frown_\epsilon (\eta r, f_r)$$
- for any $x, y : \mathit{Approx}(\mathcal{C}T)$, a, b dependent Cauchy approximations over x, y respectively, for any $\epsilon, \delta, \kappa : \mathbb{Q}_+$ if $x_\delta \approx_{\epsilon-\delta-\kappa} y_\kappa$ then
$$(x_\delta, a_\delta) \frown_{\epsilon-\delta-\kappa} (y_\kappa, b_\kappa) \implies (\lim(x), f_{x,a}) \frown_\epsilon (\lim(y), f_{y,b})$$
- for any $\epsilon : \mathbb{Q}_+$, $u, v : \mathcal{C}T$, $\zeta, \xi : u \approx_\epsilon v$ and $a : A(u)$, $b : A(v)$, for any $p : (x, a) \frown^\zeta_\epsilon (y, b)$ and $q : (x, a) \frown^\xi_\epsilon (y, b)$, an element of $p =_{\zeta=\xi} q$. Notice that this is equivalent to asking that $\frown$ takes values in mere propositions.

Under these hypotheses, we deduce functions

$$f : \prod_{x:\mathcal{C}T} A(x)$$

$$g : \prod_{x,y:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} \prod_{\zeta:x\approx_\epsilon y} (x, f(x)) \frown_\epsilon^\zeta (y, f(y))$$

which compute on constructors as

$$f(\eta q) :\equiv f_q$$

$$f(\lim(x)) :\equiv f_{x,(f,g)[x]}$$

where $(f, g)[x]$ denotes the dependent Cauchy approximation over x obtained by the application of f and g to the Cauchy approximation x.

$(\mathcal{C}T, \approx)$-induction principle specializes to two distinct induction principles for $\mathcal{C}T$ and $\approx$ by choosing special families A, B, and it reduces to a $(\mathcal{C}T, \approx)$-recursion principle in the case of a non-dependent type $A : \mathcal{U}$. The latter, in particular, will be of fundamental importance to characterize *recursively* the family of relations $\approx$ over $\mathcal{C}T$ and consequently prove that $\mathcal{C}T$ is a Cauchy-complete premetric space.

Let it firstly be derived the principle of **$\mathcal{C}T$-induction**. The constructors for $\mathcal{C}T$ suggest that the completion space $\mathcal{C}T$ is freely generated by the elements of the form ηt, for a $t : T$, or $\lim(x)$, for a $x : \mathit{Approx}(\mathcal{C}T)$; for any dependent family $A : \mathcal{C}T \to \mathcal{U}$ one would like to construct $f : \prod_{x:\mathcal{C}T} A(x)$ by defining it only on the "generators". This is indeed possible, as long as A does not have too much structure (it will turn out that it needs to be a family of mere proposition): if one defines $\frown$ constantly equal to 1, i.e. $\frown :\equiv 1$, the requirements of the induction principle on $\frown$ become trivial, and in order to construct the desired f it is necessary to give:

- for any $q : T$ an element $f_q : A(\eta q)$;
- for any $x : \mathit{Approx}(\mathcal{C}T)$ and any $a : \prod_{\epsilon:\mathbb{Q}_+} A(x_\epsilon)$, an element $f_{x,a} : A(\lim(x))$;
- for any $u, v : \mathcal{C}T$ such that $\prod_{\epsilon:\mathbb{Q}_+} u \approx_\epsilon v$, for any $a : A(u)$ and $b : A(v)$, an element of $u =^A_{\mathrm{eq}_{\mathcal{C}T}(u,v)} b$

Under these hypotheses, we obtain $f : \prod_{x:\mathcal{C}T} A(x)$ that computes on $\mathcal{C}T$-constructors as $f(\eta q) :\equiv f_q$, $f(\lim(x)) :\equiv f_{x,f(x)}$.

Observation 4.9. The third condition is satisfied if and only if $A : \mathcal{C}T \to \mathcal{U}$ is a family of mere proposition. To see this, suppose it satisfied by a type family $A : \mathcal{C}T \to \mathcal{U}$ and fix $u : \mathcal{C}T$. Taking $v \equiv u$ and $a : A(u)$, for hypothesis we have that $a =^A_{\mathrm{eq}_{\mathcal{C}T}(u,u)} a$ and that for any other $b : A(u)$ $a =^A_{\mathrm{eq}_{\mathcal{C}T}(u,u)} b$. From the definition of the dependent path type, we conclude that $a = b$, that is, $A(u)$ is a mere prposition. This means that $\mathcal{C}T$-induction allows to prove properties of $\mathcal{C}T$, and that a property of $\mathcal{C}T$ is valid for any element if and only if it holds for T and for limits of Cauchy approximations in $\mathcal{C}T$.

We can also recover the priciple of **$\approx$-induction**, which shows that the family $\approx$: $\mathbb{Q}_+ \to \mathcal{C}T \to \mathcal{C}T \to \mathrm{Prop}$ is inductively generated by *its* constructors. Assume we want to prove a certain property $B : \prod_{x,y:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} (x \approx_\epsilon y) \to \mathcal{U}$ of elements of $\mathcal{C}T$ ϵ-close, then we have to construct an element of $\prod_{x,y:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} \prod_{\zeta:x\approx_\epsilon y} B(x, y, \epsilon)$. This can be obtained by $(\mathcal{C}T, \approx)$-induction by taking A constantly equal to $\mathbf{1}$: in this way, $\frown :\equiv \prod_{x,y:\mathcal{C}T} 1 \to 1 \to \prod_{\epsilon:\mathbb{Q}_+} B(x,y,\epsilon)$ becomes a family indexed by $\mathbb{Q}_+$ of relations between elements ϵ-close; we may write $u \frown_\epsilon v$ instead $(u,*) \frown_\epsilon (v,*)$. The required data to apply induction reduce to the following:

- for any $p, q : T$, $\epsilon : \mathbb{Q}_+$ if $p \approx_\epsilon q$ then $\eta p \frown_\epsilon \eta q$;
- for any $q : T$, $y : Approx(\mathcal{C}T)$ $\epsilon, \delta : \mathbb{Q}_+$, if $\eta q \approx_{\epsilon-\delta} y_\delta$ and $\eta q \frown_{\epsilon-\delta} y_\delta$ then $\eta q \frown_\epsilon \lim(y)$;
- for any $r : T$, $x : Approx(\mathcal{C}T)$ $\epsilon, \delta : \mathbb{Q}_+$, if $x_\delta \approx_{\epsilon-\delta} \eta r$ and $x_\delta \frown_{\epsilon-\delta} \eta r$ then $\lim(x) \frown_\epsilon \eta r$;
- for any $x, y : Approx(\mathcal{C}T)$, $\epsilon, \delta, \kappa : \mathbb{Q}_+$, if $x_\delta \approx_{\epsilon-\delta-\kappa} y_\kappa$ and $x_\delta \frown_{\epsilon-\delta-\kappa} y_\kappa$ then $\lim(x) \frown_\epsilon \lim(y)$.

Under these conditions one concludes that

$$\prod_{u,v:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} (u \approx_\epsilon v) \to (u \frown_\epsilon v)$$

We can finally state the principle of $(\mathcal{C}\mathbf{T}, \approx)$**-recursion**, the non-dependent version of the induction principle that prescribes how to construct functions out of $\mathcal{C}T$ into non-dependent types $A : \mathcal{U}$. Given $A : \mathcal{U}$, the sole $\mathcal{C}T$-recursion is not enough to construct a map $f : \mathcal{C}T \to A$, since such a function not only has to be defined on ηT and limits in $\lim Approx(\mathcal{C}T)$, but also needs to respect the separatedness relation induced by $\approx$, for f, as any function, must respect equality, i.e. for any $u, v : \mathcal{C}T$ such that $\prod_{\epsilon:\mathbb{Q}_+} u \approx_\epsilon v$, $f(u) = f(v)$. The solution is given by joining $\mathcal{C}T$ and $\approx$-recursion principles and defining, at the same time, f and a family of relations on A that specifies an arbitrary way in which f acts on ϵ-close elements of $\mathcal{C}T$, which we can then prove to be the case by a simultaneous induction with the definition of f. Therefore, suppose given $A : \mathcal{U}$; to construct a function $f : \mathcal{C}T \to A$ by $(\mathcal{C}\mathbf{T}, \approx)$**-recursion** it is necessary to:

- define a family of relations on A $\frown : A \to A \to \mathbb{Q}_+ \to \mathcal{U}$, with the admissibility condition of separatedness: for any $a, b : A$ it must be inhabited the type$(\prod_{\epsilon:\mathbb{Q}_+} a \frown_\epsilon b) \Rightarrow (a = b)$;
- construct $f(\eta q) : A$ for any $q : T$;
- construct $f(\lim(x)) : A$ for any $x : Approx(\mathcal{C}T)$, assuming as inductive hypothesis that $f(x_\epsilon)$ has been defined for any $\epsilon : \mathbb{Q}_+$ and that $\lambda\epsilon.f(x_{epsilon}) : \mathbb{Q}_+ \to A$ is Cauchy approximation with respect to $\frown$.

To conclude we still have to verify other four conditions:

- for any $q, r : T$, $\epsilon : \mathbb{Q}_+$, if $q \approx_\epsilon r$ then $f(\eta q) \frown_\epsilon f(\eta r)$;
- "right continuity": for any $q : T$ and any $y : Approx(\mathcal{C}T)$, if there exists $\delta : \mathbb{Q}_+$ such that $\eta q \approx_{\epsilon-\delta} y_\delta$ and $f(\eta q) \frown_{\epsilon-\delta} f(y_\delta)$, assuming $\lambda\phi.f(y_\phi)$ a Cauchy approximation with respect to $\frown$, then $f(\eta q) \frown_\epsilon f(\lim(y))$;
- "left continuity": for any $r : T$ and any $x : Approx(\mathcal{C}T)$,if there exists $\delta : \mathbb{Q}_+$ ssuch that $x_\delta \approx_{\epsilon-\delta} f(\eta r)$ and $f(x_\delta) \frown_{\epsilon-\delta} f(\eta r)$, assuming $\lambda\phi.f(x_\phi)$ a Cauchy approximation with respect to $\frown$, then $f(\lim(x)) \frown_\epsilon f(\eta r)$;
- for any $x, y : Approx(\mathcal{C}T)$, if there exist $\delta, \phi : \mathbb{Q}_+$ such that $x_\delta \approx_{\epsilon-\delta-\phi} y_\phi$ and $f(x_\delta) \frown_{\epsilon-\delta-\phi} f(y_\phi)$, assuming $\lambda\theta.f(x_\theta)$ and $\lambda\theta.f(y_\theta)$ Cauchy approximations with respect to $\frown$, then $f(\lim(x)) \frown_\epsilon f(\lim(y))$.

Under these hypotheses, the $(\mathcal{C}T, \approx)$-recursion principle gives a function

$$f : \mathcal{C}T \to A$$

with $f(\eta q)$ and $f(\lim(x))$ defined by judgmental equalities (the computational rule). In addiction, from $(\mathcal{C}T, \approx)$-induction we obtain that

$$\prod_{u,v:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} (u \approx_\epsilon v) \to (f(u) \frown_\epsilon f(v))$$

which means that actually the premetric of $\mathcal{C}T$ transports on A through f, as it is required by its very construction.

Summing up, specifications of the $(\mathcal{C}T, \approx)$-induction make possible to prove mere properties of elements of $\mathcal{C}T$ ($\mathcal{C}T$-induction), properties of ϵ-close elements of $\mathcal{C}T$ ($\approx$-induction) and to construct functions out of $\mathcal{C}T$ into non-dependent types, as long as they respect the completion's structure (($\mathcal{C}T, \approx$)-recursion). Our main aim is to prove the correcteness of the Cauchy completion. These principles will be everything needed in order to show the above results.

3. Properties of the completion

We want to prove that the relation $\approx : \mathcal{C}T \to \mathcal{C}T \to \mathbb{Q}_+ \to \mathrm{Prop}$ is a premetric on $\mathcal{C}T$ which extends the one on T. The main difficulty is proving that triangularity and roundedness apply to $\approx$, and in order to do so we characterize recursively $\approx$, so that it computes on constructors; we do this in Theorem 4.10. Reflexivity and symmetry are instead easy to prove, by $\mathcal{C}T$-induction and $\approx$-induction respectively. (Details can be found in [16] as well as in [4]). Finally, separatedness of $\approx$ is precisely the third $\mathcal{C}T$-constructor.

THEOREM 4.10. *There exists a family of mere relations*

$$B_{\_}(-,-) : \mathcal{C}T \to \mathcal{C}T \to \mathbb{Q}_+ \to \mathrm{Prop}$$

such that for any $q, r : T$, $x, y : Approx(\mathcal{C}T)$, $\epsilon : \mathbb{Q}_+$:

$$B_\epsilon(\eta q, \eta r) :\equiv q \approx_\epsilon r$$

$$B_\epsilon(\eta q, \lim(y)) :\equiv \left\| \sum_{\delta:\mathbb{Q}_+} B_{\epsilon-\delta}(\eta q, y_\delta) \right\|$$

$$B_\epsilon(\lim(x), \eta r) :\equiv \left\| \sum_{\delta:\mathbb{Q}_+} B_{\epsilon-\delta}(x_\delta, \eta r) \right\|$$

$$B_\epsilon(\lim(x), \lim(y)) :\equiv \left\| \sum_{\delta,\xi:\mathbb{Q}_+} B_{\epsilon-\delta-\xi}(x_\delta, y_\xi) \right\|$$

In addition, for any $u, v, w : \mathcal{C}T$ the following types are inhabited:

$$B_\epsilon(u, v) \leftrightarrow \left\| \sum_{\theta:\mathbb{Q}_+} B_{\epsilon-\theta}(u, v) \right\|$$

$$B_\epsilon(u, v) \to (v \approx_\delta w) \to B_{\epsilon+\delta}(u, w)$$

$$(u \approx_\epsilon v) \to B_\delta(v, w) \to B_{\epsilon+\delta}(u, w)$$

Proof. Let us begin with an overview of the argument used to prove the theorem. We define $B_{\_}(-,-) : \mathcal{C}T \to \mathcal{C}T \to \mathbb{Q}_+ \to \text{Prop}$ by double $(\mathcal{C}T, \approx)$-recursion. Firstly, we choose as codomain a certain subset Balls of $\mathcal{C}T \to \mathbb{Q}_+ \to \text{Prop}$ that captures the properties of roundedness and triangularity on the second component, and then we apply $(\mathcal{C}T, \approx)$-recursion to define $B_{\_}(-,-) : \mathcal{C}T \to \text{Balls}$ on its first component. We give the definition of $B_{\_}(-,-) : \mathcal{C}T \to \text{Balls}$ on $\mathcal{C}T$ constructors by using a nested $(\mathcal{C}T, \approx)$-recursion, this time choosing as codomain a subset Upper of $\mathbb{Q}_+ \to \text{Prop}$ to incorporate the "missing side" of the property of triangularity. In order to apply the recursion, we endow the two subsets Balls and Upper with families of closeness relations $\smile$ and $\frown$ respectively, which mirror the action of B on elements ϵ-close.
We can now develop the details. First of all, we define the codomain Balls and the family $\smile$: Balls $\to$ Balls $\to \mathbb{Q}_+ \to$ Prop.

Definition 4.11 (Concentric balls). A family $B : \mathcal{C}T \to \mathbb{Q}_+ \to \text{Prop}$ is a concentric ball if it is inhabited the type $\text{isBall}(B) :\equiv \mathcal{R}(B) \times \mathcal{T}(B)$, where we call

$$\mathcal{R}(B) :\equiv \prod_{\epsilon:\mathbb{Q}_+} \prod_{y:\mathcal{C}T} B_\epsilon y \leftrightarrow \left\| \sum_{\delta:\mathbb{Q}_+} (\delta < \epsilon) \times B_\delta y \right\|$$

the ball roundedness property, and

$$\mathcal{T}(B) :\equiv \prod_{\epsilon,\phi:\mathbb{Q}_+} \prod_{y,z:\mathcal{C}T} y \approx_\epsilon z \to B_\delta y \to B_{\epsilon+\delta} z$$

the ball triangularity property. We define the type of concentric balls as

$$\text{Balls} :\equiv \sum_{B:\mathcal{C}T\to\mathbb{Q}_+\to\text{Prop}} \text{isBall}(B)$$

We endow Balls with a family of mere relations $\smile$: Balls $\to$ Balls $\to \mathbb{Q}_+ \to$ Prop, defined as

$$B^1 \smile_\epsilon B^2 :\equiv \prod_{y:\mathcal{C}T} \prod_{\delta:\mathbb{Q}_+} (B^1_\delta y \to B^2_{\epsilon+\delta}) \times (B^2_\delta y \to B^1_{\epsilon+\delta})$$

It is easy to check that $\mathcal{R}(B)$ and $\mathcal{T}(B)$ are mere propositions, and hence also $\text{isBall}(B)$. Therefore, we may identify an element u : Balls with its first component $\text{pr}_1(u) : \mathcal{C}T \to \mathbb{Q}_+ \to \text{Prop}$.

Claim 1. *The family* $\smile$: Balls $\to$ Balls $\to \mathbb{Q}_+ \to$ Prop *is a premetric on* Balls. *In particular, it is separated, and hence it satisfies the admissibility condition of* $(\mathcal{C}T, \approx)$*-recursion.*

Suppose claim 1 is proven; we have to define $B_{\_}(-,-) : \mathcal{C}T \to \text{Balls}$ on the remaining two $\mathcal{C}T$-constructors.

- for any $q : T$, an element $B_{\_}(\eta q, -)$: Balls;
- for any x : *Approx*$(\mathcal{C}T)$, an element $B_{\_}(\lim(x), -)$, inductively assuming to have defined, for any $\epsilon : \mathbb{Q}_+$, the element $B_{\_}(x_\epsilon, -)$: Balls and that $\lambda\epsilon.B_{\_}(x_\epsilon, -)$ is a Cauchy approximation with respect to $\smile$.

Recall that an element B : Balls is a family $\text{pr}_1(B) : \mathcal{C}T \to \mathbb{Q}_+ \to \text{Prop}$ together with a proof of $\text{isBall}(\text{pr}_1(B)))$. We define $\text{pr}_1(B)$ (which we identify with B) by a nested recursion with codomain Upper, a certain subtype of $\mathbb{Q}_+ \to$ Prop which captures roundedness and triangularity in the first component; we endow Upper with a family of mere relations $\frown$: Upper $\to$ Upper $\to \mathbb{Q}_+ \to$ Prop.

DEFINITION 4.12 (Upper cut). An upper cut is a predicate on $\mathbb{Q}_+$, i.e. $U : \mathbb{Q}_+ \to$ Prop, which is upward rounded, i.e. it is inhabited the type

$$\text{isUpper}(U) :\equiv \prod_{\epsilon:\mathbb{Q}_+} U_\epsilon \leftrightarrow \left\| \sum_{\delta:\mathbb{Q}_+} (\delta < \epsilon) \times U_\delta \right\|$$

The type of upper cuts is defined as

$$\text{Upper} :\equiv \sum_{U:\mathbb{Q}_+\to\text{Prop}} \text{isUpper}(U)$$

We endow Upper with a family of relations $\frown$: Upper $\to$ Upper $\to \mathbb{Q}_+ \to$ Prop as

$$U^1 \frown_\epsilon U^2 :\equiv \prod_{\delta:\mathbb{Q}_+} (U^1_\delta \to U^2_{\epsilon+\delta}) \times (U^2_\delta \to U^1_{\epsilon+\delta})$$

Observe that isUpper(U) is a mere proposition, and so again we identify an element u : Upper with $\text{pr}_1(u) : \mathbb{Q}_+ \to$ Prop.

CLAIM 2. *The family* $\frown$: Upper $\to$ Upper $\to \mathbb{Q}_+ \to$ Prop *is a premetric on* Upper. *In particular, it is separated, and hence it satisfies the admissibility condition of* $(\mathcal{C}T, \approx)$*-recursion.*

Assuming to have proven claim 2, we are ready to define the initial data required by the external recursion. Let $q : T$ be fixed and define $B_{\_}(\eta q, -) : \mathcal{C}T \to$ Upper by recursion:

- for any $r : T$ we set
$$B_{\_}(\eta q, \eta r) :\equiv \lambda\epsilon. q \approx_\epsilon r$$
- if $x : Approx(\mathcal{C}T)$ and we inductively assume to have defined $B_{\_}(\eta q, x_\epsilon)$: Upper for any $\epsilon : \mathbb{Q}_+$ and that they form a Cauchy approximation in Upper with respect to $\frown$, we define
$$B_{\_}(\eta q, \lim(x)) :\equiv \lambda\epsilon.\exists(\delta : \mathbb{Q}_+).B_{\epsilon-\delta}(\eta q, x_\delta)$$

The elements defined stay in Upper, namely they are rounded. In fact, roundedness of $B_{\_}(\eta q, \eta r)$ follows, by definition, from the same property of $\approx$ on T, while we show roundedness of $B_{\_}(\eta q, \lim(x))$ as follows. Suppose that for a certain $\epsilon : \mathbb{Q}_+$ we have $B_\epsilon(\eta q, \lim(x))$, then by definition there exists $\delta : \mathbb{Q}_+$ such that $B_{\epsilon-\delta}(\eta q, x_\delta)$: Upper,(of type Upper for inductive hypothesis), and so there exists $\xi : \mathbb{Q}_+$ such that $B_{\epsilon-\delta-\xi}(\eta q, x_\delta)$. Since $\epsilon - \xi < \epsilon$ we obtain the thesis. At this point, in order to apply $(\mathcal{C}T, \approx)$-recursion, we need to show that our definition of $B_{\_}(\eta q, -)$ satisfies the required conditions on $\frown$.

CLAIM 3. *For any* $q : T$*, the following facts hold.*

(i) For any $r, s : T$*,* $\epsilon : \mathbb{Q}_+$*, if* $r \approx_\epsilon s$ *then* $B_{\_}(\eta q, \eta r) \frown B_{\_}(\eta q, \eta s)$*;*
(ii) For any $r : T$ *and* $y : Approx(\mathcal{C}T)$*, if there exists* $\delta : \mathbb{Q}$ *such that* $\eta r \approx_{\epsilon-\delta} y_\delta$ *and* $B_{\_}(\eta q, \eta r) \frown_{\epsilon-\delta} B_{\_}(\eta q, y_\delta)$*, assuming* $\lambda\epsilon.B_{\_}(\eta q, y_\epsilon)$ *a Cauchy approximation with respect to* $\frown$*, then* $B_{\_}(\eta q, \eta r) \frown_\epsilon B_{\_}(\eta q, \lim(x))$*;*

(iii) *For any* $s : T$ *and* $x : Approx(\mathcal{C}T)$*, if there exists* $\delta : \mathbb{Q}_+$ *such that* $x_\delta \approx_{\epsilon-\delta} \eta s$ *and* $B_\_(\eta q, x_\delta) \frown_{\epsilon-\delta} B_\_(\eta q, \eta s)$*, assuming* $\lambda\epsilon.B_\_(\eta q, x_\epsilon)$ *a Cauchy approximation with respect to* $\frown$*, then* $B_\_(\eta q, \lim(x)) \frown_\epsilon B_\_(\eta q, \eta s)$*;*

(iv) *For any* $x, y : Approx(\mathcal{C}T)$*, if there exist* $\delta, \phi : \mathbb{Q}_+$ *such that* $x_\delta \approx_{\epsilon-\delta-\phi} y_\phi$ *and* $B_\_(\eta q, x_\delta) \frown_{\epsilon-\delta-\phi} B_\_(\eta q, y_\phi)$*, assuming* $\lambda\epsilon.B_\_(\eta q, x_\epsilon)$ *and* $\lambda\epsilon.B_\_(\eta q, y_\epsilon)$ *Cauchy approximations with respect to* $\frown$*, then* $B_\_(\eta q, \lim(x)) \frown_\epsilon B_\_(\eta q, \lim(y))$*.*

For what concerns the definition of $B_\_(\lim(x), -)$, by recursion we can assume to have already defined the elements $B_\_(x_\epsilon, -)$: Balls and that they form a Cauchy approximation in Balls with respect to $\smile$.

- for any $r : T$, we define

$$B_\_(\lim(x), \eta r) :\equiv \lambda\epsilon.\exists(\delta : \mathbb{Q}_+).B_{\epsilon-\delta}(x_\delta, \eta r)$$

- for any y : *Approx*($\mathcal{C}T$), we inductively assume defined for any $\epsilon : \mathbb{Q}_+$ $B_\_(\lim(x), y_\epsilon)$ and that $\lambda\epsilon.B_\_(\lim(x), y_\epsilon)$ forms a Cauchy approximation in Upper with respect to $\frown$ and we set

$$\lambda\epsilon.\_(\lim(x), \lim(y)) :\equiv \exists(\delta, \phi : \mathbb{Q}_+).B_{\epsilon-\delta-\phi}(x_\delta, y_\phi)$$

As before, in the first place we must verify that the objects just defined are of type Upper, i.e. that they have the property of roundedness, and then we show that the four conditions on $\frown$ hold. In both cases it will be essential the inductive hypothesis of the external recursion, namely the fact that for any $\delta : \mathbb{Q}_+$ we have $B_\_(x_\delta, -)$: Balls, which ensures the right triangularity property.

Claim 4. *For any* x, y : *Approx*($\mathcal{C}T$)*, and* $q : T$ *we have that* $B_\_(\lim(x), \lim(y))$ *and* $B_\_(\lim(x), \eta q)$ *are rounded.*

Claim 5. *For any* x : *Approx*($\mathcal{C}T$)*, the object* $B_\_(\lim(x), -)$ *satisfies the four conditions on* $\frown$ *required by* $(\mathcal{C}T, \approx)$*-recursion.*

If we assume the previous claims, then from the inner recursion we deduce elements $B_\_(\eta q, -), B_\_(\lim(x), -) : \mathcal{C}T \to \mathbb{Q}_+ \to \text{Prop}$. These objects are valid base cases for the external recursion, because they are concentric balls, i.e. we have that $\text{isBall}(B_\_(\eta q, -))$ and $\text{isBall}(B_\_(\lim(x), -))$ for any $q : T$ and any x : *Approx*($\mathcal{C}T$). In fact, roundedness follows by definition of Upper, while from the inner recursion we get that for any $w : \mathcal{C}T$

$$\prod_{u,v:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} (u \approx_\epsilon v) \to B_\_(w, u) \frown_\epsilon B_\_(w, v)$$

and as one can see by expanding out the definition of $\frown$, this means that

$$\prod_{\delta:\mathbb{Q}_+} (u \approx_\epsilon v) \to (B_\delta(w, u) \to B_{\epsilon+\delta}(w, v)) \times (B_\delta(w, v) \to B_{\epsilon+\delta}(w, u))$$

which is exactly the triangularity property required.
We can hence complete the external recursion, namely the one to define $B_\_(-, -)$ on the first component, by verifying the four required conditions on $\smile$. Given that $B_\_(u, -)$ computes on the constructors of $\mathcal{C}T$ if we fix $u : \mathcal{C}T$, we can inductively assume the second component to be a canonical element, namely a term of ηT or a limit point. In total, we have to check eight conditions, whose proofs are completely analogous to the previous ones and hence left to the readers. □

We now proceed to prove the claims assumed in the theorem. Proofs of the properties of a premetric

PROOF. (*claim 1*) To prove separatednedd of $\smile$ we proceed as follows. Let B^1, B^2 : Balls such that

$$\prod_{\epsilon:\mathbb{Q}_+} B^1 \smile_\epsilon B^2$$

We need to show that $B^1 = B^2$: by function extensionality this is equivalent to showing that for any $\epsilon : \mathbb{Q}_+$, $y : \mathcal{C}T$ $B^1_\epsilon y = B^2_\epsilon y$, and for Univalence it is sufficient to prove that

$$B^1_\epsilon y \leftrightarrow B^2_\epsilon y$$

By symmetry of the indexes we show just one direction. Suppose given $B^1_\epsilon y$. By ball roundedness it merely exists $\delta < \epsilon$ such that $B^1_\delta y$. Given that B^1 and B^2 are $(\epsilon - \delta)$-close,we have B^2_ϵ. □

PROOF. (*claim 2*) Completely analogous to the proof that $\smile$ is a premetric on Balls. □

PROOF. (*claim 3*) In the following four points we assume that $\epsilon, \phi : \mathbb{Q}_+$.

(i) Let $r, s : T$ such that $r \approx_\epsilon s$; assuming $B\_\phi(\eta q, \eta r)$ we have to show that $B\_\phi + \epsilon(\eta q, \eta s)$, but this follows from triangularity of $\approx$ in T and the definition given of $B\_(\eta q, \eta s)$. Similarly the other direction.

(ii) Let $r : T$ and $y : Approx(\mathcal{C}T)$, and let $\xi : \mathbb{Q}_+$ be such that $B_\xi(\eta q, \eta r)$, then by $B\_(\eta q, \eta r) \frown_{\epsilon-\delta} B\_(\eta q, y_\delta)$ we have $B_{\xi+\epsilon-\delta}(\eta q, y_\delta)$, hence by definition $B_{\xi+\epsilon}(\eta q, \lim(x))$. For the other implication, assume that $B_\xi(\eta q, \lim(y))$, by definition there exists $\kappa : \mathbb{Q}_+$ such that $B_{\xi-\kappa}(\eta q, y_\kappa)$. By inductive hypothesis in the codomain we have a Cauchy approximation, so $B_{\xi+\delta}(\eta q, y_\delta)$ and by $B\_(\eta q, \eta r) \frown_{\epsilon-\delta} B\_(\eta q, y_\delta)$ we get $B_{\xi+\epsilon}(\eta q, \eta r)$.

(iii) By symmetry of $\frown$ and $\approx$, the proof is analogous to the one of the previous point.

(iv) Let $x, y : Approx(\mathcal{C}T)$, our goal is to show that

$$B\_(\eta q, \lim(x)) \frown_\epsilon B\_(\eta q, \lim(y))$$

by means of inductive hypotheses. By symmetry we show just that if $B_\xi(\eta q, \lim(x))$ then $B_{\xi+\epsilon}(\eta q, \lim(y))$. By definition, from $B_\xi(\eta q, \lim(x))$ it follows the existence of a $\theta : \mathbb{Q}_+$ such that $B_{\xi-\theta}(\eta q, x_\theta)$. By definition of (dependent) Cauchy approximation we have $B_{\xi+\delta}(\eta q, x_\delta)$ and by $B\_(\eta q, x_\delta) \frown_{\epsilon-\delta-\phi} B\_(\eta q, y_\phi)$ we have $B_{\xi+\epsilon-\phi}(\eta q, y_\phi)$, namely $B_{\xi+\epsilon}(\eta q, \lim(y))$, as wanted.

□

PROOF. (*claim 4*) Suppose it exists an $\epsilon : \mathbb{Q}_+$ such that $B_\epsilon(\lim(x), \eta r)$, then by definition it means that there exists $\delta : \mathbb{Q}_+$ such that $B_{\epsilon-\delta}(x_\delta, \eta r)$. By inductive hypothesis, $B\_(x_\delta, \eta r)$: Upper, so by roundedness of cuts there exists $\xi : \mathbb{Q}_+$ so that $B_{\epsilon-\delta-\xi}(x_\delta, \eta q)$, hence by definition $B_{\epsilon-\xi}(\lim(x), \eta r)$.
If $B_\epsilon(\lim(x), \lim(y))$, then by definition there exist δ, ϕ such that $B_{\epsilon-\delta-\phi}(x_\delta, y_\phi)$, in Upper for inductive hypothesis. By roundedness of cuts there exists $\xi : \mathbb{Q}_+$ such that $B_{\epsilon-\delta-\phi-\xi}(x_\delta, y_\phi)$. Given that $y_\phi \approx_{\phi+\frac{\xi}{3}} y_{\frac{\xi}{3}}$, by triangularity of $B\_(x_\kappa, -)$,

in Balls for any κ by inductive hypothesis, we obtain $B_{\epsilon-\delta-\frac{2}{3}\xi}(x_\delta, y_{\frac{\xi}{3}})$, and so the thesis, given that $\epsilon - \delta - \frac{2}{3}\xi = (\epsilon - \frac{\xi}{3}) - \frac{\xi}{3} - \delta$. □

PROOF. *claim 5*

(i) Assume $r, s : T$ such that $r \approx_\epsilon s$, we show that if $B_\phi(\lim(x), \eta r)$ then $B_{\phi+\epsilon}(\lim(x), \eta s)$, since the other direction is symmetric. If $B_\phi(\lim(x), \eta r)$, there exists $\delta : \mathbb{Q}_+$ such that $B_{\phi-\delta}(x_\delta, \eta r)$: by ball triangularity $B_{\phi-\delta+\phi}(\lim(x), \eta s)$, i.e. $B_{\phi+\epsilon}(\lim(x), \eta s)$.

(ii) Let $r : T$, $y : \mathit{approx}(\mathcal{C}T)$ such that $\eta r \approx_{\epsilon-\delta} y_\delta$. If $B_\phi(\lim(x), \eta r)$, then $B_{\phi-\theta}(x_\theta, \eta r)$ for some $\theta : \mathbb{Q}_+$, hence by triangularity $B_{\phi-\theta+\epsilon-\delta}(x_\theta, y_\delta)$, thesis. Conversely, if $B_\phi(\lim(x), \lim(y))$, then there exist γ, ψ such that $B_{\phi-\gamma-\psi}(x_\gamma, y_\psi)$. Having a Cauchy approximation, by triangularity $B_{\phi-\gamma+\delta}(x_\gamma, y_\delta)$ and so by inductive hypothesis and triangularity $B_\phi(\lim(x), \eta r)$. We notice that no other inductive hypotheses apart that of $\eta r \approx_{\epsilon-\delta} y_\delta$ have been necessary.

(iii) Analogous to the previous point, by symmetry of $\approx$ and $\frown$.

(iv) To show

$$B_{\_}(\lim(x), \lim(y)) \frown_\epsilon B_{\_}(\lim(x), \lim(z))$$

it is sufficient to prove one implication. Hence, assume $B_\phi(\lim(x), \lim(y))$, namely that there exist $\theta, \zeta : \mathbb{Q}_+$ such that $B_{\phi-\theta-\zeta}(x_\theta, y_\zeta)$. By triangularity $B_{\phi-\theta+\delta}(x_\theta, y_\delta)$, and again by triangularity $B_{\phi-\theta+\epsilon-\xi}(x_\theta, z_\xi)$, as wanted.

□

We can finally characterize the family $\approx : \mathcal{C}T \to \mathcal{C}T \to \mathbb{Q}_+ \to \mathrm{Prop}$ by means of the family just defined.

THEOREM 4.13.

$$\prod_{u,v:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} u \approx_\epsilon v = B_\epsilon(u, v)$$

PROOF. Given that both $u \approx_\epsilon v$ and $B_\epsilon(u, v)$ are mere propositions, by univalence it suffices to show they are logically equivalent, i.e.$u \approx_\epsilon v \leftrightarrow B_\epsilon(u, v)$. For both directions, it will be essential the recursive definition of $B_{\_}(-,-)$, namely how it computes on $\mathcal{C}T$-constructors.

$\to$) By $\approx$-induction on the family$C(u, v, \epsilon) :\equiv B_\epsilon(u, v)$. If $u \equiv \eta q$ and $v \equiv \eta r$ for some $q, r : T$, then $q \approx_\epsilon r$ and $B_\epsilon(\eta q, \eta r)$ are judgmentally equal terms. If $u \equiv \eta q$ and $v \equiv \lim(x)$ for some $q : T$, $x : \mathit{Approx}(\mathcal{C}T)$, from the hypotheses $\eta q \approx_{\epsilon-\delta} x_\delta$ and $B_{\epsilon-\delta}(\eta q, x_\delta)$ we deduce $B_\epsilon(\eta q, \lim(x))$; the same happens if u is a limit and v a term of type ηT. Finally, if both u, v are limits, i.e. $u \equiv \lim(x)$, $v \equiv \lim(y)$ for some $x, y : \mathit{Approx}(\mathcal{C}T)$, again the hypotheses $x_\delta \approx_{\epsilon-\delta-\kappa} y_\kappa$ and $B_{\epsilon-\delta-\kappa}(x_\delta, y_\kappa)$ lead by definition to $B_\epsilon(\lim(x), \lim(y))$.

$\leftarrow$) By $\mathcal{C}T$-induction on the family $A : \mathcal{C}T \to \mathcal{U}$, defined as

$$A(u) :\equiv \prod_{v:\mathcal{C}T} \prod_{\epsilon:\mathbb{Q}_+} B_\epsilon(u, v) \to u \approx_\epsilon v$$

In order to provide terms $f_q : A(\eta q)$ and $f_{\lim(x)} : A(\lim(x))$, we proceed with a second $\mathcal{C}T$-induction on $A(\eta q)$ and $A(\lim(x))$ respectively. Let $q : T$ be fixed, we need to define f_q on ηr, $r : T$ and on $\lim(x)$, $x : Approx(\mathcal{C}T)$, assuming all the inductive hypotheses allowed. But $B_{\_}(-,-)$ computes on constructors, hence

$$B_\epsilon(\eta q, \eta r) \equiv q \approx_\epsilon r$$

and by the first $\approx$-constructor we get $\eta q \approx_\epsilon \eta r$, while

$$B_\epsilon(\eta q, \lim(x)) \equiv \exists(\delta : \mathbb{Q}_+).B_{\epsilon-\delta}(\eta q, x_\delta)$$

By induction, we have $B_{\epsilon-\delta}(\eta q, x_\delta) \to \eta q \approx_{\epsilon-\delta} x_\delta$, and so by the second $\approx$-constructor we obtain $\eta q \approx_\epsilon \lim(x)$. Similarly, let $x : Approx(\mathcal{C}T)$ be fixed, we need to define $f_{\lim(x)} : A(\lim(x))$. Let it be observed that

$$B_\epsilon(\lim(x), \eta r) \equiv \exists(\delta : \mathbb{Q}_+).B_{\epsilon-\delta}(x_\delta, \eta r)$$

which, by the external induction, implies that $x_\delta \approx_{\epsilon-\delta} \eta r$ and hence $\lim(x) \approx_\epsilon \eta r$ (by the third $\approx$-constructor). Finally,

$$B(\lim(x), \lim(y)) \equiv \exists(\delta, \kappa).B_{\epsilon-\delta-\kappa}(x_\delta, y_\kappa)$$

by induction it is inhabited the type $x_\delta \approx_{\epsilon-\delta-\kappa} y_\kappa$ and so it $\lim(x) \approx_\epsilon \lim(y)$ by the fourth $\approx$-constructor. □

Observation 4.14. Univalence has been essential to obtain a propositional equality between the two types instead of a weaker equivalence. Indeed, Univalence Axiom is also necessary to show an equivalence between them, since univalence was needed to prove separatedness of the closeness relations defined on Balls and Upper.

Corollary 4.15.

$$\eta q \approx_\epsilon^{\mathcal{C}T} \eta r = q \approx_\epsilon^T r$$
$$\eta q \approx_\epsilon \lim(y) = \exists(\delta : \mathbb{Q}_+).\eta q \approx_{\epsilon-\delta} y_\delta$$
$$\lim(x) \approx_\epsilon \eta r = \exists(\delta : \mathbb{Q}_+).x_\delta \approx_{\epsilon-\delta} \eta r$$
$$\lim(x) \approx_\epsilon \lim(y) = \exists(\delta, \kappa).x_\delta \approx_{\epsilon-\delta-\kappa} y_\kappa$$

Corollary 4.16. *Roundedness and triangularity apply to $\approx$: for any $u, v, w : \mathcal{C}T$*

$$u \approx_\epsilon v = \exists(\theta : \mathbb{Q}_+).u \approx_{\epsilon-\theta} v$$
$$u \approx_\epsilon v \to v \approx_\delta w \to u \approx_{\epsilon+\delta} w$$

Now that we have shown that $(\mathcal{C}T, \approx)$ is indeed a premetric space, we want to prove that the completion is also Cauchy complete. Let it be noted that this is not an idle question: even though the Cauchy completion has the lim map, *a priori* it needs not to be Cauchy complete, since the elements $x_\epsilon : \mathcal{C}T$ of a Cauchy approximation might not be the canonical ηq or $\lim(y)$; in other words, not necessarily the lim map computes on constructors.

Theorem 4.17. *$\mathcal{C}T$ is Cauchy complete. In particular, for any $x : Approx(\mathcal{C}T)$, $\lim(x)$ is precisely the limit of x.*

Proof. It is sufficient to show that

$$\prod_{u:\mathcal{C}T} \prod_{x:Approx(\mathcal{C}T)} \prod_{\epsilon,\delta:\mathbb{Q}_+} u \approx_\epsilon x_\delta \to u \approx_{\epsilon+\delta} \lim(x)$$

Indeed, if the statement above holds, then from $x_\epsilon \approx_{\epsilon+\frac{\delta}{2}} x_{\frac{\delta}{2}}$ it follows that $x_\epsilon \approx_{\epsilon+\delta} \lim(x)$, hence for any $\epsilon, \delta : \mathbb{Q}_+$, $x_\epsilon \approx_{\epsilon+\delta} \lim(x)$, q.e.d. The claim is proved by $\mathcal{C}T$-induction. □

We can summarize the results obtained so far in the next theorem.

THEOREM 4.18. *Let $(T, \approx^T)$ be a premetric space. Then its Cauchy-completion $(\mathcal{C}T, \approx)$ is a Cauchy-complete premetric space, whose closeness relation extends the one on T and it is Cauch.*

4. Idempotence of the completion

Another essential property it would be desirable to apply to the completion operator is idempotence: completing a premetric space T already Cauchy-complete leads to a type $\mathcal{C}T$ equal to T. That is indeed the case, and it follows from properties of *continuous* functions on $\mathcal{C}T$, that are characterized based only on their behaviour on the base elements ηq, and *Lipschitz-functions*, which can be lifted from maps between premetric spaces to functions between their completions.

DEFINITION 4.19 (Lipschitz function). Let A, B be two premetric spaces. A map $f : A \to B$ is said to be *L-Lipschitz*, $L : \mathbb{Q}_+$, if

$$\prod_{\epsilon:\mathbb{Q}_+} \prod_{x,y:A} x \approx^A_\epsilon y \to f(x) \approx^B_{L*\epsilon} f(y)$$

When $L \equiv 1$ f is said to be *non-expanding*.

THEOREM 4.20 (Unary Lipschitz extension). *Let $A, T : \mathcal{U}$ be premetric spaces, with A Cauchy-complete, and $f : T \to A$ L-Lipschitz. Then there exists a unique $\overline{f} : \mathcal{C}T \to A$ L-Lipschitz which extends f, i.e. such that*

$$\prod_{x:T} \overline{f}(\eta x) \equiv f(x)$$

PROOF. We define $\overline{f} : \mathcal{C}T \to A$ by $\mathcal{C}T$-recursion, defining the family of mere relations $\frown: A \to A \to \mathbb{Q}_+ \to \text{Prop}$ as the action we want f to have on ϵ-close elements:

$$a \frown_\epsilon b :\equiv a \approx_{L*\epsilon} b$$

Separatedness of $\frown$ follows from that of $\approx$. We need to define $\overline{f}$ on the constructors. For any $q : T$, $\overline{f}(\eta q) :\equiv f(q)$, while for any $x : \mathit{Approx}(\mathcal{C}T)$ $\overline{f}(\lim(x)) :\equiv \lim(\lambda\epsilon.\overline{f}(x_{\frac{\epsilon}{L}}))$, were indeed dove in effetti $\lambda\epsilon.\overline{f}(x_{\frac{\epsilon}{L}}$ is a Cauchy approximation in A, since by inductive hypothesis $\lambda\epsilon.\overline{f}(x_\epsilon)$ is so with respect to $\frown$. Then, we need to verify the four conditions. If $\eta q \approx_\epsilon \eta r$, $\overline{f}(\eta q) \frown_\epsilon \overline{f}(\eta r)$ follows from lipschitzianity of f and definitions of $\overline{f}$ and $\frown$. If $\eta q \approx_{\epsilon-\delta} x_\delta$ and $\overline{f}(\eta q) \frown_{\epsilon-\delta} \overline{f}(x_\delta)$, instead, expressing $\frown$ in terms of $\approx$ and using roundedness and triangularity of the latter we get$\overline{f}(\eta q) \approx_{L*\epsilon} \lim(\lambda\delta.\overline{f}(x_{\frac{L*\delta}{L}})) \equiv \overline{f}(\eta q) \frown_\epsilon \overline{f}(\lim(x))$. Symmetrically one can verify the third condition. Finally, if $x, y : \mathit{Approx}(\mathcal{C}T)$ are such that $x_\delta \approx_{\epsilon-\delta-\theta} y_\theta$ e $\overline{f}(x_\delta) \frown_{\epsilon-\delta-\theta} \overline{(}y_\theta)$, by roundedness and triangularity we get $\lim(\overline{f}(\lambda\delta.x_{\frac{L*\delta}{L}})) \frown_{\epsilon-\delta-\theta} \lim(\overline{f}(\lambda\theta.y_{\frac{L*\theta}{L}})) \equiv \overline{f}(\lim(x)) \frown_\epsilon \overline{f}(\lim(y))$. □

Let it be observed that the completion operator is a functor in the category of premetric spaces with Lipschitz functions. Its action on morphisms is given by the lifting map

$$\text{map} : (X \to Y) \implies (\mathcal{C}X \to \mathcal{C}Y)$$
$$\lambda f.\,\text{map}(f) :\equiv \overline{\eta_Y \circ f}$$

It is worth mentioning Gilbert's work on the monadic structure that the completion has on this category, [4].

Definition 4.21 (Continuous function). Let $A, B : \mathcal{U}$ be premetric spaces. A function $f : A \to B$ is said to be *continuous* if

$$\prod_{\epsilon:\mathbb{Q}_+} \prod_{x:A} \exists(\delta : \mathbb{Q}_+). \prod_{y:A} x \approx_\delta^A y \to f(x) \approx_\epsilon^B f(y)$$

Theorem 4.22. *Let $A : \mathcal{U}$ be a premetric space and $f, g : \mathcal{C}T \to A$ continuous functions such that*

$$\prod_{q:T} f(\eta q) = g(\eta q)$$

Then

$$\prod_{u:\mathcal{C}T} .f(u) = g(u)$$

Proof. By induction on u, since $f(u) = g(u)$ is a mere proposition by separatedness of $\approx^A$. The case $u \equiv \eta q$ and $q : T$ is the hypothesis.
Suppose instead $u \equiv \lim(x)$, $x : \mathit{Approx}(\mathcal{C}T)$. By separatedness it is sufficient to show

$$\prod_{\epsilon:\mathbb{Q}_+} f(\lim(x)) \approx_\epsilon g(\lim(x))$$

and this follows from continuity of f and g. □

Repeated application of the above theorem make possible to deal with multiple variables functions. For instance, if $f, g : \mathcal{C}T_1 \to \mathcal{C}T_2 \to A$ are continuous in both arguments and they coincide on T_1, T_2 then they are equal. Several properties of the completion may be proved by means of equalities between continuous functions; in those cases, namely when we can prove a property by checking it only on base points of ηT, we may talk of *proof by continuity*. It is easy to see that Lipschitz functions are continuous.

Theorem 4.23. *If T is Cauchy-complete then*

$$\mathcal{C}T = T$$

Proof. The identity of T is non-expanding and T is Cauchy-complete, hence id_T can be extended to a non-expanding map $\overline{\mathrm{id}_T} : \mathcal{C}T \to T$. We claim it is an equivalence, with inverse $\eta_T : T \to \mathcal{C}T$.
$\overline{\mathrm{id}_T} \circ \eta_T = \mathrm{id}_T$, since for any $u : T$ $\overline{\mathrm{id}_T}(\eta_T(u)) \equiv \mathrm{id}_T(u) = u$;
$\eta_T \circ \overline{\mathrm{id}_T} = \mathrm{id}_{\mathcal{C}T}$ by continuity. Indeed, $\eta_T \circ \overline{\mathrm{id}_T}$ is continuous and for any $u : T$ it holds that

$$\mathrm{id}_{\mathcal{C}T}(\eta_T(u)) = \eta_T(u) = \eta_T \circ \overline{\mathrm{id}_T}(\eta_T(u)) \equiv \eta_T(u)$$

□

Observation 4.24. As with the proof that $\approx$ was a premetric on $\mathcal{C}T$, in order to obtain a path between T and $\mathcal{C}T$ istead of a weaker equivalence it was essential the use of the Univalence Axiom, without which, however, we could not even have obtained an equivalence. In fact, inhabitedness of $\eta_T \circ \overline{id_T} = id_{\mathcal{C}T}$ is shown by continuity, which needs the fact that $\mathcal{C}T$ is a premetric space.

5. Cauchy reals' algebraic structure

For the greater expressiveness power, when talking about real numbers we will denote η as rat, i.e. $\mathrm{rat} : \mathbb{Q} \to \mathbb{R}_c$. By extending non-expanding maps on rationals to maps on reals, we are able to endow $\mathbb{R}_c$ with the structure of abelian group, with $0 :\equiv \mathrm{rat}\, 0$, and with the structure of a lattice, and we define partial and strict orders on $\mathbb{R}_c$ in terms of the max map:

$$u \le v :\equiv (\max(u, v) = v)$$

$$u < v :\equiv \exists(q, r : \mathbb{Q}).(q < r) \times (u \le \mathrm{rat}(q)) \times \mathrm{rat}(r) \le v$$

Axioms of groups and properties that make $\le$,$<$ order relations transfer from $\mathbb{Q}$ by continuity, since they are expressible in terms of equalities between continuous functions.

The archimedean principle for $\mathbb{R}_c$ directly follows from the definition of the strict order $<$.

Theorem 4.25 (Archimedean principle for $\mathbb{R}_c$).

$$\prod_{u,v:\mathbb{R}_c} (u < v) \to \exists(q : \mathbb{Q}).(u < q < v)$$

It is also possible to characterize further the premetric on $\mathbb{R}_c$ in terms of the lifted absolute value:

Theorem 4.26. *For all $u, v : \mathbb{R}_c$ and $\epsilon : \mathbb{Q}_+$*

$$u \approx_\epsilon v \simeq |u - v| < \mathrm{rat}(\epsilon)$$

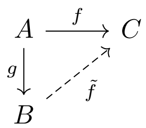
Unlike addition, multiplication cannot be lifted to an operation on $\mathbb{R}_c$, for it is not L-Lipschitz on each variable for any $L : \mathbb{Q}$; in order to equip $\mathbb{R}_c$ with multiplicative structure, it is necessary to recur to a defininition of multiplication *by surjection*, in the following sense: if one variable stays fixed and we impose the other variable to range over a bounded interval, then multiplication is Lipschitz in the free variable, and can hence be lifted; after defining multiplication on bounded intervals of reals, we join them together to cover $\mathbb{R}_c$.

Definition 4.27 (Definition by surjection). Let A, B, C be sets, $f : A \to C$, $g : A \to B$ such that

(i) g is surjective;
(ii) f respects the equivalence relation $\sim_g$ on A induced by$g$, i.e.

$$\prod_{x,y:A} .g(x) = g(y) \to f(x) = f(y)$$

Then $B \simeq A/\sim_g$ and there exists $\tilde{f}$ that makes the following diagram commute:



We may say that $\tilde{f}$ is defined by surjection from f and g.

Definition 4.28 (Interval). For $a, b : \mathbb{Q}$ (respectively $a, bs\mathbb{R}_c$) we define the *interval* as the space

$$[a, b] :\equiv \sum_x a \leq x \leq b$$

premetric with closeness relation induced by the first component. As usual, we identify an element $x : [a, b]$ with $\mathrm{pr}_1(x)$.

We set $A :\equiv \sum_{a:\mathbb{Q}_+}[-\mathrm{rat}(a), \mathrm{rat}(a)]$ and $B :\equiv C :\equiv \mathbb{R}_c$. Let $y : \mathbb{R}_c$, we define by surjection the multiplication for y. $\mathbb{R}_c$ is a vector space over $\mathbb{Q}$: multiplication by $q : \mathbb{Q}$ is $|q| + 1$-Lipschitz and can be lifted to $\mu_q : \mathbb{R}_c \to \mathbb{R}_c$. Let a be a rational such that $y : [-\mathrm{rat}(a), \mathrm{rat}(a)]$, then it is well-defined and a-Lipschitz the map

$$\mu_{a,y} : \mathbb{Q} \to \mathbb{R}_c$$
$$\mu_{a,y}(q) :\equiv \mu_q(y)$$

which can be lifted to $\overline{\mu_{a,y}} : \mathbb{R}_c \to \mathbb{R}_c$. By continuity, if $y : [-\mathrm{rat}(b), \mathrm{rat}(b)]$ for another $b : \mathbb{Q}$, then $\overline{\mu_{a,y}} = \overline{\mu_{b,y}}$. We define $f :\equiv \mu_{y-,-} : \sum_{a:\mathbb{Q}_+}[-\mathrm{rat}(a), \mathrm{rat}(a)] \to \mathbb{R}_c$ as $\mu_{y-,-}(x) :\equiv \overline{\mu_{y_1,y_2}}(x)$; it respects the equivalence relation induced by the surjective map $g : \sum_{a:\mathbb{Q}_+}[-\mathrm{rat}(a), \mathrm{rat}(a)] \to \mathbb{R}_c, g(x) :\equiv \mathrm{pr}_2(x)$, inducing the function $\widetilde{\mu_{y_{-,-}}} : \mathbb{R}_c \to \mathbb{R}_c$, which coincides over rationals with the extension of $\overline{\mu_{a,y}}$ for any convenient $a : \mathbb{Q}$. We denote this induced function as $*y$ and we call it **multiplication by y**, i.e. $*y :\equiv \widetilde{\mu_{y_{-,-}}}$

Definition 4.29 (multiplication). The operation of **multiplication** on $\mathbb{R}_c$ is the function

$$* : \mathbb{R}_c \to \mathbb{R}_c \to \mathbb{R}_c$$
$$x * y :\equiv \widetilde{\mu_{y_{-,-}}}(x)$$

The properties required to make $(\mathbb{R}_c, +, -, *, 0, 1)$ an abelian ring with unit follow from continuity of multiplication as a two-variables-function, as shown in [4]. In order to make $\mathbb{R}_c$ become a field, we have to define multiplicative inversion. Being in a constructive setting, it is convenient to define an aparteness relation as follows.

Definition 4.30 (Apartness). We define the apartness relation on $\mathbb{R}_c$ as the dependent family $\# : \mathbb{R}_c \to \mathbb{R}_c \to \mathrm{Prop}$, $x\#y :\equiv \|x < y + y < x\|$.

Apartness is a constructive version of inequality. Indeed, we have that if $x\#y$ then $\neg(x = y)$. The converse holds if one assumes the excluded middle, but it is not provable constructively.

Theorem 4.31. *$u : \mathbb{R}_c$ is invertible if and only if $u\#0$*

Proof. If $u : \mathbb{R}_c$ has an inverse $v : \mathbb{R}_c$, by the archimedean property there is a $q : \mathbb{Q}_+$ such that $|v| < q$. Hence $1 = |u*v| < |u|*v < |u|*q$ and so $\frac{1}{q} < |u|$, i.e. $u\#0$. For the opposite direction it is possible to define by surjection the multiplicative inversion for reals separated from 0. Such a construction can be found in [16]. □

We can state in a compact manner what proven so far.

Theorem 4.32. *$\mathbb{R}_c$ is a complete archimedean ordered field which is Cauchy complete.*

As expected from their (higher) inductive nature, we can characterize Cauchy reals with the universal property of being initial among complete archimedean fields.

THEOREM 4.33. *Cauchy reals embeds in any other archimedean ordered field Cauchy complete.*

PROOF. Let F be a field satisfying the hypothesis. There exists a canonical embedding $\mathbb{Q} \to F$, hence without loss of generality we assume $\mathbb{Q} \subseteq F$. Given that limits are unique when they exist and that F is complete, there is an operator $\lim : Approx(F) \to F$ which maps Cauchy approximations in F to their limits. The embedding

$$e : \mathbb{R}_c \to F$$

is defined by $(\mathbb{R}_c, \approx)$-recursion, setting for any $q : \mathbb{Q}$, $x : Approx(\mathcal{CT})$

$$e(\mathrm{rat}(q)) :\equiv q$$
$$e(\lim(x)) :\equiv \lim(e \circ x)$$

A suitable $\frown$ on F is given by

$$a \frown_\epsilon b :\equiv |a - b| < \epsilon$$

(where as usual $|-| : F \to F$ is defined as $|a| :\equiv \max(a, -a)$); $\frown$ is separated because F is archimedean. The rest of the clauses for the application of $(\mathbb{R}_c, \approx)$-recursion are easily checked, together with the fact that the resulting e is an embedding of ordered fields which fixes the rationals. $\square$

To conclude this section, we take into account some natural questions that may arise after this presentation of Cauchy reals in Homotopy Type Theory, which mostly develop around the issue of the relationship between the definition of $\mathbb{R}_c$ as HIIT and the classical one as quotient of $Approx(\mathbb{Q})$. Firstly, even if $\mathbb{R}_c$ may not be a quotient of Cauchy sequences of rationals, it is nevertheless a quotient of the set of Cauchy sequences of reals, as expressed by the following lemma.

LEMMA 4.34. $\lim : Approx(\mathbb{R}_c) \to \mathbb{R}_c$ *is surjective, in the sense that every real merely is a limit point:*

$$\prod_{u:\mathbb{R}_c} \exists (x : Approx(\mathcal{CT})). u = \lim(x)$$

In particular, we have that

$$\mathbb{R}_c \simeq Approx(\mathbb{R}_c)/\sim$$

where $\sim$ *is the equivalence relation on* $Approx(\mathbb{R}_c)$ *induced by* lim. *In other words, for any set* A *and any map* $f : Approx(\mathbb{R}_c) \to A$ *which respects coincidence of Cauchy approximations, in the sense that*

$$\prod_{x,y:Approx(\mathbb{R}_c)} . \lim(x) = \lim(y) \implies f(x) = f(y)$$

we have that f *factors uniquely through* lim.

PROOF. We show surjectivity of lim by $\mathbb{R}_c$-induction; the other properties follow from theorem 10.1.5 of the HoTT Book [16]. $\square$

More in general, one may ask whether it would be available in HoTT a construction of the real field as a certain quotient of $Approx(\mathbb{Q})$. In this respect, it has been proven ([9]) that there is no way to show Cauchy-completeness of the quotient Cauchy reals within IZF_{Ref} (Intuitionistic Zermelo Frenkel set theory where the Collection schema is replaced by the Reflection schema); for what concerns type theory, though, one may still hope in a way to canonically choose representatives of the equivalence classes (as it was the case for $\mathbb{Z}$ and $\mathbb{Q}$) and by means of such *normalizability* to avoid principles as the Axiom of Choice or the law of excluded middle. However, it has been proven that, in the language of [8], "there is no definable normalization function on the set of Cauchy sequences in any extension of basic MLTT [Martin Löf Type Theory] which admits the standard property that definable functions are continuous. "

CHAPTER 5

Dedekind reals

In this chapter we present the construction of Dedekind real numbers in Homotopy Type Theory and later we compare the type of Dedekind reals $\mathbb{R}_d$ with the type of Cauchy reals $\mathbb{R}_c$. Despite the fact that, in general, the two approaches do not give rise to equivalent types (i.e. equal types by univalence), there exists a condition under which they do, namely that every Dedekind real merely comes equipped with a *locator*; since classical assumptions such as LEM and AC imply this sufficient condition, we get that HoTT's development of real numbers is entirely compatible with the one of classical logic.

The traditional definition of two-sided Dedekind cuts states that a *Dedekind cut* consists in a pair $(\mathtt{L}, \mathtt{U})$ of subsets of rationals $\mathtt{L}, \mathtt{U} \subseteq \mathbb{Q}$, called the *lower* and *upper cut* respectively, which are:

(i) *inhabited*: there are $q \in \mathtt{L}$ and $r \in \mathtt{U}$;
(ii) *rounded*: $q \in \mathtt{L} \leftrightarrow \exists (r \in \mathbb{Q}). q < r \land r \in \mathtt{L}$
$r \in \mathtt{U} \leftrightarrow \exists (q \in \mathbb{Q}). q \in \mathtt{U} \land q < r$;
(iii) *disjoint*: $\neg(q \in \mathtt{L} \land q \in \mathtt{U})$;
(iv) *located*: $q < r \implies q \in \mathtt{L} \lor r \in \mathtt{U}$.

Dedekind reals are then defined as the set $\mathbb{R}_d :\equiv \{(\mathtt{L}, \mathtt{U}) | \mathtt{L} \text{ and } \mathtt{U} \text{ are Dedekind cuts}\}$.

Despite some technical difficulties, the same construction can be carried out in Homotopy Type Theory too.

The main obstacle is that by defining cuts to be dependent families $\mathtt{L}, \mathtt{U} : \mathbb{Q} \to \mathrm{Prop}$ we get a construction which raises universe levels. In fact, by typical ambiguity the notation Prop stands for $\mathrm{Prop}_{\mathcal{U}_i}$ for some universe $\mathcal{U}_i$. Once we decide to fix a universe $\mathcal{U}_i$, hence considering just $\mathrm{Prop}_{\mathcal{U}_i}$, the type of Dedekind reals will reside in the next universe $\mathcal{U}_{i+1}$, a property of reals in $\mathcal{U}_{i+2}$ and so on, the increase of universe levels being due to the fact that $\mathrm{Prop}_{\mathcal{U}_j} : \mathcal{U}_{j+1}$.

One way to circumvent this problem is to work under the simplifying assumption that a single type of propositions Ω is sufficient for all purposes related to Dedekind constructions. More precisely, we require Ω to be closed under countable conjunctions, disjunction and existential quantifiers. The assumption will be further discussed in 2.

Definition 5.1 (Dedekind cut, Dedekind reals). Let Ω be a type of propositions closed under countable conjunctions, disjunction and existential quantifiers, which moreover is a set, i.e. $\mathrm{isSet}(\Omega)$. A Dedekind cut is a pair $(\mathtt{L}, \mathtt{U})$ of mere predicates $\mathtt{L}, \mathtt{U} : \mathbb{Q} \to \Omega$ such that the followings are inhabited:

(i) $\text{isInhab}(\mathtt{L},\mathtt{U}) :\equiv \left\|\sum_{q:\mathbb{Q}} \mathtt{L}(q)\right\| \times \left\|\sum_{r:\mathbb{Q}} \mathtt{U}(r)\right\|$

(ii) $\text{isRound}(\mathtt{L}) \times \text{isRound}(\mathtt{U})$, where

$$\text{isRound}(\mathtt{L}) :\equiv \prod_{q,r:\mathbb{Q}} \left(\mathtt{L}(q) \leftrightarrow \left\|\sum_{r:\mathbb{Q}} (q<r) \times \mathtt{L}(r)\right\|\right)$$

$$\text{isRound}(\mathtt{U}) :\equiv \prod_{q,r:\mathbb{Q}} \left(\mathtt{U}(r) \leftrightarrow \left\|\sum_{q:\mathbb{Q}} (q<r) \times \mathtt{U}(q)\right\|\right)$$

(iii) $\text{Disj}(\mathtt{L},\mathtt{U}) :\equiv \prod_{q:\mathbb{Q}} \neg(\mathtt{L}(q) \times \mathtt{U}(q))$

(iv) $\text{isLocated}(\mathtt{L},\mathtt{U}) :\equiv \prod_{q,r:\mathbb{Q}} (q<r) \to \|\mathtt{L}(q) + \mathtt{U}(r)\|$

We let $\text{isCut}(\mathtt{L},\mathtt{U})$ denote the conjunction of these conditions, namely

$\text{isCut}(\mathtt{L},\mathtt{U}) :\equiv \text{isInhab}(\mathtt{L},\mathtt{U}) \times \text{isRound}(\mathtt{L}) \times \text{isRound}(\mathtt{U}) \times \text{Disj}(\mathtt{L},\mathtt{U}) \times \text{isLocated}(\mathtt{L},\mathtt{U})$

The type of Dedekind reals is

$$\mathbb{R}_d :\equiv \sum_{(\mathtt{L},\mathtt{U}):(\mathbb{Q}\to\Omega)\times(\mathbb{Q}\to\Omega)} \text{isCut}(\mathtt{L},\mathtt{U})$$

Since $\text{isCut}(\mathtt{L},\mathtt{U})$ is a mere proposition, for it is a conjunction of mere propositions, we may identify an element of $\mathbb{R}_d$ with its first component.

OBSERVATION 5.2. $\mathbb{R}_d$ is a set.

PROOF. Let $u, v : \mathbb{R}_d$ be two Dedekind reals and $\overline{h}, \overline{k} : u = v$ be two paths between them; we need to show that $\overline{h} = \overline{k}$. By Theorem 2.7.2 of [16], we have that

$$u = v \simeq \sum_{p:\text{pr}_1(u) =_{(\mathbb{Q}\to\Omega)\times(\mathbb{Q}\to\Omega)} \text{pr}_1(v)} p_*(\text{pr}_2(u)) =_{\text{isCut}(\text{pr}_1(v))} \text{pr}_2(v)$$

By this characterization, the paths $\overline{h}$ and $\overline{k}$ correspond to two objects of the type on the right, say h, k, which by Σ-induction we may assume to be canonical, i.e. $h \equiv (h_1, h_2)$ and $k \equiv (k_1, k_2)$. Now, Ω is a set by assumption, so by function extensionality we have that $\mathbb{Q} \to \Omega$ is a set, hence also $(\mathbb{Q} \to \Omega) \times (\mathbb{Q} \to \Omega)$. From this, it follows that $h_1 = k_1$. But $\text{isCut}(\text{pr}_1(v))$ is a mere proposition, hence a set, and so $h_2 = k_2$. We have proven that $h \equiv (h_1, h_2) = (k_1, k_2) \equiv k$, and this implies $\overline{h} = \overline{k}$, as wanted. □

1. Dedekind reals' algebraic structure

The construction of the algebraic and order-theoretic structure of Dedekind reals is essentially the usual one performed in set theory. For this reason, we briefly sketch an outline of how $\mathbb{R}_d$ can be endowed with the structure of an archimedean totally ordered field, and we then investigate some issues that naturally arise when there comes to translate Dedekind cuts' definition in a valid and working one of Homotopy Type Theory.

We write $\mathtt{L}_x, \mathtt{U}_x$ for the lower and upper cut of a real number $x : \mathbb{R}_d$. First of all, note that we have the inclusion $\mathbb{Q} \subseteq \mathbb{R}_d$ since any rational can be identified with the cut $(\mathtt{L}_\mathtt{q}, \mathtt{U}_q)$ defined as

$$\mathtt{L}_\mathtt{q}(r) :\equiv r < q$$
$$\mathtt{U}_q(r) :\equiv q < r$$

Addition is defined as

$$\mathtt{L}_{x+y}(q) :\equiv \left\| \sum_{r,s:\mathbb{Q}} \mathtt{L}_x(r) \times \mathtt{L}_y(s) \times q = r + s \right\|$$
$$\mathtt{U}_{x+y} :\equiv \left\| \sum_{r,s:\mathbb{Q}} \mathtt{U}_x(r) \times \mathtt{U}_y(s) \times q = r + s \right\|$$

and the additive inverse by

$$\mathtt{L}_{-x}(q) :\equiv \left\| \sum_{r:\mathbb{Q}} \mathtt{U}_x(r) \times q = -r \right\|$$
$$\mathtt{U}_{-x}(q) :\equiv \left\| \sum_{r:\mathbb{Q}} \mathtt{L}_x(r) \times q = -r \right\|$$

Multiplication's formulation is related to multiplication of intervals in interval arithmetic and is defined by

$$\mathtt{L}_{x\cdot y}(q) :\equiv \left\| \sum_{a,b,c,d:\mathbb{Q}} \mathtt{L}_x(a) \times \mathtt{U}_x(b) \times \mathtt{L}_y(c) \times \mathtt{U}_y(d) \times q < \min(a \cdot c, a \cdot d, b \cdot c, b \cdot d) \right\|$$
$$\mathtt{U}_{x\cdot y}(q) :\equiv \left\| \sum_{a,b,c,d:\mathbb{Q}} \mathtt{L}_x(a) \times \mathtt{U}_x(b) \times \mathtt{L}_y(c) \times \mathtt{U}_y(d) \times \max(a \cdot c, a \cdot d, b \cdot c, b \cdot d) < q \right\|$$

At this point we have a commutative ring with unit $(\mathbb{R}_d, 0, 1, +, -, \cdot)$. To treat multiplicative inverses in a constructive setting, where the apartness relation will be needed, we must first introduce order, which we define as

$$(x \leq y) :\equiv \prod_{q:\mathbb{Q}} \mathtt{L}_x(q) \to \mathtt{L}_y(q)$$
$$(x < y) :\equiv \left\| \sum_{q:\mathbb{Q}} \mathtt{U}_x(q) \times \mathtt{L}_y(q) \right\|$$

Lemma 5.3. *For all $x : \mathbb{R}_d$ and $q : \mathbb{Q}$, $\mathtt{L}_x(q) \leftrightarrow (q < x)$ and $\mathtt{U}_x(q) \leftrightarrow (x < q)$.*

Proof. If $\mathtt{L}_x(q)$ then by roundedness there merely is $r > q$ such that $\mathtt{L}_x(r)$, and since $\mathtt{U}_q(r)$ it follows that $q < x$. Conversely, if $q < x$ then there is $r : \mathbb{Q}$ such that $\mathtt{U}_q(r)$ and $\mathtt{L}_x(r)$, hence $\mathtt{L}_x(q)$ because $\mathtt{L}_x$ is a lower set. The other half of the proof is symmetric. □

On the grounds of this lemma we may indifferently write $\mathtt{L}_x(q)$ or $(q < x)$, and $\mathtt{U}_x(q)$ or $(x < q)$. The relation $\leq$ is a partial order, and $<$ is transitive and irreflexive. Assuming excluded middle, we get linearity, i.e.

$$\mathrm{LEM} \to \|x < y + y \leq x\|$$

Without excluded middle, we get *weak linearity.*

Proposition 5.4. $<$ *is weakly linear, i.e. for any* $x, y, z : \mathbb{R}_d$ *we have*

$$x < y \to \|(x < z) + (z < y)\|$$

Proof. Suppose $x < y$. Then there merely exists $q : \mathbb{Q}$ such that $\mathtt{U}_x(q)$ and $\mathtt{L}_y(q)$. By roundedness there merely exist $r, s : \mathbb{Q}$ such that $r < q < s, \mathtt{U}_x(r)$ and $\mathtt{L}_y(s)$. Then by locatedness $\mathtt{L}_z(r)$ or $\mathtt{U}_z(s)$: in the first case we get $x < z$ and in the second $z < y$ □

Observation 5.5. The truncation in the statement above, namely $\|(x < z) + (z < y)\|$, arises from locatedness of $(\mathtt{L}_z, \mathtt{U}_z)$. In fact, recall from the induction principle of truncation that case analysis on $\|\mathtt{L}_z r + \mathtt{U}_z(r)\|$ can be done only when attempting to prove a mere propositions.

Weak linearity can be regarded as linearity "up to a small numerical error". In fact, for any $\epsilon : \mathbb{Q}$, $\epsilon > 0$, by taking $x \equiv u - \epsilon$ and $y \equiv u + \epsilon$ we get

$$\|(u - \epsilon < z) + (z < u + \epsilon)\|$$

This is consistent with the computational nature of type theory: "since it is unreasonable to expect that we can actually compute with infinite precision, we should not be surprised that we can decide $<$ only up to whatever finite precision we have computed" ([16], pp 377).

As with the Cauchy reals, invertibility is characterized by the apartness relation.

Theorem 5.6. *A Dedekind real* $x : \mathbb{R}_d$ *is invertible if and only if* $x \# 0$.

Proof. If $x \cdot y = 1$, then there merely exist $a, b, c, d : \mathbb{Q}$ such that $a < x < b$, $c < y < d$ and $0 < \min(ac, ad, bc, bd)$. From $0 < ac$ and $0 < bc$ it follows that a, b, c are either all positive or all negative. Hence either $0 < a < x$ or $x < b < 0$, so that $x \# 0$. Conversely, if $x \# 0$ then the following Dedekind cut is inhabited and defines the inverse of x:

$$\mathtt{L}_{x^{-1}}(q) :\equiv \left\| \sum_{r:\mathbb{Q}} \mathtt{U}_x(r) \times \|(0 < r \times qr < 1) + (r < 0 \times 1 < qr)\| \right\|$$

$$\mathtt{U}_{x^{-1}}(q) :\equiv \left\| \sum_{r:\mathbb{Q}} \mathtt{L}_x(r) \times \|(0 < r \times qr > 1) + (r < 0 \times 1 > qr)\| \right\|$$

□

By definition of $<$, it follows also the archimedean property of $\mathbb{R}_d$.

Theorem 5.7 (Archimedean principle for $\mathbb{R}_d$). *For all* $x, y : \mathbb{R}_d$, *if* $x < y$ *then there merely exists* $q : \mathbb{Q}$ *such that* $x < q < y$.

Summing up, we have proven the following:

Theorem 5.8. *The Dedekind reals* $\mathbb{R}_d$ *form an ordered archimedean field.*

2. Formal issues

In the definition of a Dedekind cut we made the assumption of a type of propositions Ω which is a set and is closed under countable disjunctions and conjunctions and existential quantifiers over $\mathbb{Q}$. We may justify this in four ways:

(i) By identifying Ω with the ambiguous Prop and tracking the universe levels that appear in the construction. This choice, though, does not solve the problem of the raising of universe levels.
(ii) By assuming what is known as *Propositional resizing axiom*, which states that for every i, $\text{Prop}_{\mathcal{U}_i} = \text{Prop}_{\mathcal{U}_{i+1}}$, and setting $\Omega \equiv \text{Prop}_{\mathcal{U}_0}$, namely the lowest level to which every $\text{Prop}_{\mathcal{U}_i}$ collapses.
(iii) By assuming LEM: that would imply that for every universe $\text{Prop}_{\mathcal{U}_i} = \mathbf{2}$, hence $\Omega \equiv \mathbf{2}$. This option is not really desirable, though, since we are trying to develop a constructive version of Dedekind reals in HoTT without any additional axiom.
(iv) By asking for a minimal requirement to make the construction work. In particular, it is sufficient ([16, pp.375]) to define Ω as the initial σ-frame, since the condition for a family over $\mathbb{Q}$ of mere propositions to be a Dedekind cut is expressible by conjunctions, disjunctions and existential quantifiers (essentially by Σ^1_0-formulas) over $\mathbb{Q}$, which is a countable set. We mention that in [16] it is suggested to construct Ω as an higher inductive-inductive type.

We observe that in all of the above cases Ω is a set, which is essential to guarantee, by function extensionality, that $\mathbb{R}_d$ be a set.

The second issue we need to deal with is the translation of the definition of Dedekind cuts into a valid and working one in Homotopy Type Theory's language, because it raises the question of where to truncate (propositionally) the type-theoretic versions of the properties of cuts. In fact, while there was essentially one way to place truncations in the translations of the first three properties, the locatedness condition still leaves a certain margin of discretion. While in definition 5.1 locatedness is stated as

$$\text{isLocated}(\mathtt{L}, \mathtt{U}) :\equiv \prod_{q,r:\mathbb{Q}} (q < r) \to \|\mathtt{L}(q) + \mathtt{U}(r)\|$$

nothing prevents it from being formulated in this other way:

$$\text{isLocated}_1(\mathtt{L}, \mathtt{U}) :\equiv \prod_{q,r:\mathbb{Q}} \|(q < r) \to \mathtt{L}(q) + \mathtt{U}(r)\|$$

Fortunately, the following lemma makes the two formulations equivalent.

Proposition 5.9. *Let P be a decidable proposition, i.e. it is inhabited the type $P + \neg P$, and let X be any type. Then $\|P \to X\|$ is equivalent to $P \to \|X\|$.*

Proof. Firstly, we prove that for any mere proposition P : Prop we have $\|P \to X\| \to (P \to \|X\|)$. Since P, $\|X\|$ are mere propositions, so is $P \to \|X\|$, hence we can apply the recursion principle for truncation and show that

$$(P \to X) \to (P \to \|X\|)$$

But this is true, as for any $f : (P \to X)$ we can construct an element $g : P \to \|X\|$ by defining $g(p) :\equiv |f(p)|$ for any $p : P$.

Conversely, we will show that $(P+\neg P) \to ((P \to \|X\|) \to (\|P \to X\|))$, and by the induction principle for coproduct this means showing separately that

$$P \to ((P \to \|X\|) \to (\|P \to X\|))$$

and

$$\neg P \to ((P \to \|X\|) \to (\|P \to X\|))$$

Assume P; then we need to show that $\|X\| \to \|P \to X\|$. But it is always the case that $X \to P \to X$, and since $Q \to \|Q\|$ for any type $Q : \mathcal{U}$ we have $X \to \|P \to X\|$. Now, given that $\|P \to X\|$ is by definition a mere proposition, from the induction principle for truncations it follows that $\|X\| \to \|P \to X\|$, as wanted. Assume instead that $\neg P$ holds. Let $f : P \to \|X\|$, we can suppose $p : P$, then we have an element of $\mathbf{0}$, and from its induction principle it follows (anything, in particular that) $\|P \to X\|$. □

In our case, decidability of the order on $\mathbb{Q}$ results in

$$\text{isLocated}(\mathtt{L}, \mathtt{U}) \simeq \text{isLocated}_1(\mathtt{L}, \mathtt{U})$$

However, a few strengthenings of the locatedness property are possible, and in order to study them and their consequences we introduce the following definition.

Definition 5.10 (Locator). A locator for $x : \mathbb{R}_d$ is a function

$$l : \prod_{q,r:\mathbb{Q}} q < r \to (q < x) + (x < r)$$

We denote with locator(x) the type of locators on x.

Observe that in the definition of isLocated(x) the disjoint sum $(q < x) + (x < r)$ of locator(x) has been replaced with its truncation $\|(q < x) + (x < r)\|$, and that was sufficient to guarantee that isLocated(x) is a mere proposition. By contrast, locator(x) may have different inhabitants ([1]), so that equipping a real number x with a locator results in a *structure* on x, rather than a property.

Lemma 5.11. *For any* $x : \mathbb{R}_d$ $\|\text{locator}(x)\| \to \text{isLocated}(x)$.

Proof. Since locator(x) $\to$ isLocated(x) and isLocated(x) is a mere proposition, by the recursion principle for truncation we get $\|\text{locator}(x)\| \to \text{isLocated}(x)$. □

However, the converse does not hold in general, at least if we do not assume any nonconstructive principle such as LEM or AC. Remind from 3.11 that one of the equivalent formulations of the type-theoretic Axiom of Choice is the following:

$$\left(\prod_{x:X} \|\Upsilon(x)\|\right) \to \left\|\prod_{x:X} \Upsilon(x)\right\|$$

where X is a set and $\Upsilon : X \to \mathcal{U}$ a family of sets (i.e. $\Upsilon(x)$ is a set for any $x : X$).

Observation 5.12.

$$\text{isLocated}(x) \to \|\text{locator}(x)\|$$

is an instance of AC.

To see this, it suffices to choose $X \equiv \mathbb{Q} \times \mathbb{Q}$ and $\Upsilon(q, r) :\equiv (q < r) \to \mathtt{L}_x(q) + \mathtt{U}_x(r)$. Clearly, X is a set, since cartesian product preserves sets. On the other hand, we

show that $(q < r) \to \mathtt{L}_x(q) + \mathtt{U}_x(r)$ is a set for any $(q, r) : \mathbb{Q} \times \mathbb{Q}$ by proving that $\mathtt{L}_x(q) + \mathtt{U}_x(r)$ is a set. This , together with the fact that a mere proposition is also a set and that $\to$ preserves sets, will give the thesis. Suppose $a, b : \mathtt{L}_x(q) + \mathtt{U}_x(r)$ and $h, k : a = b$, we have to show that $p = q$. By the induction principle for coproducts we can proceed by case analysis on a, b. If $a \equiv \mathrm{inl}(a')$ and $b \equiv \mathrm{inl}(b')$ with $a', b' : \mathtt{L}_x(q)$, then $(a = b) \equiv (\mathrm{inl}(a') = \mathrm{inl}(b')) = (a' = b')$ by characterization of the identity type of coproduct ([16, p. 2.12]), so h, k correspond to, let's say, $h', k' : a' = b'$. But $\mathtt{L}_x(q)$ is a mere proposition, hence a set, and so $h' = k'$, from which we get $h = k$. The same if $a \equiv \mathrm{inr}(a''), b \equiv \mathrm{inr}(b'')$. Instead, if we have $a \equiv \mathrm{inl}(a')$, $a' : \mathtt{L}_x(q)$ and $b \equiv \mathrm{inr}(b')$, $b' : \mathtt{U}_x(q)$, then $a = b \simeq \mathbf{0}$, hence we deduce $h = k$. Symmetrically if $a \equiv \mathrm{inr}(a')$ and $b \equiv \mathrm{inl}(b')$.
Finally, the fact that function type preserves sets can be shown in the following way: $f, g : A \to B$ with A, B sets, and $\overline{p}, \overline{q} : f = g$. By function extensionality $\overline{p}$ and $\overline{q}$ correspond respectively to $p, q : \prod_{a:A} f(a) =_B g(a)$. For any $a : A$, $p(a) = q(a)$ because $p(a), q(a) : f(a) =_B g(a)$ and B is a set. So by function extensionality $p = q$ hence $\overline{p} = \overline{q}$.

As we will see in the next section, requiring from every Dedekind real to *merely* have a locator, i.e. $\prod_{x:\mathbb{R}_d} \|\mathrm{locator}(x)\|$, turns out to be sufficient to guarantee the coincidence of Dedekind reals and Cauchy reals, whereas without any additional assumption we just have $\mathbb{R}_c \subseteq \mathbb{R}_d$.

However, it wouldn't have been possible in any case to expect *all* Dedekind reals to come equipped with a locator, since that would imply an instance of the nonconstructive principle of omniscience.

Definition 5.13 (WLPO). The weak limited principle of omniscience (WLPO) is the following consequence of the law excluded middle:

$$\mathrm{WLPO} :\equiv \prod_{P:\mathbb{N}\to\mathrm{Prop}} (P + \neg P) \to \neg \left\| \sum_{n:\mathbb{N}} P(n) \right\| + \neg\neg \left\| \sum_{n:\mathbb{N}} P(n) \right\|$$

Namely, for every decidable predicate $P : \mathbb{N} \to \mathrm{Prop}$ on naturals, we can decide $\neg \|\sum_{n:\mathbb{N}} P(n)\|$.

Lemma 5.14. *Suppose that* $\prod_{x:\mathbb{R}_d} \mathrm{locator}(x)$*, then we can define a non-constant function* $f : \mathbb{R}_d \to \mathbf{2}$.

Proof. Let $g : \prod_{x:\mathbb{R}_d} \mathrm{locator}(x)$. Since $0 < 1$, for any $x : \mathbb{R}_d$ we have that $g(x)(0, 1) : 0 < x + x < 1$, hence either $g(x)(0, 1) \equiv \mathrm{inl}(u_x)$ or $g(x)(0, 1) \equiv \mathrm{inr}(v_x)$. For any $x : \mathbb{R}_d$ we define $f(x)$ by case analysis in the following way:

$$f(x) = \begin{cases} 1_{\mathbf{2}} & \text{if } g(x)(0,1) \equiv \mathrm{inl}(u_x) \\ 0_{\mathbf{2}} & \text{if } g(x)(0,1) \equiv \mathrm{inr}(v_x) \end{cases}$$

□

Lemma 5.15. *If there exists a strongly non-constant function* $\mathbb{R}_d \to \mathbf{2}$*, then WLPO holds.*

A proof of the previous lemma can be found in [1, p. 21].

3. Dedekind completeness

We now prove that Deedekind reals are *Dedekind complete*, in the following sense. We obtained $\mathbb{R}_d$ as the type of Dedekind cuts on $\mathbb{Q}$, but the construction of Dedekind cuts works with any archimedean ordered field F, and it leads to an archimedean ordered field $\overline{F}$, the Dedekind completion of F, with F contained as a subfield. Completeness of Dedekind reals consists in the fact that when this construction is applied to $\mathbb{R}_d$, it returns $\mathbb{R}_d$ itself, – or better, a type *equal* to it. This is a consequence of the finality property of $\mathbb{R}_d$ among archimedean ordered fields (admissible for Ω).

DEFINITION 5.16 (Embedding). Let $f : A \to B$, then f is an embedding if fory every $x, y : A$ the function $\mathsf{ap}_f : (x =_A y) \to (f(x) =_B f(y))$ is an equivalence.

OBSERVATION 5.17. By Lemma 3.6, if A and B are sets, then f is an embedding just when for every $x, y : A$ we have $(f(x) =_B f(y)) \to (x =_A y)$, since the two identity types are both mere propositions.

DEFINITION 5.18. An ordered field F is admissible for Ω when the strict order $<$ on F is a map $<: F \to F \to \Omega$.

OBSERVATION 5.19. If F is admissible for Ω then so is its Dedekind completion $\overline{F}$.

PROOF. The strict order on $\overline{F}$ is defined by

$$((\mathtt{L}, \mathtt{U}) < (\mathtt{L}', \mathtt{U}')) :\equiv \left\| \sum_{q:\mathbb{Q}} \mathtt{U}(q) \times \mathtt{L}'(q) \right\|$$

Since $\mathtt{U}(q)$ and $\mathtt{L}'(q)$ are elements of Ω, the statement holds as long as Ω is closed under conjunctions and countable existentials, which we assumed from the outset. □

THEOREM 5.20. *Every archimedean ordered field F which is admissible for Ω is a subfield of $\mathbb{R}_d$, i.e. there exists a field embedding*

$$F \to \mathbb{R}_d$$

which preserves and reflects the order.

PROOF. Let F be an archimedean ordered field. Since F is admissible for Ω, we can define for every $x : F$ $\mathtt{L}_x, \mathtt{U}_x : \mathbb{Q} \to \Omega$ by

$$\mathtt{L}_x(q) :\equiv (q < x) \quad \text{and} \quad \mathtt{U}_x(q) :\equiv (x < q)$$

Then $(\mathtt{L}_x, \mathtt{U}_x)$ is a Dedekind cut. Indeed, the cuts are inhabited and rounded because F is archimedean and $<$ is transitive, disjoint because $<$ is irreflexive, and located because $<$ is a weak linear order. Let $e : F \to \mathbb{R}_d$ be the map $e(x) :\equiv (\mathtt{L}_x, \mathtt{U}_x)$. The claim is that e is a field embedding which preserves and reflects the order. Evidently, for every $q : \mathbb{Q}$ $e(q) = q$. In addition, we have that for all $x, y : F$

$$x < y \leftrightarrow \left\| \sum_{q:\mathbb{Q}} x < q < y \right\| \leftrightarrow \left\| \sum_{q:\mathbb{Q}} \mathtt{U}_x(q) \times \mathtt{L}_{\mathtt{x}}(\mathtt{q}) \right\| \leftrightarrow e(x) < e(y)$$

so e preserves and reflects the order. That $e(x+y) = e(x) + e(y)$ holds because, for all $q : Q$,

$$q < x + y \leftrightarrow \left\| \sum_{r,s:\mathbb{Q}} r < x \times s < y \times q = r + s \right\|$$

Similarly, it can be shown that e preserves multiplication. □

COROLLARY 5.21. *The Dedekind reals are Dedekind complete: for every real-valued Dedekind cut* $(\mathtt{L}, \mathtt{U})$ *there is a unique* $x : \mathbb{R}_d$ *such that* $\mathtt{L}(y) = (y < x)$ *and* $\mathtt{U}(y) = (x < y)$*. In other words,* $\overline{\mathbb{R}_d} = \mathbb{R}_d$*.*

PROOF. By observation 5.19, the Dedekind completion $\overline{\mathbb{R}_d}$ of $\mathbb{R}_d$ is admissible for Ω, so by Theorem 5.20 we have an embedding $\overline{\mathbb{R}_d} \to \mathbb{R}_d$, as well as an embedding $\mathbb{R}_d \to \overline{\mathbb{R}_d}$. But these embeddings must be isomorphisms, because their compositions are order-preserving field homomorphisms which fix the dense subfield $\mathbb{Q}$, which means that they are the identity. The corollary now follows from the fact that $\overline{\mathbb{R}_d} \to \mathbb{R}_d$ is an isomorphism. □

Finally, not only is $\mathbb{R}_d$ Dedekind complete, but it is also Cauchy complete.

THEOREM 5.22. $\mathbb{R}_d$ *is Cauchy complete, namely every Cauchy approximation in* $\mathbb{R}_d$ *has a limit.*

PROOF. Observe that we are showing existence, not mere existence, of the limit. Given a Cauchy approximation $x : \mathbb{Q}_+ \to \mathbb{R}_d$, define

$$\mathtt{L}_y(q) :\equiv \left\| \sum_{\epsilon,\theta:\mathbb{Q}_+} \mathtt{L}_{x_\epsilon}(q + \epsilon + \theta) \right\|$$

$$\mathtt{U}_y(q) :\equiv \left\| \sum_{\epsilon,\theta:\mathbb{Q}_+} \mathtt{L}_{x_\epsilon}(q - \epsilon - \theta) \right\|$$

It can be easily proven that $\mathtt{L}_y$ and $\mathtt{U}_y$ are inhabited, rounded, disjointed and located, hence they determine a Dedekind real $y : \mathbb{R}_d$. By density of $\mathbb{Q}$ in $\mathbb{R}_d$ (i.e. archimedean property) it follows that y is the limit of x. □

4. Cauchy and Dedekind reals: a comparison

We have carried out two constructions of real numbers, obtaining Dedekind reals $\mathbb{R}_d$ and Cauchy reals $\mathbb{R}_c$: while the former have been defined essentially following Dedekind's original prescriptions, the latter have been introduced by means of Higher Inductive-Inductive Types, a characteristic feature of Homotopy Type Theory. In set theory, by assuming classical non-constructive principles such as AC or LEM, $\mathbb{R}_d$ and $\mathbb{R}_c$ (defined as a quotient) result in isomorphic fields, which are hence identified. However, the axiomatization of real numbers is not categorical without such suppositions, and in general Dedekind reals and Cauchy reals do not coincide, as can be seen in [10]. In this section we investigate the relationship between HoTT's $\mathbb{R}_c$ and $\mathbb{R}_d$ and we present a sufficent condition for their coincidence, namely for inhabitedness of $\mathbb{R}_c = \mathbb{R}_d$.

The main aspects under which we may compare HoTT's Dedekind and Cauchy reals are the following:

(i) Characterization in terms of universal properties: $\mathbb{R}_c$ is initial while $\mathbb{R}_d$ is final among complete archimedean ordered fields.
(ii) Conditions under which they coincide.
   (a) Dedekind reals are Cauchy complete, hence by initiality of $\mathbb{R}_d$, or equivalently by finality of $\mathbb{R}_d$, we have that
   $$\mathbb{R}_c \hookrightarrow \mathbb{R}_d$$
   where $\hookrightarrow$ stands for an embedding of ordered fields that fixes the rationals. We may say $\mathbb{R}_c \subseteq \mathbb{R}_d$.
   (b)
   $$\left(\prod_{x:\mathbb{R}_d} \|\text{locator}(x)\|\right) \to (\mathbb{R}_c = \mathbb{R}_d)$$
   (c)
   $$\|\text{AC}_{\aleph_0} + \text{LEM}\| \to \prod_{x:\mathbb{R}_d} \|\text{locator}(x)\|$$

Regarding universal properties that apply to the two constructions, we have proven initiality of Cauchy reals among Cauchy complete archimedean ordered fields in Theorem 4.33, while finality of Dedekind reals among archimedean ordered fields admissible for Ω has been proven in Theorem 5.20.

The following observation, together with the fact that $\mathbb{R}_d$ is Cauchy complete, allows to obtain an embedding of $\mathbb{R}_c$ into $\mathbb{R}_d$, either by finality of $\mathbb{R}_d$ or initiality of $\mathbb{R}_c$.

Observation 5.23. $\mathbb{R}_c$ is an archimedean ordered field which is admissible for Ω. Indeed, if Ω is the initial σ-frame it follows by $\mathbb{R}_c$-induction, otherwise it is immediate.

Lemma 5.24. *There exists an embdedding of ordered fields*
$$\mathbb{R}_c \to R_d$$
which fixes rationals.

Lemma 5.25. *If for any* $x\colon \mathbb{R}_d$ *there merely exists* $c : \text{locator}(x)$*, i.e. if*
$$\prod_{x:\mathbb{R}_d} \|\text{locator}(x)\|$$
then $\mathbb{R}_c = \mathbb{R}_d$

Proof. We already know that $\mathbb{R}_c$ embeds into $\mathbb{R}_d$, so it suffices to show that every Dedekind real merely is the limit of a Cauchy sequence of rational numbers. Consider any $x : \mathbb{R}_d$. By assumption there merely exists $c : \text{locator}(x)$, and by inhabitation of cuts there merely exist $a, b : \mathbb{Q}$ such that $a < x < b$. We construct a sequence $f : \mathbb{N} \to \sum_{(q,r):\mathbb{Q}\times\mathbb{Q}} q < r$ by recursion:

(i) Set $f(0) :\equiv ((a, b), p_0)$, where $p_0 : a < b$.
(ii) Suppose $f(n)$ is already defined as $((q_n, r_n), p)$, where $p : q_n < r_n$. Define $s :\equiv (2q_n+r_n)/3$ and $t :\equiv (q_n+2r_n)/3$. Then $c(s, t)$ decides between $s < x$ and $x < t$. If it decides $s < x$ then there exists (unique up to propositional

equality) $p_{n+1} : s < r_n$ and we set $f(n+1) :\equiv ((s, r_n), p_{n+1})$, otherwise $f(n+1) :\equiv ((q_n, t), p'_{n+1})$, where $p'_{n+1} : (q_n, t)$.

Let us write (q_n, r_n) for $f(n)$ and omit the witness that $q_n < r_n$. Then it is easy to see that $q_n < x < r_n$ and $|q_n - r_n| \leq (2/3)^n \cdot |q_0 - r_0|$ for all $n : \mathbb{N}$. Therefore $q_0, q_1, \ldots$ and $r_0, r_1, \ldots$ are both Cauchy sequences converging to the Dedekind cut x. Hence we have obtained the sequence of embeddings $\mathbb{R}_d \to \mathit{Approx}(\mathbb{Q}) \to \mathit{Approx}(\mathbb{R}_c) \to \mathbb{R}_c$. □

The previous lemma implies that the Cauchy and Dedekind reals' constructions in HoTT are entirely compatible with results of classical logic: in particular, by assuming classical principles such as the axiom of choice or the law of excluded middle, $\mathbb{R}_c$ and $\mathbb{R}_d$ result in equal types.

Corollary 5.26. *If excluded middle or countable choice holds then $\mathbb{R}_c$ and $\mathbb{R}_d$ are equivalent, hence $\mathbb{R}_c = \mathbb{R}_d$ by univalence.*

Proof. If excluded middle hols, then $(x < y) \to (x < z) + (z < y)$ can be proved: either $x < z$ or $\neg(x < z)$. In the former case we are done, while in the latter we get $z < y$ because $z \leq x < y$. Therefore, we get $c : \mathrm{locator}(x)$ and we can apply Lemma 5.25.
Suppose countable choice holds. The set $\mathcal{S} :\equiv \sum_{(q,r)\in\mathbb{Q}\times\mathbb{Q}} q < r$ is equivalent to $\mathbb{N}$, so we may apply countable choice to the statement that x is located,

$$\prod_{(q,r):\mathcal{S}} \|(q < x) + (x < r)\|$$

Note that $\|(q < x) + (x < r)\|$ is expressible as an existential statement

$$\left\| \sum_{b:\mathbf{2}} (b = 0_{\mathbf{2}} \to q < x) \times (b = 1_{\mathbf{2}} \to x < r) \right\|$$

The curried form of the choice function is then precisely a term of type $\mathrm{locator}(x)$, so we can again apply Lemma 5.25. □

Bibliography

[1] Auke B. Booij. “Extensional constructive real analysis via locators”. In: *MSCS Special Issue on Homotopy Type Theory and Univalent Foundations (to appear)* (2020).

[2] Cyril Cohen et al. “Cubical Type Theory: a constructive interpretation of the univalence axiom”. In: *CoRR* abs/1611.02108 (2016). arXiv: `1611.02108`. URL: `http://arxiv.org/abs/1611.02108`.

[3] Martín Hötzel Escardó and Alex K. Simpson. “A Universal Characterization of the Closed Euclidean Interval.” In: *LICS*. IEEE Computer Society, 2001, pp. 115–125. ISBN: 0-7695-1281-X.

[4] Gaëtan Gilbert. “Formalising Real Numbers in Homotopy Type Theory”. In: *CoRR* abs/1610.05072 (2016). arXiv: `1610.05072`. URL: `http://arxiv.org/abs/1610.05072`.

[5] Martin Hofmann. “Extensional concepts in intensional type theory”. PhD thesis. School of informatics at The University of Edinburgh, 1995.

[6] Chris Kapulkin and Peter LeFanu Lumsdaine. “The Simplicial Model of Univalent Foundations (after Voevodsky)”. In: *Journal of the European Mathematical Society (to appear)* (2018). arXiv: `1211.2851 [math.LO]`.

[7] J. Ledent and F. Wiedijk. “Modeling set theory in homotopy type theory”. In: 2014.

[8] Nuo Li. “Quotient types in type theory”. PhD thesis. University of Nottingham, 2015.

[9] Robert Lubarsky. “On the Cauchy Completeness of the Constructive Cauchy Reals”. In: *Mathematical Logic Quarterly* (2007). arXiv: `1510.00639 [math.LO]`.

[10] Robert Lubarsky and Michael Rathjen. “On the Constructive Dedekind Reals”. In: *Logic and Analysis, 1, No. 2* (2008). arXiv: `1510.00641 [math.LO]`.

[11] Thomas Streitcher Martin Hofmann. “The groupoid interpretation of type theory”. In: *Twenty-five years of constructive type theory*. Ed. by Jan M. Smith Giovanni Sambin. 1998.

[12] Per Martin-Löf. “An Intuitionistic Theory of Types: Predicative Part”. In: *Studies in Logic and the Foundations of Mathematics*. 1975, pp. 73–118.

[13] *Introduction to homotopy type theory*. Carnegie Mellon University. 2018.

[14] Bertrand Russell. “Mathematical Logic as Based on the Theory of Types”. In: *American Journal of Mathematics* (1908).

[15] Kristina Sojakova. “Higher Inductive Types as Homotopy-InitialAlgebras”. Carnegie Mellon University, 2016.

[16] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: `https://homotopytypetheory.org/book`, 2013.