

Equazioni alle Derivate Parziali

EQUAZIONI DEL PRIMO ORDINE

EDP

$$u(x_1, \dots, x_n) \leadsto f(x, u, \nabla u, D^2 u, \dots) = 0$$

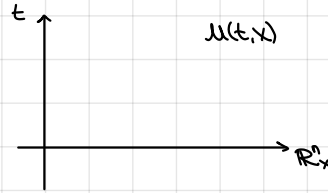
EDO

$$u(t) \leadsto f(t, u, u', \dots, u^{(k)}) = 0$$

Equazione del trasporto

$$(EDP) \begin{cases} \partial_t u + \sum_{j=1}^n \underbrace{C_j(t, x_1, \dots, x_n)}_{\text{funzioni assegnate}} \frac{\partial u}{\partial x_j} = 0 \\ u(0, x_1, \dots, x_n) = u_0(x_1, \dots, x_n) \end{cases}$$

$C(t, x_1, \dots, x_n) \in \mathbb{R}^n$ campo vettoriale



$\leadsto$ sistema di EDO

$$(EDO) \begin{cases} \dot{x}_j(t, x_0) = C_j(t, x(t, x_0)) \\ x_j(0, x_0) = x_{0,j} \end{cases}$$

def. le curve $(t, x(t, x_0)) \in \mathbb{R} \times \mathbb{R}^n$ si chiamano curve caratteristiche.

Caso $n=1$:

$$\begin{cases} \partial_t u + C(x) \frac{\partial u}{\partial x} = 0 \\ u(0, x) = u_0(x) \\ \dot{x}(t, x_0) = C(t, x(t, x_0)) \\ x(0, x_0) = x_0 \end{cases}$$



(Hp) le soluzioni $x(t, x_0)$ sono globali $\forall x_0 \in \mathbb{R}^n$

proposizione Se $u(t, x)$ è soluzione di EDP, allora
 $\frac{d}{dt} u(t, x(t, x_0)) = 0 \quad \forall x_0 \in \mathbb{R}^n$.

DIMOSTRAZIONE

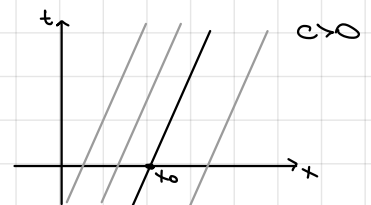
$$\frac{d}{dt} u(t, x(t, x_0)) = \partial_t u(t, x(t, x_0)) + \sum_{j=1}^n \frac{\partial u}{\partial x_j} \dot{x}_j = \left[\partial_t u + \sum_{j=1}^n \frac{\partial u}{\partial x_j} C_j \right](t, x(t, x_0)) = 0 \quad \square$$

corollario $u(t, x(t, x_0)) = u(0, x(0, x_0)) = u(0, x_0) = u_0(x_0) \quad \forall t, x_0$
 $u(t, x) = u_0(x_0) \quad \text{dove } x(t, x_0) = x$

esempio Risolvere $\begin{cases} \partial_t u + C \partial_x u = 0 \\ u(0, x) = u_0(x) \end{cases} \quad C \in \mathbb{R}$

$$x(t, x_0) : \begin{cases} \dot{x}(t, x_0) = C \\ x(0, x_0) = x_0 \end{cases} \Rightarrow \begin{aligned} x(t, x_0) &= ct + x_0 \\ x_0 &= x - ct \end{aligned}$$

Quindi $u(t, x) = u_0(x - ct)$



esempio
$$\begin{cases} \partial_t u + x \partial_x u = 0 \\ u(0, x) = u_0(x) \end{cases}$$

$$\begin{cases} \dot{x}(t, x_0) = x(t, x_0) \Rightarrow x(t, x_0) = x_0 e^t \\ x(0, x_0) = x_0 \end{cases}$$

Quindi $u(t, x) = u_0(x e^{-t})$

esempio
$$\begin{cases} \partial_t u + t \partial_x u = 0 \\ u(0, x) = u_0(x) \end{cases}$$

$$\begin{cases} \dot{x}(t, x_0) = t \Rightarrow x(t, x_0) = \frac{t^2}{2} + x_0 \\ x(0, x_0) = x_0 \end{cases}$$

Quindi $u(t, x) = u_0(x - \frac{1}{2}t^2)$

Leggi di conservazione

$$\begin{cases} \partial_t u + \partial_x \Phi(u) = 0 & (t, x) \in \mathbb{R} \times \mathbb{R} \quad \text{con } \Phi: \mathbb{R} \rightarrow \mathbb{R} \\ u(0, x) = u_0(x) \end{cases}$$

oppure $\partial_t u + \Phi'(u) \partial_x u = 0$

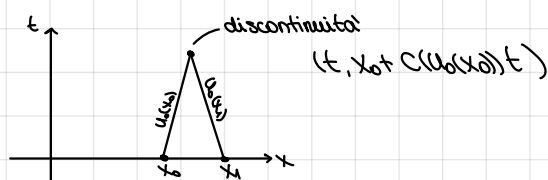
Ponendo $c(u) = \Phi'(u)$, abbiamo che equivale a

$$\begin{cases} \partial_t u + c(u) \partial_x u = 0 \\ u(0, x) = u_0(x) \end{cases}$$

esempio $\Phi(u) = \frac{u^2}{2}$
 $\partial_t u + u \partial_x u = 0$ equazione di Burgers

$$\begin{cases} \dot{x}(t, x_0) = c(u(t, x(t, x_0))) \\ x(0, x_0) = x_0 \end{cases} \Rightarrow u(t, x(t, x_0)) = u_0(x_0)$$

$$\Rightarrow \begin{cases} \dot{x}(t, x_0) = c(u_0(x_0)) \\ x(0, x_0) = x_0 \end{cases} \Rightarrow x(t, x_0) = x_0 + c(u_0(x_0))t$$



Oss Se $c(u) = u$ e $u_0(x) \uparrow$, le caratteristiche non si intersecano per $t \geq 0$

teorema

Se $u_0 \in C^1$, $u_0 \in L^\infty$, $u'_0 \in L^\infty$, $f(u) \in C^2$, allora
 $\exists!$ soluzione $u(t, x)$ definita su $[0, T) \times \mathbb{R}$
 Inoltre possiamo prendere $T = \frac{1}{\max |C'(u_0(x_0)) u'_0(x_0)|}$

DIMOSTRAZIONE

Abbiamo provato che le curve caratteristiche sono $(t, x_0 + tC(u_0(x_0)))$

$$\Rightarrow u(t, x_0 + tC(u_0(x_0))) = u_0(x_0)$$

Il teorema è provato se dimostro che

$$x_0 \xrightarrow{h_t} x_0 + tC(u_0(x_0)) \text{ è invertibile per } t \ll 1$$

$$u(t, h_t(x_0)) = u_0(x_0) \rightarrow u(t, x) = u_0(h_t^{-1}(x))$$

$$h'_t(x_0) = 1 + t \underbrace{C'(u_0(x_0)) u'_0(x_0)}_{\text{limitato}} > 0 \text{ per } t \ll 1, \quad h_t(x_0) \xrightarrow{x_0 \rightarrow \pm\infty} \pm\infty$$

□

In generale

$$f(x, u, \nabla u) = 0 \quad (\text{EVANS, EDP})$$

$$\partial_t u = f(x, \nabla u) \quad \text{equazione di Hamilton-Jacobi}$$

Consideriamo il problema con termine forzante:

$$\begin{cases} \partial_t u + Au = F(t, x) \\ u(0, x) = f(x) \end{cases}$$

esempio $A = \sum_{j=1}^n g_j(x) \frac{\partial}{\partial x_j}$, $A = -\Delta$

Per le CDE:

$$u' + au = f(t) \quad \text{variazione delle costanti}$$

principio di Duhamel (o variazione delle costanti)

$$\begin{cases} \partial_t u + Au = 0 \\ u(0, x) = f(x) \end{cases}$$

$$S_t: f(x) \mapsto u(t, x)$$

$$S_0 = \text{Id}, \quad S_{t+s} = S_t \circ S_s \quad \forall t, s$$

esempio Se $A = \sum_{j=1}^n g_j(x) \frac{\partial}{\partial x_j}$, $S_t(f(x)) = f(\Phi_t(x))$
 dove $\begin{cases} \dot{\Phi}_t(x) = C(\Phi_t(x)) \\ \Phi_0(x) = x \end{cases}$

teorema

Sia A un operatore indipendente dal tempo e

sia S_t l'operatore di soluzione al tempo t per il problema di Cauchy.

Allora

$$\begin{cases} \partial_t u + Au = F(t, x) \\ u(0, x) = f(x) \end{cases}$$

$$\text{La soluzione } u(t, x) = S_t(f(x)) + \int_0^t S_{t-\tau} [F(\tau, x)] d\tau$$

DIMOSTRAZIONE

Cerchiamo $u(t, x) = S_t(f(x)) + w(t, x)$

$$\text{con } (*) \begin{cases} \partial_t w(t, x) + A w(t, x) = F(t, x) \\ w(0, x) = 0 \end{cases}$$

Cerchiamo $w(t, x)$ del tipo

$$S_t[v(t, x)] = w(t, x)$$

Imponiamo che w risolva $(*)$

$$\begin{cases} v(0, x) = 0 \\ \partial_t (S_t[v(t, x)]) + A S_t[v(t, x)] = F(t, x) \end{cases}$$

Ora $\partial_t S_t[f] + A S_t[f] = 0$, quindi $\partial_t S_t = -A S_t$

$$S_{t+h}[v(t+h, x)] - S_t[v(t+h, x)] = (S_{t+h} - S_t)[v(t+h, x)] + S_t[v(t+h, x)] - S_t[v(t, x)] =$$

$$\text{Quindi } \partial_t (S_t[v(t, x)]) = \partial_t S_t[v(t, x)] + S_t[\partial_t v(t, x)] \stackrel{h}{=} -A S_t[v(t, x)] + S_t[\partial_t v(t, x)]$$

$$\Rightarrow \partial_t (S_t[v(t, x)]) + A S_t[v(t, x)] = S_t[\partial_t v(t, x)] = F(t, x)$$

Quindi $v(t, x)$ deve soddisfare:

$$\begin{cases} v(0, x) = 0 \\ S_t[\partial_t v(t, x)] = F(t, x) \Rightarrow \partial_t v(t, x) = S_{-t}[F(t, x)] \end{cases}$$

$$\Rightarrow v(t, x) = \int_0^t S_{-t} [F(\tau, x)] d\tau$$

$$\Rightarrow w(t, x) = S_t \left[\int_0^t S_{-\tau} [F(\tau, x)] d\tau \right] = \int_0^t S_{t-\tau} [F(\tau, x)] d\tau$$

□

esempio Risolvere $\begin{cases} \partial_t u + \partial_x u = F(t, x) \\ u(0, x) = f(x) \end{cases}$

$$\text{Risolvo } \begin{cases} x(t, x_0) = 1 \\ x(0, x_0) = x_0 \end{cases} \Rightarrow x(t, x_0) = x_0 + t$$

$$\text{Quindi } S_t[f(x)] = f(x-t)$$

Per la parte di Duhamel abbiamo

$$\int_0^t S_{t-\tau} [F(\tau, x)] d\tau = \int_0^t F(\tau, x-(t-\tau)) d\tau$$

EQUAZIONE DELLE ONDE

Equazione di D'Alembert (corda vibrante, onde)

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & (t, x) \in \mathbb{R} \times \mathbb{R} \\ u(0, x) = f(x) \\ \partial_t u(0, x) = g(x) \end{cases}$$

$$\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x) \circ (\partial_t + \partial_x)$$

$$= (\partial_t + \partial_x) \circ (\partial_t - \partial_x)$$

$$\text{Ker}(\partial_t + \partial_x) = \text{Ker}(\partial_t^2 - \partial_x^2)$$

$$\text{Ker}(\partial_t - \partial_x) = \text{Ker}(\partial_t^2 - \partial_x^2)$$

$$\forall \varphi(x), \psi(x)$$

$$u(t, x) = \varphi(x-t) + \psi(x+t) \text{ è soluzione di } \partial_t^2 - \partial_x^2 = 0$$

Costruiamo φ e ψ tali che

$$\begin{cases} \varphi(x) + \psi(x) = f(x) \\ -\varphi'(x) + \psi'(x) = g(x) \end{cases} \Rightarrow \begin{cases} \varphi' + \psi' = f' \\ -\varphi' + \psi' = g \end{cases} \Rightarrow \begin{cases} \psi' = \frac{f' + g}{2} \\ \varphi' = \frac{f' - g}{2} \end{cases}$$

$$\text{Quindi} \quad \begin{cases} \psi(x) = \psi(0) + \frac{1}{2} \int_0^x f' + g \\ \varphi(x) = \varphi(0) + \frac{1}{2} \int_0^x f' - g \end{cases} \Rightarrow \begin{cases} \psi(x) = \psi(0) + \frac{1}{2} f(x) - \frac{1}{2} f(0) + \frac{1}{2} \int_0^x g \\ \varphi(x) = \varphi(0) + \frac{1}{2} f(x) - \frac{1}{2} f(0) - \frac{1}{2} \int_0^x g \end{cases}$$

$$\varphi(x-t) + \psi(x+t) = \cancel{\varphi(0)} + \frac{1}{2} f(x-t) - \frac{1}{2} f(0) - \frac{1}{2} \int_0^{x-t} g + \cancel{\psi(0)} + \frac{1}{2} f(x+t) - \frac{1}{2} f(0) + \frac{1}{2} \int_0^{x+t} g$$

$$= \frac{1}{2} (f(x+t) + f(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} g(\tau) d\tau$$

Se avessi due soluzioni u_1, u_2 , allora

$$\partial_t^2 (u_1 - u_2) - \partial_x^2 (u_1 - u_2) = 0$$

$$(u_1 - u_2)(0, x) = 0$$

$$\partial_t (u_1 - u_2)(0, x) = 0$$

$$\partial_t^2 w - \partial_x^2 w = 0$$

$$w(0, x) = 0$$

$$\partial_t w(0, x) = 0$$

$$\stackrel{?}{\Rightarrow} w = 0$$

Sì, lo dimostreremo in ogni dimensione (è falso per l'eq. del calore)
l'eq. è stabile per piccole perturbazioni.

Duhamel per le onde

$$\partial_t u + Au = F$$

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = F(t, x) \\ u(0, x) = f(x) \\ \partial_t u(0, x) = g(x) \end{cases}$$

$$\begin{cases} \partial_t^2 w - \partial_x^2 w = F(t, x) \\ w(0, x) = 0 \\ \partial_t w(0, x) = 0 \end{cases}$$

soluzione particolare

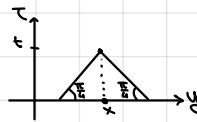
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = F(t, x) \\ u(0, x) = f(x) \\ \partial_t u(0, x) = g(x) \end{cases}$$

$$\longleftrightarrow \begin{cases} \partial_t u = 0 \\ \partial_t u - \partial_x^2 u = F(t, x) \\ u(0, x) = f(x) \\ u(0, x) = g(x) \end{cases}$$

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} - \underbrace{\begin{pmatrix} 0 & \text{Id} \\ \partial_x^2 & 0 \end{pmatrix}}_A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ F(t, x) \end{pmatrix}$$

$$S_t(f(x), g(x)) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy$$

$$\int_0^t S_{t-\tau}(0, F(\tau, x)) d\tau = \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} F(\tau, y) dy d\tau$$



proposizione

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = F(t, x) \\ u(0, x) = f(x) \\ \partial_t u(0, x) = g(x) \end{cases}$$

$$\text{Allora } u(t, x) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy + \frac{1}{2} \iint_{T(t, x)} F(\tau, y) d\tau dy$$

• Equazione delle onde con dato al bordo sull'asse t

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & (t, x) \in \mathbb{R} \times \mathbb{R}_+ \\ u(0, x) = f(x) & \text{con } f(0) = 0 \\ \partial_t u(0, x) = g(x) & g(0) = 0 \\ u(t, 0) = 0 \end{cases}$$

Considero $\tilde{f}, \tilde{g}$ estensioni dispari

$$\tilde{f}(x) = \begin{cases} f(x) & \text{se } x \geq 0 \\ -f(-x) & \text{se } x < 0 \end{cases}, \quad \tilde{g}(x) = \begin{cases} g(x) & \text{se } x \geq 0 \\ -g(-x) & \text{se } x < 0 \end{cases}$$

$$\begin{cases} \partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} = 0 & (t, x) \in \mathbb{R} \times \mathbb{R} \\ \tilde{u}(0, x) = \tilde{f}(x) \\ \partial_t \tilde{u}(0, x) = \tilde{g}(x) \end{cases}$$

$$\Rightarrow \tilde{u}(t, x) = \frac{\tilde{f}(x+t) + \tilde{f}(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \tilde{g}(y) dy$$

$$\tilde{u}|_{\mathbb{R} \times (0, +\infty)} \text{ soddisfa } \tilde{u}|_{\mathbb{R} \times (0, +\infty)}(t, 0) = 0$$

unicità per l'equazione delle onde

$$(*) \begin{cases} \partial_t^2 u - \Delta_x u = 0 & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0, x) = f(x) \\ \partial_t u(0, x) = g(x) \end{cases} \quad \text{con } u \in C^2(\mathbb{R} \times \mathbb{R}^n)$$

teorema (*) ammette al più una soluzione.

DIMOSTRAZIONE

Se per assurdo u_1, u_2 fossero due soluzioni, allora $w = u_1 - u_2$ soddisfa

$$(**) \begin{cases} \partial_t^2 w - \Delta_x w = 0 \\ w(0, x) = 0 \\ \partial_t w(0, x) = 0 \end{cases}$$

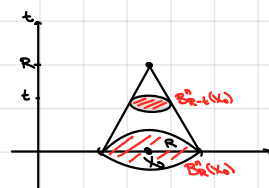
Concludo se $w \equiv 0$

**Proposizione:
monotonia
dell'energia locale**

Sia u soluzione di (*) e siano $R > 0, x_0 \in \mathbb{R}^3$. Allora

$$\int_{B_{R-t}(x_0)} e(u(t, x)) dx \leq \int_{B_R(x_0)} |\nabla_x \varphi|^2 + |\varphi|^2 dx$$

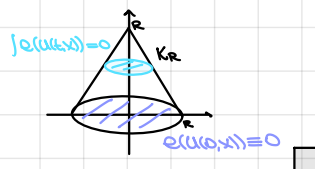
dove $e(u(t, x)) = (\partial_t u(t, x))^2 + |\nabla_x u(t, x)|^2 = |\nabla_{t,x} u(t, x)|^2$



Dico che se w soddisfa (**), allora $w|_{K_R} \equiv 0 \quad \forall R$

$$\Rightarrow \nabla_{t,x} w(t, x) = 0 \text{ su } K_R \Rightarrow w \text{ costante su } K_R$$

$$\text{ma } w(0, x) = 0 \Rightarrow w \equiv 0 \text{ su } K_R$$



DIMOSTRAZIONE

Strategia 1: $t \mapsto \int_{B_{R-t}(x_0)} |\nabla_{t,x} u|^2 dx$
 $\dot{e}(t) \leq 0$

Strategia 2: Moltiplico (*) per $\partial_t u$

$$\partial_t u (\partial_t^2 u - \Delta_x u) = 0$$

$$\frac{1}{2} \partial_t (\partial_t u)^2 - \sum_{j=1}^3 \partial_t u \partial_{x_j}^2 u = 0$$

$$\frac{1}{2} \partial_t (\partial_t u)^2 - \sum_{j=1}^3 \partial_{x_j} (\partial_t u \partial_{x_j} u) + \sum_{j=1}^3 \partial_t \partial_{x_j} u \partial_{x_j} u = \frac{1}{2} \partial_t (\partial_t u)^2 - \sum_{j=1}^3 \partial_{x_j} (\partial_t u \partial_{x_j} u) + \frac{1}{2} \sum_{j=1}^3 \partial_t (\partial_{x_j} u)^2$$

Quindi $\text{div}_{x,t} \vec{F}(x, t) = 0$ dove $\vec{F}(x, t) = (-\partial_t u \partial_{x_1} u, \dots, -\partial_t u \partial_{x_n} u, \frac{1}{2} |\nabla_{t,x} u|^2)$

Prendo il dominio K (porzione di cono tra $B_R(x_0)$ e $B_{R-t}(x_0)$)

Per il Teorema della divergenza

$$0 = \int_K \text{div} \vec{F} dx dt = \int_{\partial K} \vec{F} \cdot \vec{\nu}_{\text{ext}} ds$$

Ora $\partial K = B_t \cup B_2 \cup L$, B_1 base inferiore, B_2 base superiore

$$\int_{B_1} \vec{F} \cdot \vec{\nu}_{\text{ext}} ds = -\frac{1}{2} \int_{B_R(x_0)} e(u(0, x)) dx = -\frac{1}{2} \int_{B_R(x_0)} (|\nabla_x \varphi|^2 + |\varphi|^2) dx$$

$$\int_{B_2} \vec{F} \cdot \vec{\nu}_{\text{ext}} ds = \frac{1}{2} \int_{B_{R-t}(x_0)} |\nabla_{t,x} u(t, x)|^2 dx$$

$$-\frac{1}{2} \int_{B_R(x_0)} |\nabla_x \varphi|^2 + |\varphi|^2 dx + \frac{1}{2} \int_{B_{R-t}(x_0)} |\nabla_{t,x} u(t, x)|^2 dx + \int_L \vec{F} \cdot \vec{\nu}_{\text{ext}} ds = 0$$

Per concludere, basta provare che $\int_L \vec{F} \cdot \vec{\nu}_{\text{ext}} ds \geq 0$

Mostriamo che $\vec{F} \cdot \vec{\nu}_{\text{ext}} \geq 0$ su L



$$\vec{\nu}_{\text{ext}} = \left(\frac{1}{\sqrt{2}} \frac{x - x_0}{\|x - x_0\|}, \frac{1}{\sqrt{2}} \right)$$

$$\{ \|x - x_0\| + t - R = 0 \} \quad \nabla_{t,x} G = \dots$$

$$\sqrt{2} \vec{F} \cdot \vec{\nu}_{\text{ext}}(t, x) = -\partial_t u \partial_{x_1} u \frac{(x - x_0)_1}{\|x - x_0\|} - \dots - \partial_t u \partial_{x_n} u \frac{(x - x_0)_n}{\|x - x_0\|} + \frac{1}{2} |\nabla_{t,x} u|^2 =$$

$$= -\partial_t u (\nabla_x u \cdot \frac{(x - x_0)}{\|x - x_0\|}) + \frac{1}{2} \|\nabla_{t,x} u\|^2$$

$$\partial_t u (\nabla_x u \cdot \frac{(x - x_0)}{\|x - x_0\|}) \stackrel{(*)}{\leq} |\partial_t u| \cdot |\nabla_x u| \leq \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} \|\nabla_x u\|^2$$

□

onde 3d: formula di Kirchhoff

$$(*) \begin{cases} \partial_t^2 u - \Delta_x u = 0 & (t, x) \in \mathbb{R} \times \mathbb{R}^3 \\ u(0, x) = f(x) \\ \partial_t u(0, x) = g(x) \end{cases}$$

teorema La funzione $u(t, x) = t \int_{S_t(x)} g \, d\sigma + \partial_t (t \int_{S_t(x)} f \, d\sigma)$
dove $S_t(x) = \{y \in \mathbb{R}^n : \|y - x\| = t\}$
è soluzione di (*).

Oss Basta provare la formula per $f=0$.

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = 0 \\ u_1(0, x) = 0 \\ \partial_t u_1(0, x) = g(x) \end{cases} \quad \begin{cases} \partial_t^2 u_2 - \Delta u_2 = 0 \\ u_2(0, x) = 0 \\ \partial_t u_2(0, x) = f(x) \end{cases}$$

Allora $\partial_t u_2$ risolve

$$\begin{cases} \partial_t^2 (\partial_t u_2) - \Delta (\partial_t u_2) = 0 \\ \partial_t u_2(0, x) = f(x) \\ \partial_t (\partial_t u_2(0, x)) = \Delta u_2(0, x) = 0 \end{cases}$$

Quindi $u(t, x) = u_1(t, x) + \partial_t u_2(t, x)$ risolve (*)

Lemma Sia $h(x) \in C^1$:
 $\frac{d}{dr} \int_{B_r(x_0)} h(x) \, dx = \int_{S_r(x_0)} h(y) \, d\sigma$

DIMOSTRAZIONE

wlog $x_0 = 0$

$$\int_{B_r(0)} h(x) \, dx = \int_0^r \left(\int_{S_\lambda(0)} h(y) \, d\sigma \right) d\lambda$$

Caso particolare della formula di coarea ($g = |x|$):

$$\text{date } h, g, \quad \int_{\mathbb{R}^n} h(x) |g(x)| \, dx = \int_{-\infty}^{\infty} \left(\int_{|g|=\lambda} h \, d\sigma \right) d\lambda$$

DIMOSTRAZIONE (2)

$$\int_{B_r} h(x) \, dx = \int_{B_r} h(r y) \, r^n \, dy, \text{ quindi}$$

$$\begin{aligned} \frac{d}{dr} \int_{B_r} h(x) \, dx &= n r^{n-1} \int_{B_r} h(r y) \, dy + \int_{B_r} y \cdot \nabla h(r y) \, r^n \, dy = \\ &= n r^{n-1} \int_{B_r} h(r y) \, dy + \frac{1}{r} \int_{B_r} y \cdot \nabla h(y) \, dy = \frac{n}{r} \int_{B_r} h(y) \, dy + \frac{1}{r} \int_{B_r} \operatorname{div}(y h(y)) \, dy - \frac{n}{r} \int_{B_r} h(y) \, dy = \\ &= \frac{1}{r} \int_{\partial B_r} y h(y) \cdot \nu_{\text{ext}} = \frac{1}{r} \int_{\partial B_r} \frac{y}{r} \cdot y h(y) \, d\sigma = \int_{\partial B_r} h(y) \, d\sigma \end{aligned}$$

□

Lemma di Darboux

Dato $h(x)$, consideriamo $M_r h(x) = \int_{S_r(x)} h(y) \, d\sigma_y$, $x \in \mathbb{R}^3$, $r \in \mathbb{R}^+$.
 $\partial_r^2 M_r h(x) + \frac{2}{r} \partial_r M_r h(x) = M_r(\Delta h)(x)$
($\partial_r^2 + \frac{2}{r} \partial_r$ è il Laplaciano per funzioni radiali).

DIMOSTRAZIONE

$$r^2 \partial_r^2 M_r h + 2r \partial_r M_r h = r^2 M_r(\Delta h)$$

$$\partial_r (r^2 \partial_r M_r h)$$

wlog $x=0$

$$\begin{aligned} \partial_r \left(\int_{S_r} h(y) \, d\sigma_y \right) &\cong \partial_r \left(\frac{1}{r^2} \int_{B_r} h(y) \, d\sigma_y \right) = \partial_r \left(\int_{B_r} h(r y) \, d\sigma_y \right) = \\ &= \int_{B_r} y \cdot \nabla h(r y) \, d\sigma_y = \frac{1}{r^2} \int_{B_r} y \cdot \nabla h(y) \, d\sigma_y = \frac{1}{r^2} \int_{B_r} \frac{y}{r} \cdot \nabla h(y) \, d\sigma_y = \frac{1}{r^2} \int_{B_r} \Delta h(y) \, dy \\ \partial_r (r^2 \partial_r M_r h) &= \partial_r \left(\int_{B_r} \Delta h(y) \, dy \right) = \int_{B_r} \Delta h(y) \, d\sigma_y \cong r^2 \int_{S_r} \Delta h(y) \, dy \end{aligned}$$

□

DIMOSTRAZIONE

$U(t, x) = t \int_{x-c(t)}^{x+c(t)} g(y) dy$ è soluzione di (*)

$$U(t, x) \rightsquigarrow \int_{x-c(t)}^{x+c(t)} U(t, y) dy \xrightarrow{r \rightarrow 0} U(t, x)$$

$U_x(t, r)$ con $r > 0$

$$U_x(t, r) = M_r U(t, x)$$

$$\partial_t^2 U - \Delta U = 0 \longrightarrow M_r \partial_t^2 U = M_r \Delta U$$

$$\partial_t^2 U_x(t, r) = \partial_t^2 M_r U \quad \partial_r^2 M_r U + \frac{2}{r} M_r U = \partial_r^2 U_x(t, r) + \frac{2}{r} U_x(t, r)$$

$$\begin{cases} \partial_t^2 U_x(t, r) - \partial_r^2 U_x(t, r) - \frac{2}{r} U_x(t, r) = 0 & (t, r) \in \mathbb{R} \times [0, +\infty) \\ U_x(0, r) = 0 \\ \partial_t U_x(0, r) = M_r g(x) \end{cases}$$

$$W_x(t, r) = r U_x(t, r)$$

$$W_x(t, 0) = 0$$

Moltiplico l'eq. di $U_x(t, r)$ per r :

$$\partial_t^2 (r U_x(t, r)) - r \partial_r^2 U_x(t, r) - 2 \partial_r U_x(t, r) = 0$$

$$\partial_t^2 W_x(t, r) - \partial_r^2 W_x(t, r) = 0$$

Quindi

$$\begin{cases} \partial_t^2 W_x(t, r) - \partial_r^2 W_x(t, r) = 0 \\ W_x(t, 0) = 0 \\ W_x(0, r) = 0 \\ \partial_t W_x(0, r) = r M_r g(x) \end{cases}$$

$$\widetilde{r M_r g(x)} = \begin{cases} r M_r g(x) & \text{se } r > 0 \\ r M_r g(x) & \text{se } r < 0 \end{cases}$$

$$W_x(t, r) = \frac{1}{2} \int_{r-t}^{r+t} \widetilde{r M_r g(x)} ds \rightsquigarrow U_x(t, r) = \frac{1}{2r} \int_{r-t}^{r+t} \widetilde{r M_r g(x)} ds$$

Siccome interessa $r \rightarrow 0$, posso supporre $r < t$, e quindi

$$U_x(t, r) = \frac{1}{2r} \int_{r-t}^{r+t} \widetilde{r M_r g(x)} ds = \frac{1}{2r} \underbrace{\int_{r-t}^{t-r} \widetilde{r M_r g(x)} ds}_{=0} + \frac{1}{2r} \int_{t-r}^{t+r} \widetilde{r M_r g(x)} ds = \frac{1}{2r} \int_{t-r}^{t+r} M_s g(x) ds$$

$$U_x(t, r) = \int_{t-r}^{t+r} M_s g(x) ds \xrightarrow{r \rightarrow 0} t M_t g(x) = t \left(\int_{x-c(t)}^{x+c(t)} g(y) dy \right)$$

□

onde 2d: metodo di discesa

$$\begin{cases} \partial_t^2 u - \partial_x^2 u - \partial_y^2 u = 0 & (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \\ u(0, x, y) = 0 \\ \partial_t u(0, x, y) = g(x, y) \end{cases}$$

$w(t, x, y, z) = u(t, x, y)$ soddisfa

$$\begin{cases} \partial_t^2 w - \partial_x^2 w - \partial_y^2 w - \partial_z^2 w = 0 \\ w(0, x, y, z) = 0 \\ \partial_t w(0, x, y, z) = g(x, y) \end{cases}$$

Applico Kirchhoff: $w(t, x, y, z) = t \left(\int_{S_{x, y, 0}^+(t)} g(u, v) d\sigma \right)$

$$= t \frac{1}{4\pi t^2} \int_{S_{x, y, 0}^+(t)} g(u, v) d\sigma = \frac{1}{4\pi t} \int_{S_{x, y, 0}^+(t)} g(u, v) d\sigma$$

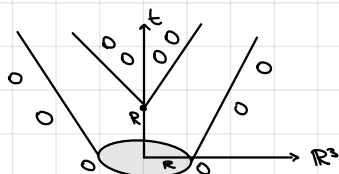
$$w(t, x, y, z) = \frac{1}{2\pi t} \int_{S_{x, y, 0}^+(t)} g(u, v) d\sigma$$

$B_t^+(x, y) \xrightarrow{(u, v) \mapsto (u, v, \sqrt{t^2 - (x-u)^2 - (y-v)^2})} S_{x, y, 0}^+(t)$

$$= \frac{1}{2\pi t} \int_{B_t^+(x, y)} g(u, v) \sqrt{1 + |\nabla_{u, v} \varphi|^2} du dv = \frac{1}{2\pi t} \int_{B_t^+(x, y)} g(u, v) \sqrt{1 + \frac{(x-u)^2}{t^2 - (x-u)^2 - (y-v)^2} + \frac{(y-v)^2}{t^2 - (x-u)^2 - (y-v)^2}} du dv =$$

$$= \frac{1}{2\pi t} \int_{B_t^+(x, y)} g(u, v) \sqrt{\frac{t^2}{t^2 - (x-u)^2 - (y-v)^2}} du dv = \frac{1}{2\pi} \int_{B_t^+(x, y)} g(u, v) \frac{du dv}{\sqrt{t^2 - (x-u)^2 - (y-v)^2}} = \frac{t^2}{2} \int_{B_{x, y}(t)} \frac{g(u, v)}{\sqrt{t^2 - (x-u)^2 - (y-v)^2}} du dv$$

Fenomeno di Huygens



FUNZIONI ARMONICHE

DEF. u è armonica se $\Delta u = 0$, con $\Delta = \sum_{j=1}^n \partial_{x_j}^2$

Vogliamo studiare il problema di Dirichlet

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

Il problema di Cauchy

$$\begin{cases} \partial_t^2 u + \partial_x^2 u = 0 & (t, x) \in \mathbb{R} \times \mathbb{R} \\ u(0, x) = f(x) \\ \partial_t u(0, x) = g(x) \end{cases}$$

è mal posto nel senso di Hadamard

proposizione $\forall \varepsilon > 0 \exists f_\varepsilon(x), g_\varepsilon(x), u_\varepsilon(t, x), t_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} +\infty$ t.c.

$$(i) \begin{cases} \partial_t^2 u_\varepsilon + \partial_x^2 u_\varepsilon = 0 \\ u_\varepsilon(0, x) = f_\varepsilon(x) \\ \partial_t u_\varepsilon(0, x) = g_\varepsilon(x) \end{cases}$$

$$(ii) \|f_\varepsilon\|_{\infty} + \|g_\varepsilon\|_{\infty} \xrightarrow{\varepsilon \rightarrow 0} 0$$

$$(iii) |u_\varepsilon(t_\varepsilon, 0)| \rightarrow +\infty$$

DIMOSTRAZIONE

Separazione delle variabili: cerchiamo $u_\varepsilon(t, x) = h_\varepsilon(x) \varrho_\varepsilon(t)$

$$h_\varepsilon''(x) \varrho_\varepsilon(t) + h_\varepsilon(x) \varrho_\varepsilon''(t) = 0, \text{ pongo } \begin{cases} \varrho_\varepsilon''(t) = \lambda_\varepsilon \varrho_\varepsilon(t) \\ h_\varepsilon''(x) = -\lambda_\varepsilon h_\varepsilon(x) \end{cases} \quad \lambda_\varepsilon > 0$$

$$\varrho_\varepsilon(t) (h_\varepsilon''(x) + \lambda_\varepsilon h_\varepsilon(x)) = 0 \implies \begin{cases} \varrho_\varepsilon''(t) = \lambda_\varepsilon \varrho_\varepsilon(t) \\ h_\varepsilon''(x) = -\lambda_\varepsilon h_\varepsilon(x) \end{cases}$$

$$\varrho_\varepsilon(t) = a_\varepsilon e^{\sqrt{\lambda_\varepsilon} t} + b_\varepsilon e^{-\sqrt{\lambda_\varepsilon} t}$$

$$h_\varepsilon(x) = c_\varepsilon \sin(\sqrt{\lambda_\varepsilon} x) + d_\varepsilon \cos(\sqrt{\lambda_\varepsilon} x)$$

$$c_\varepsilon = b_\varepsilon = 0, \quad a_\varepsilon = 1, d_\varepsilon = \varepsilon^2$$

$$u_\varepsilon(t, x) = \varepsilon^2 e^{\sqrt{\lambda_\varepsilon} t} \cos(\sqrt{\lambda_\varepsilon} x)$$

$$\varepsilon^2 \sqrt{\lambda_\varepsilon} \rightarrow 0, \text{ basta } \lambda_\varepsilon = \frac{1}{\varepsilon^2}$$

$$\text{Quindi } u_\varepsilon(t, x) = \varepsilon^2 e^{\frac{t}{\varepsilon}} \cos\left(\frac{x}{\varepsilon}\right)$$

$$\text{Per } t_\varepsilon = \sqrt{\varepsilon}, \quad u_\varepsilon(t_\varepsilon, 0) = \varepsilon^2 e^{\frac{1}{\sqrt{\varepsilon}}} \xrightarrow{\varepsilon \rightarrow 0} +\infty$$

□

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f \end{cases} \quad \text{con } u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \\ f \in C^0(\partial\Omega)$$

unicità

$$0 = \int_\Omega \Delta u \, u = - \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \operatorname{div}(u \nabla u) = - \int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u \partial_\nu u \, ds = 0 \text{ perché } u|_{\partial\Omega} = 0$$

C'è unicità se $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ (unicità condizionata)

$$\begin{cases} \Delta u_1 = 0 \\ u_1|_{\partial\Omega} = f \\ \frac{\partial u_1}{\partial \nu}|_{\partial\Omega} = g_1 \end{cases} \quad \begin{cases} \Delta u_2 = 0 \\ u_2|_{\partial\Omega} = f \\ \frac{\partial u_2}{\partial \nu}|_{\partial\Omega} = g_2 \end{cases} \implies \begin{cases} \Delta(u_1 - u_2) = 0 \\ (u_1 - u_2)|_{\partial\Omega} = 0 \end{cases} \implies u_1 = u_2 \quad \text{se } g_1 \neq g_2$$

non ha speranza di avere soluzione

Controesempio di Hadamard

$\exists u(x,y), u: B^2(1) \rightarrow \mathbb{R}$ t.c. $\Delta u = 0$ su B , $u \in C(\bar{B})$, $u' \notin L^2(B) \Rightarrow u \notin C^1(\bar{\Omega})$

Sia $f(z)$ olomorfa su B t.c. $f \in C(\bar{B})$, $\|f'(z)\|_{L^2(B)} = +\infty$

f olomorfa $\Rightarrow f$ armonica

perché per le eq. di Cauchy-Riemann $\frac{\partial}{\partial x} f + i \frac{\partial}{\partial y} f = 0$

$$f(z) = \sum_n \frac{z^n}{n^2} : \sum \sup_B \left| \frac{z^n}{n^2} \right| = \sum \frac{1}{n^2} < +\infty \Rightarrow f \in C(\bar{B})$$

$$f'(z) = \sum_n n! \frac{z^{n-1}}{n^2} \xRightarrow{\text{Parseval}} \|f'(z)\|_{L^2}^2 = \sum_n \left\| \frac{n!}{n^2} z^{n-1} \right\|_{L^2}^2$$

sono ortogonali in $L^2(B)$

$$z^h \perp_{L^2} z^k \text{ se } h \neq k (h > k) : \int_B z^h \bar{z}^k = \int_B |z|^{2h} z^{h-k} = \int_0^{2\pi} \int_0^1 r^{2h} r^{h-k} e^{i(h-k)\theta} r dr d\theta = 0$$

$$\sum_n \left(\frac{n!}{n^2} \right)^2 \int_B |z|^{2(n-1)} dz \simeq \sum_n \left(\frac{n!}{n^2} \right)^2 \int_0^1 r^{2(n-1)} r dr \simeq \sum_n \left(\frac{n!}{n^2} \right)^2 \int_0^1 r^{2n-1} dr \simeq \sum_n \frac{(n!)^2}{n^4} \cdot \frac{1}{2n!} = +\infty$$

teorema:
principio della
media

Sia $u \in C^2(\Omega)$ con $\Delta u = 0$. Allora

$$(i) u(x) = \int_{S(x,r)} u(y) dy \quad \forall x \in \Omega, B(x,r) \subset \Omega$$

$$(ii) u(x) = \int_{B(x,r)} \Delta u(y) dy \quad \forall x \in \Omega, B(x,r) \subset \Omega$$

DIMOSTRAZIONE

(i) vale per $x=0$

$$\frac{d}{dr} \int_{S(0,r)} u(y) dy = 0 \Rightarrow \int_{S(0,r)} u(y) dy = \text{cost} = u(0)$$

$$\int_{S(0,r)} u(y) dy = c_0 \cdot \frac{1}{r^{n-1}} \int_{S(0,r)} u(y) dy = c_0 \cdot \frac{1}{r^{n-1}} \int_{S(0,r)} u(r y) r^{n-1} dy$$

$$\frac{d}{dr} \int_{S(0,r)} u(y) dy = c_0 \frac{d}{dr} \int_{S(0,r)} u(r y) dy = c_0 \int_{S(0,r)} \nabla u(r y) \cdot y dy = \frac{c_0}{r^n} \int_{S(0,r)} y \cdot \nabla u(y) dy =$$

$$= \frac{c_0}{r^n} \int_{S(0,r)} \nabla u \cdot \nabla u dy = \frac{c_0}{r^n} \int_{B(0,r)} \Delta u dy = 0$$

$$(i) \Rightarrow (ii) \quad \int_{B(0,r)} \Delta u(y) dy = \int_0^r \int_{S(0,s)} \Delta u dy ds = \int_0^r u(0) |S(0,s)| ds = u(0) \int_0^r \omega_{n-1} s^{n-1} ds =$$

$$= u(0) \omega_{n-1} \frac{r^n}{n} = u(0) |B(0,r)|$$

□

teorema

Sia Ω aperto connesso, $\Delta u = 0$ su Ω .

Allora vale una tra le seguenti:

(i) u è costante su Ω

(ii) $\inf_{\bar{\Omega}} u < u(x) < \sup_{\bar{\Omega}} u \quad \forall x \in \Omega$

DIMOSTRAZIONE

Supponiamo $L = \sup_{\bar{\Omega}} u < +\infty$

Sia $\Omega_L = \{x \in \Omega : f(x) = L\}$: è chiuso per continuità di f

Ω_L è anche aperto: se $u(x_0) = L$ per $x_0 \in \Omega$, allora

$$L = \int_{B(x_0,r)} \Delta u dy \Rightarrow u = L \text{ su } B(x_0,r)$$

□

**teorema:
principio del
massimo**

Sia Ω aperto limitato e connesso, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ f.c. $\Delta u = 0$ su Ω . Allora.

(i) $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u, \min_{\bar{\Omega}} u = \min_{\partial\Omega} u$

(forma debole)

(ii) se inoltre $\exists x_0 \in \Omega$ f.c. $u(x_0) = \max_{\bar{\Omega}} u$ oppure $u(x_0) = \min_{\bar{\Omega}} u$, allora u è costante.

(forma forte)

DIMOSTRAZIONE

Per Weierstrass $\exists \bar{x} \in \bar{\Omega}$ f.c. $u(\bar{x}) = \max_{\bar{\Omega}} u$.

Allora $u(x) \equiv u(\bar{x})$ oppure $u(x) < u(\bar{x}) \quad \forall x \in \Omega \Rightarrow \bar{x} \in \partial\Omega$ $\square$

corollario

Ω limitato e connesso.

Il problema di Dirichlet

$$\begin{cases} \Delta u = 0 & \text{su } \Omega \\ u|_{\partial\Omega} = g & \text{su } \partial\Omega \end{cases}$$

con $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, ha al più una soluzione

DIMOSTRAZIONE

Se u_1, u_2 sono due soluzioni, $w = u_1 - u_2$ soddisfa $\begin{cases} \Delta w = 0 \\ w|_{\partial\Omega} = 0 \end{cases}$, con $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

Allora $\max_{\bar{\Omega}} w = \min_{\bar{\Omega}} w = \max_{\partial\Omega} w = \min_{\partial\Omega} w = 0 \Rightarrow w \equiv 0$ su $\bar{\Omega}$. $\square$

**teorema di
Liouville**

Sia u armonica su $\mathbb{R}^n$.

Se u è limitata, allora u è costante.

DIMOSTRAZIONE

Mostriamo che $\nabla u \equiv 0$.

$$\begin{aligned} \left| \frac{\partial u}{\partial x_i}(x_0) \right| &= \left| \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i}(x) dx \right| = \left| \frac{C_0}{r^n} \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i}(x) dx \right| = \left| \frac{C_0}{r^n} \int_{\mathbb{R}^n} u \nabla \cdot \left(\frac{x}{r^n} \right) dx \right| \\ &= \frac{C_0}{r^n} \int_{\mathbb{R}^n} u dx \leq \frac{C_0}{r^n} \|u\|_{L^\infty} r^{n-1} \simeq \frac{C}{r} \xrightarrow{r \rightarrow \infty} 0 \end{aligned}$$

$\square$

Oss In realtà basta $u \geq c$

**principio del massimo
(forma debole)**

Sia $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ armonica su Ω , Ω aperto limitato.

Allora $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ e $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.

DIMOSTRAZIONE

1° passo $-\Delta u > 0 \Rightarrow \min_{\bar{\Omega}} u = \min_{\partial\Omega} u$

Per assurdo $\exists x_0 \in \Omega$ f.c. $u(x_0) = \min_{\bar{\Omega}} u \Rightarrow -\Delta u(x_0) \leq 0$ ∇

2° passo $-\Delta u = 0 \Rightarrow \min_{\bar{\Omega}} u = \min_{\partial\Omega} u$

$$u_\varepsilon(x) = u(x) - \varepsilon |x|^2, \quad -\Delta u_\varepsilon = -\Delta u + \varepsilon \Delta(|x|^2) = 2n\varepsilon > 0$$

Per il 1° passo, $\min_{\bar{\Omega}} \{u(x) - \varepsilon |x|^2\} = \min_{\partial\Omega} \{u(x) - \varepsilon |x|^2\}$

$$\downarrow \varepsilon \rightarrow 0$$

$$\min_{\bar{\Omega}} u$$

$$\downarrow \varepsilon \rightarrow 0$$

$$\min_{\partial\Omega} u$$

$\square$

- esempio**
- $-\Delta u + \sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j} u = 0$
In x_0 p.to di min, $-\Delta u \leq 0$
 - $-\Delta u + V(x)u = 0$
 - $-\frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) = 0$

Caso $n=2$: $x=(x_1, x_2)$, $A(x) \in M(2)$ simmetrica e def. positiva, con $\sup \|A(x)\| < +\infty$

$$\operatorname{div}(A \nabla u) = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j}$$

Se $A=I$, $\operatorname{div}(A \nabla u) = \Delta u$

Teorema di tipo Liouville

Sia $u \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Se $\operatorname{div}(A \nabla u) = 0$, allora u è costante.

DIMOSTRAZIONE

Sia $\varphi: \mathbb{R}^2 \rightarrow [0,1]$, $\varphi \in C^\infty$, $\varphi(x) \equiv 0$ se $|x| \geq 1$, $\varphi(x) \equiv 1$ se $|x| \leq \frac{1}{2}$

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$$

Moltiplichiamo l'eq. per $\varphi_R^2 u$

$$(i) \int_{\mathbb{R}^2} A(x) \nabla u \cdot \nabla u < +\infty$$

$$(ii) \int_{\mathbb{R}^2} A(x) \nabla u \cdot \nabla u = 0$$

$$0 = \int_{\mathbb{R}^2} \operatorname{div}(A \nabla u) \varphi_R^2 u = - \int_{\mathbb{R}^2} A \nabla u \cdot \nabla u \varphi_R^2 - 2 \int_{\mathbb{R}^2} A \nabla u \cdot \nabla \varphi_R u \varphi_R$$

$$\Rightarrow \int_{\mathbb{R}^2} A \nabla u \cdot \nabla u \varphi_R^2 = -2 \int_{\mathbb{R}^2} A \nabla u \cdot \nabla \varphi_R u \varphi_R \leq 2 \int_{\mathbb{R}^2} |A \nabla u \cdot \nabla \varphi_R| u \varphi_R \leq C \int_{\mathbb{R}^2} |A \nabla u \cdot \nabla \varphi_R| \varphi_R \leq C \int_{\mathbb{R}^2} (A \nabla u \cdot \nabla u)^{1/2} (A \nabla \varphi_R \cdot \nabla \varphi_R)^{1/2} \varphi_R \stackrel{(*)}{\leq} C \underbrace{\left(\int_{\mathbb{R}^2} (A \nabla u \cdot \nabla u) \varphi_R^2 \right)^{1/2}}_{(*)} \underbrace{\left(\int_{\mathbb{R}^2} A \nabla \varphi_R \cdot \nabla \varphi_R \right)^{1/2}}_{\leq C \left(\int_{\mathbb{R}^2} \frac{1}{R^2} \right)^{1/2}} \leq C$$

Quindi $\sup \int_{\mathbb{R}^2} (A \nabla u \cdot \nabla u) \varphi_R^2 < +\infty$

Per Boppo-Levi $\int_{\mathbb{R}^2} A \nabla u \cdot \nabla u < +\infty$

Ripetendo gli stessi passi:

$$(*) = \int_{\mathbb{R}^2} A \nabla u \cdot \nabla u \varphi_R^2 \chi_{\frac{R}{2} < |x| < R} \xrightarrow{R \rightarrow +\infty} 0 \text{ q.o. per Lebesgue}$$

$$\Rightarrow \int_{\mathbb{R}^2} A \nabla u \cdot \nabla u = 0$$

□

Applicazioni

- Superfici minime

$$\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{1+|Du|^2}} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{1+|Du|^2}} \frac{\partial u}{\partial y} \right) = 0 \quad (SM)$$

$\mathcal{A}(u) = \int \sqrt{1+|Du|^2}$: (SM) è la sua eq. di Eulero-Lagrange

teorema di Bernstein

Se $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ risolve (SM), allora $u(x,y) = a+bx+cy$

teorema

Se $u: \mathbb{R}^2 \rightarrow \mathbb{R}$, $u \in L^\infty$, risolve (SM), allora $u(x,y) = c$

DIMOSTRAZIONE

$$A = \begin{pmatrix} \frac{1}{\sqrt{1+|Du|^2}} & 0 \\ 0 & \frac{1}{\sqrt{1+|Du|^2}} \end{pmatrix}$$

□

- Congettura di De Giorgi

$$\Delta u = u^3 - u \text{ su } \mathbb{R}^2$$

Se $\exists v \in \mathbb{R}^2$ t.c. $\frac{\partial u}{\partial v} > 0$ su $\mathbb{R}^2$, allora $\exists \eta: \mathbb{R} \rightarrow \mathbb{R}$, $\exists \eta \in \mathbb{R}^2$ t.c. $u(x,y) = \eta(x,y) \cdot \eta$

teorema Sia $F: \mathbb{R} \rightarrow \mathbb{R}$ C^∞ , $\Delta u = F(u)$ su $\mathbb{R}^2$, $u \in C^2(\mathbb{R}^2)$, $\forall u \in L^\infty$, $\exists v \in \mathbb{R}^2$ t.c. $\frac{\partial u}{\partial v} > 0$ su $\mathbb{R}^2$. Allora u dipende da una sola variabile.

DIMOSTRAZIONE

$$w(x,y) = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y}, w: \mathbb{R}^2 \rightarrow \mathbb{C} \setminus \{0\}$$

$$\Delta w = F'(u)w$$

Ora $w(x,y) = p(x,y) e^{i\varphi(x,y)}$ con φ regolare

Inoltre $\varphi \in L^\infty$: se $v = (v_1, v_2)$, allora

$$0 < \frac{\partial u}{\partial v} = \frac{\partial u}{\partial x} v_1 + \frac{\partial u}{\partial y} v_2 = p(\cos \varphi v_1 + \sin \varphi v_2)$$

e si ha che, se $v_1^2 + v_2^2 \neq 0$, allora $\varphi \mapsto \cos \varphi v_1 + \sin \varphi v_2$ è negativa in almeno 2 punti φ_1, φ_2 : quindi φ è limitata e possiamo separarla in $[-\varphi_1, \varphi_2]$.

Inoltre $\varphi \equiv \varphi_0$, e in tal caso

$$\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = p(\cos \varphi_0 + i \sin \varphi_0)$$

Questo segue da:

$$\text{Equazione per } \varphi(x,y): \Delta(p e^{i\varphi}) = F'(u) p e^{i\varphi}$$

$$e^{i\varphi} \Delta p + 2 \nabla p \cdot \nabla e^{i\varphi} + p \Delta e^{i\varphi} = F'(u) p e^{i\varphi}$$

$$e^{i\varphi} \Delta p + 2i \nabla p \nabla \varphi e^{i\varphi} + p \Delta e^{i\varphi} = F'(u) p e^{i\varphi}$$

$$\Delta e^{i\varphi} = \operatorname{div}(i e^{i\varphi} \nabla \varphi) = i e^{i\varphi} \Delta \varphi - |\nabla \varphi|^2 e^{i\varphi}$$

$$\text{Quindi, } \cancel{e^{i\varphi}} \Delta p + 2i \nabla p \nabla \varphi \cancel{e^{i\varphi}} + p i \cancel{e^{i\varphi}} \Delta \varphi - p |\nabla \varphi|^2 \cancel{e^{i\varphi}} = F'(u) p \cancel{e^{i\varphi}}$$

$$\Delta p + 2i \nabla p \cdot \nabla \varphi + i p \Delta \varphi - p |\nabla \varphi|^2 = F'(u) p$$

Prendendo la parte immaginaria:

$$2 \nabla p \cdot \nabla \varphi + p \Delta \varphi = 0 \Rightarrow 2 \nabla p \nabla \varphi + p^2 \Delta \varphi = 0$$

$$\nabla p^2 \nabla \varphi + p^2 \Delta \varphi = \operatorname{div}(p^2 \nabla \varphi)$$

□

- Tecnica di Pohozaev

teorema $\begin{cases} \Delta u = \lambda u - |u|^{2^*-2} & \text{su } B_R^n, \text{ dove } 2^* = \frac{2n}{n-2}, u \in C^2(B_R) \cap C^1(\bar{B}_R) \\ u|_{\partial B_R^n} = 0 \\ \Rightarrow u \equiv 0 \end{cases}$ ($H^1(\mathbb{R}^n) \rightarrow L^{2^*}(\mathbb{R}^n)$, con $n \geq 3$), $\lambda > 0$

Oss. Se sostituisco $2^* \rightarrow 2 < p < 2^*$, allora $\exists$ soluzioni

$$\min_{\substack{\|u\|_{L^p} = 1 \\ u \in H_0^1}} \int |\nabla u|^2 + \lambda \int |u|^2$$

- Se sostituisco $B_R \rightarrow B_R \setminus B_\varepsilon$, il problema ha soluzione.

$$\min_{\substack{\|u\|_{L^p(B_R \setminus B_\varepsilon)} = 1 \\ u \in H_{\text{rad}}^1 \cap H_0^1}} \int |\nabla u|^2 + \lambda \int |u|^2 \text{ ha soluzione}$$

$$H_{\text{rad}}^1(B_R \setminus B_\varepsilon) \xrightarrow{\text{cpt}} L^2(B_R \setminus B_\varepsilon)$$

- Se $\lambda < 0$, $|\lambda| \ll 1$, allora $\exists$ soluzioni (Brezis-Nirenberg)

DIMOSTRAZIONE

$$-\Delta u + \lambda u = u|u|^{2^*-2}$$

Moltiplichiamo per u :

$$-\int_{\mathbb{R}^n} \Delta u \cdot u + \lambda \int_{\mathbb{R}^n} |u|^2 = \int_{\mathbb{R}^n} |u|^{2^*}$$

$$-\int_{\mathbb{R}^n} \operatorname{div}(\nabla u \cdot u) + \int_{\mathbb{R}^n} |\nabla u|^2 + \lambda \int_{\mathbb{R}^n} u^2$$

$$-\int_{\partial \mathbb{R}^n} \frac{\partial u}{\partial \nu} u = 0$$

$$\Rightarrow \int_{\mathbb{R}^n} |\nabla u|^2 + \lambda \int_{\mathbb{R}^n} |u|^2 = \int_{\mathbb{R}^n} |u|^{2^*}$$

Moltiplichiamo per $x \cdot \nabla u$:

$$\int_{\mathbb{R}^n} -\Delta u (x \cdot \nabla u) = \int_{\mathbb{R}^n} (u \cdot |u|^{2^*-2} - \lambda u) (x \cdot \nabla u)$$

$$\begin{aligned} \text{II} &= \int_{\mathbb{R}^n} (u|u|^{2^*-2} - \lambda u) x_k \partial_k u = \int_{\mathbb{R}^n} \partial_k \left(\frac{|u|^{2^*}}{2^*} - \lambda \frac{|u|^2}{2} \right) x_k = \\ &= \int_{\mathbb{R}^n} \partial_k \left(\frac{|u|^{2^*}}{2^*} - \lambda \frac{|u|^2}{2} \right) x_k = -n \int_{\mathbb{R}^n} \frac{|u|^{2^*}}{2^*} - \lambda \frac{|u|^2}{2} = \\ &= \underbrace{\int_{\mathbb{R}^n} \operatorname{div} \left(x \left(\frac{|u|^{2^*}}{2^*} - \lambda \frac{|u|^2}{2} \right) \right)}_{\text{I}} - n \int_{\mathbb{R}^n} \frac{|u|^{2^*}}{2^*} - \lambda \frac{|u|^2}{2} \\ &= \int_{\partial \mathbb{R}^n} x \cdot \nu \left(\frac{|u|^{2^*}}{2^*} - \lambda \frac{|u|^2}{2} \right) = 0 \end{aligned}$$

$$\begin{aligned} \text{I} &= -\int_{\mathbb{R}^n} \Delta u (x \cdot \nabla u) = -\int_{\mathbb{R}^n} \partial_i^2 u x_k \partial_k u = -\int_{\mathbb{R}^n} \partial_i (\partial_i u x_k \partial_k u) + \int_{\mathbb{R}^n} \partial_i u \partial_i (x_k \partial_k u) = \\ &= -\int_{\mathbb{R}^n} \operatorname{div}(\nabla u (x \cdot \nabla u)) + \int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} \partial_i u x_k \partial_i \partial_k u = \\ &= -\int_{\partial \mathbb{R}^n} \frac{\partial u}{\partial \nu} (x \cdot \nabla u) + \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \partial_k (|\nabla u|^2) x_k = \\ &= -R \int_{\partial \mathbb{R}^n} \left(\frac{\partial u}{\partial \nu} \right)^2 + \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \partial_k (x_k |\nabla u|^2) x_k = \\ &= -R \int_{\partial \mathbb{R}^n} \left(\frac{\partial u}{\partial \nu} \right)^2 + \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \partial_k (x_k |\nabla u|^2) - \frac{n}{2} \int_{\mathbb{R}^n} |\nabla u|^2 = \\ &= -R \int_{\partial \mathbb{R}^n} \left(\frac{\partial u}{\partial \nu} \right)^2 + \frac{2-n}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{2} \int_{\partial \mathbb{R}^n} \frac{(x \cdot \nu)}{R} |\nabla u|^2 = \\ &= -\frac{R}{2} \int_{\partial \mathbb{R}^n} \left(\frac{\partial u}{\partial \nu} \right)^2 + \frac{2-n}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \end{aligned}$$

Quindi

$$\int_{\mathbb{R}^n} |\nabla u|^2 + \lambda \int_{\mathbb{R}^n} |u|^2 = \int_{\mathbb{R}^n} |u|^{2^*}$$

$$-\frac{R}{2} \int_{\partial \mathbb{R}^n} \left(\frac{\partial u}{\partial \nu} \right)^2 + \frac{2-n}{2} \int_{\mathbb{R}^n} |\nabla u|^2 = \frac{2-n}{2} \int_{\mathbb{R}^n} |u|^{2^*} + n \frac{\lambda}{2} \int_{\mathbb{R}^n} |u|^2$$

Moltiplicando la prima per $\frac{2-n}{2}$ e sostituendo:

$$-\frac{R}{2} \int_{\partial \mathbb{R}^n} \left(\frac{\partial u}{\partial \nu} \right)^2 - \lambda \frac{2-n}{2} \int_{\mathbb{R}^n} |u|^2 - n \frac{\lambda}{2} \int_{\mathbb{R}^n} |u|^2 = 0$$

$$\Rightarrow \frac{R}{2} \int_{\partial \mathbb{R}^n} \left(\frac{\partial u}{\partial \nu} \right)^2 + \lambda \int_{\mathbb{R}^n} |u|^2 = 0 \xrightarrow{\lambda > 0} \int_{\mathbb{R}^n} |u|^2 = 0 \Rightarrow u \equiv 0$$

□

Oss Vale anche per $\lambda = 0$: $\frac{\partial u}{\partial \nu} = 0$ su $\partial \mathbb{R}^n$, ma $u \equiv 0$ su $\partial \mathbb{R}^n$

$\Rightarrow u \equiv 0$ (metodo di continuazione unica)

Gidas-Nirenberg

$$\begin{cases} -\Delta u = f(u) & \text{su } B_R \\ u|_{\partial B_R} = 0 \end{cases} \quad \text{con } f(0) = 0$$

$$u \in C^2(B_R) \cap C^1(\bar{B}_R) \quad u \geq 0$$

$u=0$ è soluzione

∃! soluzione $u > 0, u \neq 0$

Se u risolve (*) e $u > 0$, allora $u(x) = u(x/|x|)$ (stati fondamentali, ground state)
(Tecnica del moving plane)

principio del massimo debole

Sia $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ con Ω aperto limitato.

Se $-\Delta u \geq 0$ su Ω , allora $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.

Se $-\Delta u \leq 0$ su Ω , allora $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

DIMOSTRAZIONE

$$\text{Se } -\Delta u > 0 \Rightarrow \min_{\bar{\Omega}} u = \min_{\partial\Omega} u$$

$$u_\varepsilon = u - \varepsilon |x|^2, \quad -\Delta u_\varepsilon = -\Delta u + 2n\varepsilon > 0 \Rightarrow \min_{\bar{\Omega}} u_\varepsilon = \min_{\partial\Omega} u_\varepsilon$$

Si conclude passando a $\varepsilon \rightarrow 0$

□

Se consideriamo l'operatore $Lu = -\Delta u + a(x) \cdot \nabla u$, $a \in C(\bar{\Omega})$,
il Teorema vale in modo analogo.

teorema

Sia $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ con Ω aperto limitato.

Se $Lu \geq 0$ su Ω , allora $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.

Se $Lu \leq 0$ su Ω , allora $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

DIMOSTRAZIONE

Se $Lu > 0$, si ha $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$ (come per $-\Delta$)

Introduciamo ora w t.c. $Lw < 0$

$$u_\varepsilon = u - \varepsilon w$$

$$-\Delta(u - \varepsilon w) + a(x) \cdot \nabla(u - \varepsilon w) > 0$$

$$\underbrace{-\Delta u + a \cdot \nabla u}_{>0} + \varepsilon \underbrace{(\Delta w - a \cdot \nabla w)}_{>0} > 0$$

$$\Delta(e^{\lambda w}) = \lambda^2 e^{\lambda w}$$

$$a(x) \cdot \nabla w = \lambda a(x) e^{\lambda w}$$

$$-\Delta(e^{\lambda w}) + a(x) \cdot \nabla w < 0 \iff -\lambda^2 e^{\lambda w} + \lambda a(x) e^{\lambda w} < 0$$

$$\iff \lambda^2 - \lambda a(x) > 0$$

$$\lambda^2 - \lambda a(x) \geq \lambda^2 - \lambda \max |a_i| > \lambda^2 - \lambda k > 0$$

□

Lemma di Hopf

Sia $u \in C^2(B_R(x_0)) \cap C^0(\overline{B_R(x_0)})$

Se $-\Delta u \geq 0$ su $B_R(x_0)$, $\exists y_0 \in \partial B_R(x_0)$ t.c. $u(y_0) < u(x)$ $\forall x \in B_R(x_0)$ ed esiste $\frac{\partial u}{\partial \nu}(y_0)$, allora $\frac{\partial u}{\partial \nu}(y_0) < 0$.

DIMOSTRAZIONE

$$u(x) - \varepsilon W(x) = u_\varepsilon(x), \text{ con } W(x) = \begin{cases} -\log|x| & \text{se } n=2 \\ \frac{1}{|x|^{n-2}-1} & \text{se } n \geq 3 \end{cases}, -\Delta W = 0 \text{ su } \mathbb{R}^n \setminus \{0\}$$

$$W|_{\partial B_R(x_0)} = B_1$$



$$u(x) - \varepsilon W(x)|_{\partial B_1} = u(x)$$

$$\min_{\partial B_{1/2}} u(x) > u(x_0) \Rightarrow \text{per } \varepsilon \text{ abbastanza piccolo, } \min_{\partial B_{1/2}} (u - \varepsilon W) > u(x_0)$$

$$\begin{cases} -\Delta u_\varepsilon = -\Delta u + \varepsilon \Delta W = -\Delta u \geq 0 \\ u_\varepsilon|_{\partial B_1 \cup \partial B_{1/2}} \geq u(x_0) \end{cases}$$

$$\min_{\partial B_1 \cup \partial B_{1/2}} u_\varepsilon(x) \geq u(x_0)$$

Per il massimo debole, $u_\varepsilon(x) \geq u(x_0)$ su $B_1 \setminus B_{1/2}$

$$\Rightarrow \text{Se esiste } \frac{\partial (u - \varepsilon W)}{\partial \nu}(x_0) \leq 0$$

$$\Rightarrow \frac{\partial u}{\partial \nu}(x_0) \leq \varepsilon \frac{\partial W}{\partial \nu}(x_0) < 0$$

dove $\frac{\partial W}{\partial \nu}(x_0) < 0$ per calcolo diretto



Lemma

Sia $u \in C^2(B_R(x_0)) \cap C^0(\overline{B_R(x_0)})$

Se $\Delta u \geq 0$ su $B_R(x_0)$, $\exists y_0 \in \partial B_R(x_0)$ t.c. $u(y_0) < u(x)$ $\forall x \in B_R(x_0)$ ed esiste $\frac{\partial u}{\partial \nu}(y_0)$, allora $\frac{\partial u}{\partial \nu}(y_0) < 0$.

DIMOSTRAZIONE

Dobbiamo trovare w t.c. $\Delta w \geq 0$ da cosa giusta.

$$\Delta w = 0, w|_{\partial B_1} = 0, \frac{\partial w}{\partial \nu}(x_0) < 0$$

$$\text{Per il max debole, in realtà basta } \Delta w \leq 0 \Rightarrow \Delta(u - \varepsilon w) = \Delta u - \varepsilon \Delta w \leq 0$$

$$\text{Cerchiamo } w(x) = e^{-k|x|^2} - e^{-k} : w|_{\partial B_1} = 0, \frac{\partial w}{\partial \nu}(x_0) < 0$$

$$-\Delta w = -\text{div}(-2kx e^{-k|x|^2}) = 2k \text{div}(x e^{-k|x|^2}) = 2kn e^{-k|x|^2} + 2kx \cdot (-2kx) e^{-k|x|^2} = e^{-k|x|^2} (2kn - 4k^2|x|^2)$$

$$\Delta w = \Delta u - \varepsilon \Delta w = \Delta u - \varepsilon (2kn - 4k^2|x|^2) = \Delta u - 2kx \cdot \Delta w$$

$$\Delta w = e^{-k|x|^2} (2kn - 4k^2|x|^2 - 2kx \cdot \Delta w)$$

Basta scegliere k abbastanza grande t.c.

$$2kn - 4k^2|x|^2 - 2kx \cdot \Delta w \leq 0 \quad \forall x \in B_1 \setminus B_{1/2}$$

$$2kn - 4k^2|x|^2 - 2kx \cdot \Delta w \leq 2kn - k^2 + 2k \sup_{x \in B_1} |\Delta w| = -k^2 + Ck < 0 \text{ per } k \gg 1$$



Principio del massimo forte

Sia Ω aperto limitato connesso, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$

Se $Lu \geq 0$, allora (i) u è costante

oppure (ii) $u(x) > \min_{\bar{\Omega}} u \quad \forall x \in \Omega$

Se $Lu \leq 0$, allora (i) u è costante

oppure (ii) $u(x) < \max_{\bar{\Omega}} u \quad \forall x \in \Omega$

Oss Ω connesso è fondamentale

(1) (2)

DIMOSTRAZIONE

Sia $m = \min_{\bar{\Omega}} u$, $\Omega_m = \{x \in \Omega : u(x) = m\}$. Dico che $\Omega_m = \Omega$ o $\Omega_m = \emptyset$.

Se $\Omega_m \neq \Omega$, $\exists P \in \Omega \setminus \Omega_m : r = \text{dist}(P, \Omega_m)$. Considero $B(P, r)$:

$u(x) > m$ su $B(P, r)$, per la scelta di r

$\exists x_0 \in \partial B(P, r)$ t.c. $x_0 \in \Omega_m \Rightarrow u(x_0) = m \xrightarrow{\text{Hopf}} \frac{\partial u}{\partial \nu}(x_0) < 0$

$\Rightarrow \exists \varepsilon > 0$ t.c. $u(x_0 + \varepsilon \nu) < u(x_0) = m$ $\downarrow$

ACHTUNG Non sappiamo che $B(P, r) \subseteq \Omega$

Dobbiamo trovare $P \notin \Omega_m$ t.c. $\text{dist}(P, \Omega_m) < \text{dist}(P, \partial\Omega)$

Ω_m è chiuso, ma stiamo assumendo $\Omega_m \neq \Omega \Rightarrow$ per connessione, Ω_m non è aperto

$\Rightarrow \exists x_0 \in \Omega_m$ t.c. $x_0 \notin \bar{\Omega}_m \Rightarrow \exists x_k \rightarrow x_0$ con $x_k \notin \Omega_m$

Ora $\text{dist}(x_k, \Omega_m) \rightarrow 0$ e $x_k \rightarrow x_0$, ma $\text{dist}(x_0, \partial\Omega) \geq \varepsilon_0 > 0$,

quindi $\text{dist}(x_k, \partial\Omega) \geq \frac{\varepsilon_0}{2} > 0$: basta prendere $P = x_{k_0}$ per n_0 abbastanza grande

□

Lemma $-\Delta u + Vu = 0$, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $V \in L^\infty(\Omega)$.

Se $u \geq 0$, $u \neq 0$, allora

(i) $u(x) > 0$ su Ω

(ii) se $x_0 \in \partial\Omega$ ha la proprietà della sfera interna (cioè

$\exists B \subseteq \Omega$ t.c. $\partial B \cap \partial\Omega = \{x_0\}$) con $u(x_0) = 0$, allora

$\frac{\partial u}{\partial \nu}(x_0) < 0$ se esiste.

DIMOSTRAZIONE

Consideriamo $w(x) = u(x)e^{\lambda x_1}$: basta dimostrare la tesi per w con qualche λ .

Infatti $w(x) > 0 \iff u(x) > 0$ e

$$\frac{\partial w}{\partial \nu}(x_0) = \frac{\partial u}{\partial \nu}(x_0) \underbrace{(e^{\lambda x_1}(x_0))}_{=0} + \underbrace{u(x_0)}_{=0} \frac{\partial e^{\lambda x_1}}{\partial \nu}(x_0)$$

$$-\Delta w = -\Delta u e^{\lambda x_1} - \lambda^2 u e^{\lambda x_1} - 2\lambda e^{\lambda x_1} \partial_{x_1} u =$$

$$= -Vu e^{\lambda x_1} - \lambda^2 u e^{\lambda x_1} - 2\lambda \partial_{x_1}(e^{\lambda x_1} u) + 2\lambda^2 u e^{\lambda x_1} = -Vw - 2\lambda \partial_{x_1} w + \lambda^2 w$$

$$\text{Quindi } -\Delta w = \underbrace{w}_{=0} \underbrace{(-V + \lambda^2)}_{>0 \text{ per } \lambda \gg 1} - 2\lambda \partial_{x_1} w \geq -2\lambda \partial_{x_1} w$$

$$-\Delta w + 2\lambda \partial_{x_1} w \geq 0, \text{ cioè } -\Delta w + a(x) \cdot \nabla w \geq 0, \text{ con } a(x) = (2\lambda, 0, \dots, 0)$$

Ora per il minimo forte, $w(x) > 0$ in Ω

Poi, per Hopf, $\frac{\partial w}{\partial \nu}(x_0) < 0$.

□

Teorema di Gidas-Nirenberg

Sia $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(0)=0$,

$$\begin{cases} -\Delta u = f(u) & \text{su } B = \{x \in \mathbb{R}^n : \|x\| < 1\}, u \in C^1(\bar{B}) \cap C^2(B) \\ u|_{\partial B} = 0, u > 0 \text{ su } B, u \neq 0 \end{cases}$$

 Allora $u(x) = u(|x|)$.

DIMOSTRAZIONE

$\forall u \in \mathbb{R}^n$ consideriamo $P_0 = \{x \in \mathbb{R}^n : x \cdot v = 0\}$, R_0 riflessione rispetto a P_0

Ni basta provare che $u(x) = u(R_0(x)) \quad \forall x \Rightarrow u$ radiale

Infatti, se per assurdo $\exists x_1, x_2$ con $\|x_1\| = \|x_2\|$ e $u(x_1) \neq u(x_2)$

Prendiamo poi P_0 iperpiano equidistante da x_1 e $x_2 \Rightarrow u(x_1) = u(R_0(x_1)) = u(x_2)$ ∇

Posso supporre $v = e_n$

Quindi bisogna provare che

$$u(x, x_n) = u(x, -x_n) \quad \forall (x, x_n) \in B^+ = B \cap \{x_n > 0\}$$

Basta $u(x, x_n) \leq u(x, -x_n) \quad \forall (x, x_n) \in B^+$

perché se considero $\tilde{u}(x, x_n) = u(x, -x_n)$, ottengo $\tilde{u}(x, x_n) \leq u(x, x_n)$,

$$\text{cioè } u(x, -x_n) \leq u(x, x_n)$$

Se $x = (x, x_n)$ con $x_n \geq \lambda$, allora $x_1 = (x, 2\lambda - x_n)$

Sia $I = \{\lambda \in (0, 1) : u(x) - u(x_1) \leq 0 \text{ su } B \cap \{x_n > \lambda\}\}$

Passo 1 $I \neq \emptyset$

Passo 2 $\inf I = 0$

$$\Rightarrow \exists (\lambda_k) \in I \text{ t.c. } \lambda_k \rightarrow 0 : u(x) - u(x_{\lambda_k}) \leq 0 \text{ su } B \cap \{x_n > \lambda_k\}$$

$$\Rightarrow \text{Fissato } \mu \in (0, 1), \exists k_0 \text{ t.c. } \lambda_{k_0} < \mu : u(x, x_n) - u(x, 2\lambda_{k_0} - x_n) \leq 0 \text{ su } B \cap \{x_n > \lambda_{k_0}\}$$

$$\text{Passo al limite } k \rightarrow +\infty : u(x, x_n) - u(x, -x_n) \leq 0 \text{ su } B \cap \{x_n > \mu\}$$

e si conclude per arbitrarietà di μ .

Ora dico che $\frac{\partial u}{\partial x_n} < 0$ su $\partial B \cap \{x_n > 0\}$

$$-\Delta u = f(u) - f(0) = \frac{f'(\xi(x))}{\nu(x)} u(x) \quad \text{dove } \xi(x) \in [0, u(x)]$$

$$\nu(x) \in L^\infty$$

$$\text{Quindi } -\Delta u = \nu u.$$

Per il lemma (con $\Omega = B$), $\frac{\partial u}{\partial \nu} < 0$ su ∂B

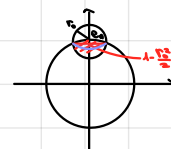
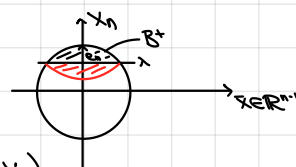
Siccome $u=0$ su ∂B , $\nabla u \cdot \nu$ su $\partial B \Rightarrow \nabla u = \mu \nu, \mu \neq 0$, anzi $\mu < 0 \Rightarrow \frac{\partial u}{\partial x_n} = \mu \nu_n < 0$ perché $\nu_n > 0$

Inoltre $u > 0$ in B e $\frac{\partial u}{\partial x_n} < 0$ su $\partial B \cap \{x_n > 0\}$

Passo 1: $\exists r_0$ t.c. $\frac{\partial u}{\partial x_n} < 0$ in $B(r_0, r_0)$

Considero $x \in R_1 - \frac{r_0^2}{2}$, con $x_n > 1 - \frac{r_0^2}{2}$

$$u(x) - u(x_{1-\frac{r_0^2}{2}}) = u(x, x_n) - u(x, 2 - r_0^2 - x_n) = \underbrace{\frac{\partial u}{\partial x_n}(x, \xi)}_{< 0} \underbrace{(2x_n - 2 + r_0^2)}_{> 0} \text{ se } x_n > 1 - \frac{r_0^2}{2}$$



Passo 2: Per assurdo $\bar{\lambda} = \inf I > 0$

Abbiamo $u(x) - u(x_{\bar{\lambda}}) \leq 0$ su $B \cap \{x_n > \bar{\lambda}\}$

$$w_{\bar{\lambda}}(x) = u(x) - u(x_{\bar{\lambda}})$$

$$-\Delta w_{\bar{\lambda}} = -(\Delta u - \Delta u(x_{\bar{\lambda}})) = f(u) - f(u(x_{\bar{\lambda}})) = f'(\xi)(u - u(x_{\bar{\lambda}})) = \lambda(x)w_{\bar{\lambda}}(x)$$

$$w_{\bar{\lambda}}|_{\partial B \cap \{x_n > \bar{\lambda}\}} < 0 \Rightarrow w_{\bar{\lambda}} \not\equiv 0.$$

Per il lemma, $\frac{\partial w_{\bar{\lambda}}}{\partial \nu} > 0 \Rightarrow \frac{\partial w_{\bar{\lambda}}}{\partial x_n} < 0$ su $\{x_n = \bar{\lambda}\} \cap B$

$$\Rightarrow \frac{\partial u}{\partial x_n}(\bar{x}, \bar{\lambda}) + \frac{\partial u}{\partial x_n}(\bar{x}, \bar{\lambda}) < 0 \Rightarrow \frac{\partial u}{\partial x_n}(\bar{x}, \bar{\lambda}) < 0$$

Se $\bar{\lambda} - \varepsilon_0 < \mu < \bar{\lambda}$, in $\{\mu < x_n \leq \bar{\lambda} + \varepsilon_0\}$

$$u(x) - u(x_{\mu}) = u(\bar{x}, x_n) - u(\bar{x}, 2\mu - x_n) = \frac{\partial u}{\partial x_n}(\bar{x}, \xi)(2x_n - 2\mu) < 0$$

Ora $\exists \mu_1 > \bar{\lambda} + c$. $u(x) - u(x_{\mu_1}) \leq 0$ in $\{x_n \geq \bar{\lambda} + \varepsilon_0\}$.

$u(x) - u(x_{\bar{\lambda}}) < -\delta_0$ con $\delta_0 > 0$, in $\{x_n \geq \bar{\lambda} + \varepsilon_0\}$ per Weierstrass

$$\text{ma } u(x) - u(x_{\mu_1}) = \underbrace{u(x) - u(x_{\bar{\lambda}})}_{< -\delta_0} + \underbrace{u(x_{\bar{\lambda}}) - u(x_{\mu_1})}_{1 \cdot 1 < \frac{\delta_0}{2} \text{ per continuit }}_{\text{per continuit }}$$

$$|u(\bar{x}, 2\bar{\lambda} - x_n) - u(\bar{x}, 2\mu_1 - x_n)| < \frac{\delta_0}{2} \text{ se } |\bar{\lambda} - \mu_1| < 1$$

□

Oss In realt , $I = (0, 1)$ e $u(r) \searrow$, $\partial_r u < 0$

Funzioni armoniche radiali

Su $\mathbb{R}^n$, se $-\Delta u = 0$, $u(x) = u(|x|)$, allora u   costante

Su $\mathbb{R}^n \setminus \{0\}$, se $-\Delta u = 0$, $u(x) = u(|x|)$, allora $u = a + b/r$, con $a, b \in \mathbb{R}$,

$$\text{e } w(x) = \begin{cases} \log|x| & \text{se } n=2 \\ \frac{1}{|x|^{n-2}} & \text{se } n \geq 3 \end{cases}$$

Se $u = u(r)$,

$$\Delta u = u'' + \frac{n-1}{r} u' = 0$$

Cerchiamo $u'' + \frac{n-1}{r} u' = 0$ su $(0, +\infty)$

Posto $u' = v$, $v' + \frac{n-1}{r} v = 0$

Moltiplico per $e^{\int \frac{n-1}{r} dr}$: $(v e^{\int \frac{n-1}{r} dr})' = 0 \Rightarrow (v r^{n-1})' = 0 \Rightarrow v r^{n-1} = a \Rightarrow v' = \frac{a}{r^{n-1}}$

$$\Rightarrow u(r) = \begin{cases} b + a \log r & \text{se } n=2 \\ b + \frac{a}{r^{n-2}} & \text{se } n \geq 3 \end{cases}$$

Consideriamo

$$\begin{cases} -\Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases} \quad u \in C^2(\Omega) \cap C^0(\bar{\Omega})$$

Lemma Se $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$, Ω regolare in $\mathbb{R}^n$, allora

$$\int_{\Omega} (\Delta u v - u \Delta v) dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} \right) d\sigma, \text{ con } \nu \text{ normale esterna.}$$

DIMOSTRAZIONE

$$\begin{aligned} \int_{\Omega} \Delta u v - u \Delta v &= \int_{\Omega} \operatorname{div}(\nabla u v) - \nabla u \nabla v - \operatorname{div}(u \nabla v) + \nabla u \nabla v = \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu}. \end{aligned} \quad \square$$

Lemma Esiste c_n t.c. se $\Delta u = 0$ su Ω , allora $\forall x \in \Omega$

$$c_n u(x) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} K(y-x) - \frac{\partial K(y-x)}{\partial \nu} u \, d\sigma$$

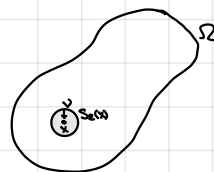
dove $K(y) = \begin{cases} -\log|y| & \text{se } n=2 \\ \frac{1}{|y|^{n-2}} & \text{se } n \geq 3 \end{cases}$ è la soluzione fondamentale.

Oss Non è vero che, fissati f e g dati al bordo per $u|_{\partial\Omega}$ e $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$, da u così costruita rispetta tali dati.

DIMOSTRAZIONE

Usando il Lemma precedente con $v = K(y-x)$ su $\Omega_\varepsilon = \Omega \setminus \bar{B}_\varepsilon(x)$:

$$\begin{aligned} 0 &= \int_{\partial\Omega_\varepsilon} \frac{\partial u}{\partial \nu} K(y-x) - u \frac{\partial K(y-x)}{\partial \nu} d\sigma = \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} K(y-x) - u \frac{\partial K(y-x)}{\partial \nu} d\sigma + \underbrace{\int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial \nu} K(y-x) - u \frac{\partial K(y-x)}{\partial \nu} d\sigma}_{\xrightarrow{\varepsilon \rightarrow 0} -c_n u(x)} \end{aligned}$$



(caso $n \geq 3$)

$$\left| \int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial \nu} K(y-x) d\sigma \right| \leq \sup_{\partial\Omega} |\nabla u| \int_{\partial B_\varepsilon(x)} \frac{1}{|y-x|^{n-2}} d\sigma = C \int_{\partial B_\varepsilon(x)} \frac{1}{\varepsilon^{n-2}} d\sigma = \frac{C}{\varepsilon^{n-2}} \omega_{n-1} \varepsilon^{n-1} = O(\varepsilon)$$

$$\int_{\partial B_\varepsilon(x)} u \frac{\partial K(y-x)}{\partial \nu} d\sigma = (n-2) \int_{\partial B_\varepsilon(x)} u \frac{y-x}{|y-x|} \cdot \frac{(y-x)}{|y-x|} \frac{1}{|y-x|^{n-1}} d\sigma = (n-2) \int_{\partial B_\varepsilon(x)} \frac{u}{|y-x|^{n-1}} d\sigma =$$

$$= (n-2) \omega_{n-1} \frac{1}{\omega_{n-1} \varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} u(x) d\sigma = (n-2) \omega_{n-1} \int_{\partial B_\varepsilon(x)} u(y) d\sigma \xrightarrow{\varepsilon \rightarrow 0} (n-2) \omega_{n-1} u(x)$$

(caso $n=2$)

$$\left| \int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial \nu} K(y-x) d\sigma \right| \leq \sup_{\partial\Omega} |\nabla u| \int_{\partial B_\varepsilon(x)} \log|y-x| d\sigma = C \int_{\partial B_\varepsilon(x)} \log \varepsilon d\sigma = C \log \varepsilon (2\pi \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$$

$$\int_{\partial B_\varepsilon(x)} u \frac{\partial K(y-x)}{\partial \nu} d\sigma = \int_{\partial B_\varepsilon(x)} u(y) \frac{y-x}{|y-x|} \cdot \left(+ \frac{1}{|y-x|} \right) \frac{y-x}{|y-x|} d\sigma = + \frac{\omega_1}{2\pi \varepsilon} \int_{\partial B_\varepsilon(x)} u(y) d\sigma \longrightarrow \omega_1 u(x) \quad \square$$

Funzioni di Green

Def. Dato $\Omega \subset \mathbb{R}^n$, si definisce funzione di Green associata a Ω la funzione $H: \Omega \times \Omega \ni (x, y) \mapsto h_x(y) \in \mathbb{R}$ dove $h_x(y)$ è definita come segue (se esiste):

$$\begin{cases} \Delta_y h_x(y) = 0 & \text{su } \Omega \\ h_x(y) = K(y-x) & \text{su } \partial\Omega \end{cases}$$

Applicando il primo lemma alla coppia u e $h_x(y)$, ottengo

$$0 = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} h_x(y) - u \frac{\partial h_x(y)}{\partial \nu} d\sigma = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} K(y-x) - u \frac{\partial h_x(y)}{\partial \nu} d\sigma$$

cioè

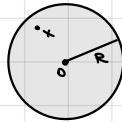
$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} K(y-x) d\sigma = \int_{\partial\Omega} u \frac{\partial h_x(y)}{\partial \nu} d\sigma$$

proposizione Se $\Delta u = 0$ in Ω , $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, allora

$$(n-2)\omega_{n-1} u(x) = \int_{\partial\Omega} u \left(\frac{\partial h_x(y)}{\partial \nu} - \frac{\partial K}{\partial \nu}(y-x) \right) d\sigma = \int_{\partial\Omega} u G(x, y) d\sigma$$

Caso $\Omega = B_R(0)$

$$\begin{cases} \Delta h_x(y) = 0 & \text{su } B_R(0) \\ h_x(y) = K(y-x) & \text{su } \partial B_R(0) \end{cases}$$



$h_x(y) = \alpha K(y-x^*)$ con $|x^*| > R$, $\alpha = \alpha(x)$, $x^* = x^*(x)$

Bisogna solo imporre la condizione al bordo.

$(n \geq 3)$ $\alpha K(y-x^*) = K(y-x) \quad \forall y \in \partial B_R(0)$

$$\frac{\alpha}{|y-x^*|^{n-2}} = \frac{1}{|y-x|^{n-2}} \quad \forall y \in \partial B_R(0)$$

$$\alpha |y-x|^{n-2} = |y-x^*|^{n-2} \quad \forall y \in \partial B_R(0) \iff \alpha^{\frac{2}{n-2}} |y-x|^2 = |y-x^*|^2 \quad \forall y \in \partial B_R(0)$$

Pongo $a = \alpha^{\frac{2}{n-2}}$: $a^2 |y-x|^2 = |y-x^*|^2 \quad \forall y \in \partial B_R(0)$, cioè

$$a^2 |y|^2 + a^2 |x|^2 - 2a^2 x \cdot y = |y|^2 + |x^*|^2 - 2y \cdot x^* \quad \forall y \in \partial B_R(0)$$

$$2y \cdot (x^* - a^2 x) = |y|^2 (1 - a^2) + |x^*|^2 - a^2 |x|^2 \quad \forall y \in \partial B_R(0)$$

$$2y \cdot (x^* - a^2 x) = R^2 (1 - a^2) + |x^*|^2 - a^2 |x|^2 \quad \forall y \in \partial B_R(0)$$

$$\text{Quindi } x^* = a^2 x, \quad R^2 (1 - a^2) + a^4 |x|^2 - a^2 |x|^2 = 0$$

$$\implies R^2 (1 - a^2) + a^2 |x|^2 (a^2 - 1) = 0 \iff (a^2 - 1) (a^2 |x|^2 - R^2) = 0$$

$$\implies a^2 = \frac{R^2}{|x|^2} \implies a = \frac{R}{|x|} \implies \alpha = a^{\frac{n-2}{2}} = \left(\frac{R}{|x|} \right)^{\frac{n-2}{2}}$$

$$x^* = \frac{R^2}{|x|^2} x$$

$$\text{Quindi } h_x(y) = \frac{R^{n-2}}{|x|^{n-2}} \frac{1}{|y - \frac{R^2}{|x|^2} x|^{n-2}} = \frac{R^{n-2}}{|x|^{n-2}} \frac{|x|^{2(n-2)}}{|n^2 y - R^2 x|^{n-2}} = \left(\frac{R|x|}{|n^2 y - R^2 x|} \right)^{n-2}$$

$(n=2)$ $\alpha K(y-x^*) + \beta = K(y-x) \quad \forall y \in \partial B_R^2(0)$

$$\beta - \alpha \log |y-x^*| = -\log |y-x| \quad \forall y \in \partial B_R^2(0)$$

$$\log \left(e^{\beta} \frac{|y-x|}{|y-x^*|} \right) = 0 \iff e^{\beta} \frac{|y-x|}{|y-x^*|} = 1 \quad \text{su } \partial B_R^2(0) \quad \text{Sceglio } \alpha=1$$

$$e^{\beta} |y-x| = |y-x^*| \quad \forall y \in \partial B_R^2(0)$$

Come prima:

$$e^{2\beta}(R^2 + |x|^2 - 2x \cdot y) = R^2 + |x^*|^2 - 2y \cdot x^* \quad \forall y \in \partial B_R(0)$$

$$2y \cdot (x^* - e^{2\beta}x) = R^2(1 - e^{2\beta}) + |x^*|^2 - e^{2\beta}|x|^2 \quad \forall y \in \partial B_R(0)$$

$$\Rightarrow x^* = e^{2\beta}x, \quad R^2(1 - e^{2\beta}) + e^{2\beta}(e^{2\beta} - 1)|x|^2 = 0$$

$$\Rightarrow \frac{(e^{2\beta} - 1)(e^{2\beta}|x|^2 - R^2)}{\neq 0} = 0 \quad \Rightarrow e^{2\beta} = \frac{R^2}{|x|^2} \quad \Rightarrow \beta = \log \frac{R}{|x|}$$

$$x^* = \frac{R^2}{|x|^2}x$$

Quindi

$$h_x(y) = \log\left(\frac{R}{|x|}\right) - \log\left|y - \frac{R^2}{|x|^2}x\right| = \log\left(\frac{R|x|}{|x|^2y - R^2x}\right)$$

Complessivamente:

$$h_x(y) = \begin{cases} \log\left(\frac{R|x|}{|x|^2y - R^2x}\right) & \text{se } n=2 \\ \left(\frac{R|x|}{|x|^2y - R^2x}\right)^{n-2} & \text{se } n \geq 3 \end{cases}$$

Ora calcoliamo $\partial_y h_x(y) - \partial_y K(y-x)$ per $y \in \partial B_R(0)$

$$(n \geq 3) \text{ Calcoliamo } \frac{\partial}{\partial y} \left(\frac{R|x|}{|x|^2y - R^2x} \right)^{n-2} - \frac{\partial}{\partial y} \frac{1}{|y-x|^{n-2}}$$

$$\frac{\partial}{\partial y} K(y-x) = \frac{y}{R} \cdot \nabla_y K(y-x) = \frac{y}{R} \cdot \frac{y-x}{|y-x|^{n-1}} \cdot \frac{2-n}{|y-x|^{n-1}} = (2-n) \frac{y}{R} \cdot \frac{y-x}{|y-x|^n}$$

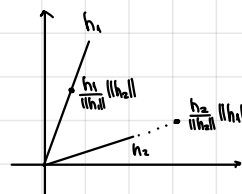
$$\frac{\partial}{\partial y} h_x(y) = \frac{y}{R} \cdot \nabla_y h_x(y) = \frac{y}{R} \cdot \frac{(|x|^2 - R^2x)}{|y|^2 - R^2x} \cdot \frac{2-n}{|y|^2 - R^2x|^{n-1}} (R|x|)^{n-2} = (2-n)|x|^2 (R|x|)^{n-2} \frac{y}{R} \cdot \frac{(|x|^2 - R^2x)}{|y|^2 - R^2x|^n}$$

Oss Se H è Hilbert, $h_1, h_2 \in H$, allora

$$\|h_1 - h_2\|^2 = \left\| \frac{h_1}{\|h_1\|} \|h_2\| - \frac{h_2}{\|h_2\|} \|h_1\| \right\|^2$$

In fatti

$$\|h_1\|^2 + \|h_2\|^2 - 2h_1 \cdot h_2 = \frac{\|h_1\|^2 \|h_2\|^2}{\|h_1\|^2} + \frac{\|h_1\|^2 \|h_2\|^2}{\|h_2\|^2} - 2h_1 \cdot h_2$$



$$\text{Ora } |y-x| = \left| \frac{y}{R}|x| - \frac{x}{|x|}R \right| = \frac{|y|^2 - R^2x|}{R|x|}$$

$$= (2-n)|x|^2 (R|x|)^{n-2} \frac{y}{R} \cdot \frac{(|x|^2 - R^2x)}{(R|x|)^n |y-x|^n} = \frac{(2-n)}{R^2} \frac{y}{R} \cdot \frac{|x|^2 - R^2x}{|y-x|^n}$$

$$\frac{\partial}{\partial y} h_x(y) - \frac{\partial}{\partial y} K(y-x) = \frac{(2-n)}{|y-x|^n} \frac{y}{R} \cdot \left(\frac{|x|^2 - R^2x}{R^2} - y+x \right) = \frac{2-n}{|y-x|^n} \frac{y}{R} \cdot \left(\frac{|x|^2 - R^2x - R^2y + R^2x}{R^2} \right) =$$

$$= \frac{2-n}{|y-x|^n} (|x|^2 - R^2) \frac{y}{R} \cdot \frac{y}{R^2} = \frac{(n-2)(R^2 - |x|^2)}{R|y-x|^n}$$

Nucleo di Poisson

Quindi la candidata soluzione è

$$u(x) = \frac{R^2 - |x|^2}{\omega_{n-1}R} \int_{\partial B_R} \frac{u(y)}{|y-x|^n} d\sigma$$

Verifichiamo che $u(x)$ funziona

teorema

Per ogni $\phi \in C(\partial B_R)$ la funzione

$$U(x) := \frac{R^2 - |x|^2}{\omega_{n-1} R} \int_{\partial B_R} \frac{\phi(y)}{|y-x|^n} d\sigma$$

soddisfa

$$\begin{cases} \Delta U = 0 & \text{su } B_R \\ U|_{\partial B_R} = \phi \end{cases}$$

DIMOSTRAZIONE

1. $\Delta U = 0$ su B_R

Posso derivare sotto il segno di integrale perché $x \in B_R, y \in \partial B_R$

$$\Delta_x \frac{R^2 - |x|^2}{|y-x|^n} \stackrel{?}{=} 0$$

$$\Delta_x (R^2 - |x|^2) = -2n$$

$$\Delta_x \frac{1}{|x-y|^n} = \frac{n(n+1)}{|x-y|^{n+2}} - \frac{n(n-1)}{|x-y|^{n+2}} = \frac{2n}{|x-y|^{n+2}}$$

$$\frac{d}{dn} \frac{1}{n} = -\frac{1}{n^2}, \quad \frac{d^2}{dn^2} \frac{1}{n} = \frac{2}{n^3}$$

$$2 \nabla_x (R^2 - |x|^2) \nabla_x \frac{1}{|x-y|^n} = 4n x \cdot \frac{x-y}{|x-y|} \frac{1}{|x-y|^{n+1}} = \frac{4n|x|^2 - 4n x \cdot y}{|x-y|^{n+2}}$$

$$-\frac{2n}{|y-x|^n} + \frac{2n}{|y-x|^{n+2}} (R^2 - |x|^2) + \frac{4n|x|^2 - 4n x \cdot y}{|x-y|^{n+2}}$$

$$\frac{-2n(R^2 + |x|^2 - 2x \cdot y) + 2nR^2 - 2n|x|^2 + 4n|x|^2 - 4n x \cdot y}{|y-x|^{n+2}} = 0$$

2. $\forall y_0 \in \partial B_R \quad \lim_{x \rightarrow y_0} U(x) = \phi(y_0)$

$$U(x) - \phi(y_0) \xrightarrow{x \rightarrow y_0} 0$$

$$\text{Claim: } \frac{1}{\omega_{n-1} R} \int_{\partial B_R} \frac{R^2 - |x|^2}{|y-x|^n} d\sigma = 1$$

Assumendola, possiamo scrivere

$$U(x) - \phi(y_0) = \frac{1}{\omega_{n-1} R} \int_{\partial B_R} \frac{R^2 - |x|^2}{|y-x|^n} (\phi(y) - \phi(y_0)) d\sigma$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ t.c. } |y-y_0| < \delta \Rightarrow |\phi(y) - \phi(y_0)| < \varepsilon$$

$$\frac{1}{\omega_{n-1} R} \int_{\partial B_R \cap B(y_0, \delta)} \frac{(R^2 - |x|^2)}{|y-x|^n} |\phi(y) - \phi(y_0)| d\sigma \leq \varepsilon \frac{1}{\omega_{n-1} R} \int_{\partial B_R \cap B(y_0, \delta)} \frac{R^2 - |x|^2}{|y-x|^n} d\sigma \leq \varepsilon$$

$$\frac{1}{\omega_{n-1} R} \int_{\partial B_R \setminus B(y_0, \delta)} \frac{(R^2 - |x|^2)}{|y-x|^n} |\phi(y) - \phi(y_0)| d\sigma \leq \frac{2\|\phi\|_\infty}{\omega_{n-1} R} \int_{\partial B_R \setminus B(y_0, \delta)} \frac{R^2 - |x|^2}{|y-x|^n} d\sigma$$

$$\text{Dico che } |y-x| \geq |y-y_0| - |y_0-x| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} \quad \text{su } x \in B(y_0, \frac{\delta}{2})$$

$$\leq \frac{2\|\phi\|_\infty (R^2 - |x|^2)}{\omega_{n-1} R} \left(\frac{2}{\delta}\right)^n$$

$$\text{Posso ora prendere } N \vee R \text{ t.c. } \frac{2\|\phi\|_\infty (R^2 - |x|^2)}{\omega_{n-1} R} \left(\frac{2}{\delta}\right)^n < \varepsilon$$

$$\text{Proviamo ora il claim: } \frac{1}{\omega_{n-1} R} \int_{\partial B_R} \frac{R^2 - |x|^2}{|y-x|^n} d\sigma = 1 \quad \forall x \in B_R$$

$U(x)$ è armonica e $U(x) = U(x)$ $\Rightarrow$ U costante: $U(x) = U(0)$

$$\text{Per } x=0: \frac{R^2}{\omega_{n-1} R} \int_{\partial B_R} \frac{1}{|y|^n} d\sigma = \frac{R}{\omega_{n-1}} \frac{1}{R^n} \omega_{n-1} R^n = 1$$

U è radiale: O trasformazione ortogonale su $\mathbb{R}^n$

$$\int_{\partial B_R} \frac{1}{|y-x|^n} d\sigma = \int_{\partial B_R} \frac{1}{|y-Ox|^n} d\sigma$$

$$\int_{\partial B_R} \frac{1}{|y-Ox|^n} d\sigma \xrightarrow{O} \int_{\partial B_R} \frac{1}{|y|^n} d\sigma$$

$$\text{Se } g: \partial B_R \rightarrow \mathbb{R}, \quad \int_{\partial B_R} g(Oy) d\sigma = \int_{\partial B_R} g(y) d\sigma$$



Rimozione di singolarità

Se $\varphi \in H(B_R \setminus \{0\}) \cap L^\infty(B_R \setminus \{0\})$, allora $\varphi \in H(B_R)$.

$$\Delta u = 0 \implies u(x) = \int_{S_{r(x)}} u \, ds$$

$$\Delta u = 0 \iff u \in C \text{ e } u(x) = \int_{S_{r(x)}} u \, ds$$

$\hookrightarrow u = u * P_\varepsilon$: questo dimostra che le funzioni armoniche sono C^∞

$$\Delta u = u * \Delta P_\varepsilon = 0$$

$$u * P_\varepsilon(x) = \int_{B_\varepsilon} u(y) P_\varepsilon(x-y) \, dy$$

$$\forall x \exists r(x) > 0 \text{ t.c. } u(x) = \int_{S_{r(x)}} u \, ds \quad \forall r \in [0, r(x)] \implies u \text{ armonica}$$

Teorema di estensibilità

Sia $u \in C^2(\Omega \setminus \{x_0\})$, $\Delta u = 0$ su $\Omega \setminus \{x_0\}$ con $u = \begin{cases} 0 \left(\frac{1}{|x-x_0|^{n-2}} \right) & \text{se } n \geq 3 \\ 0 \left(\log(x-x_0) \right) & \text{se } n = 2 \end{cases}$
Allora $u \in C^2(\Omega)$, quindi $\Delta u = 0$ su Ω .

DIMOSTRAZIONE

Wlog $x_0 = 0$, $B(0, r) \subset \Omega$. Suppongo $\Omega = B(0, r)$

Dico che posso supporre $u|_{\partial B(0, r)} = 0$:

$$\begin{cases} \Delta u = 0 & \text{su } B(0, r) \\ u|_{\partial B(0, r)} = u \end{cases}$$

Per Poisson, $\exists!$ soluzione

$$\implies W = u - v \text{ è t.c. : } \begin{cases} W|_{\partial B(0, r)} = 0 \\ \Delta W = 0 \text{ su } B(0, r) \setminus \{0\} \\ W = \begin{cases} 0 \left(\frac{1}{|x-x_0|^{n-2}} \right) & \text{se } n \geq 3 \\ 0 \left(\log(x-x_0) \right) & \text{se } n = 2 \end{cases} \end{cases}$$

($n \geq 3$)

$$\text{Introduciamo } W_\varepsilon(x) = W(x) - \varepsilon \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) = W(x) - \underbrace{\frac{\varepsilon}{|x|^{n-2}}}_{< 0 \text{ se } |x| < \delta_\varepsilon} + \frac{\varepsilon}{r^{n-2}}$$

$$\text{Per il principio del massimo, } W_\varepsilon(x) \leq \frac{\varepsilon}{r^{n-2}} \quad \forall \delta_\varepsilon < |x| < r$$

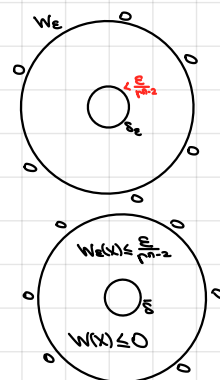
$$\downarrow \varepsilon \rightarrow 0 \quad \downarrow \varepsilon \rightarrow 0$$

$$W(x) \leq 0$$

$$\implies W(x) \leq 0 \text{ su } B(0, r) \setminus \{0\}$$

Lo stesso vale per $-W$: $-W(x) \leq 0$ su $B(0, r) \setminus \{0\}$

$$\implies W(x) = 0 \text{ su } B(0, r) \setminus \{0\} \implies W(x) = 0 \text{ su } B(0, r)$$



□

DEF $u \in C(\Omega)$ soddisfa il principio della media forte (MF) se

$$u(x) = \int_{S_{r(x)}} u \, ds \quad \forall S_{r(x)} \subset \Omega$$

$u \in C(\Omega)$ soddisfa il principio della media debole (MD) se

$$u(x) = \int_{S_{r(x)}} u \, ds \quad \forall r < r_0(x)$$

teorema Se u è armonica, vale (MF).
Se $u \in C(\Omega)$ e vale (MF), allora u è armonica.

DIMOSTRAZIONE

Passo 1 $u \in C^\infty(\Omega)$

Dico che $u(x) = u * p_\varepsilon(x)$ (Posso supporre $\Omega = \mathbb{R}^n$, altrimenti lavoro con Ω_ε)

$$\begin{aligned} u * p_\varepsilon(x) &= \int_{\mathbb{R}^n} u(x-y) p_\varepsilon(y) dy = \int_0^\infty \left(\int_{\Omega_\varepsilon} u(x-y) p_\varepsilon(y) ds \right) dr = \int_0^\infty p_\varepsilon(r) \left(\int_{\Omega_\varepsilon} u(x-y) ds \right) dr = \\ &= \int_0^\infty p_\varepsilon(r) |\Omega| u(x) dr = u(x) \int_0^\infty p_\varepsilon(r) n \omega_n r^{n-1} dr = u(x) \end{aligned}$$

Passo 2 $\Delta u = 0$

$\Delta u = u * \Delta p_\varepsilon = 0$ perché (stessi conti di sopra):

$$u * \Delta p_\varepsilon(x) = u(x) \int_0^\infty \Delta p_\varepsilon(r) n \omega_n r^{n-1} dr = 0$$

□

Lemma Se w soddisfa (MD) su un connesso, allora vale una tra le seguenti:
(i) w è costante
(ii) w non ammette max e min interni

DIMOSTRAZIONE

$M = \max_{\bar{\Omega}} w$, allora $\{u(x) = M\}$ è chiuso per continuità e aperto per (MD)

$$\Rightarrow \{u(x) = M\} = \emptyset \text{ o } \Omega$$

□

teorema Se $u \in C(\Omega)$ e vale (MD), allora u è armonica.

DIMOSTRAZIONE

Dico che u è armonica su $B(x_0, r) \subset \Omega$

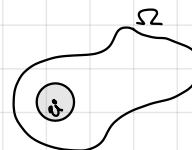
$$\begin{cases} \Delta u = 0 & \text{su } B(x_0, r) \\ u|_{\partial B(x_0, r)} = u|_{\partial B(x_0, r)} & \exists! \end{cases}$$

Dico che $u = \bar{u}$ su $B(x_0, r)$

$w = u - \bar{u}$ soddisfa (MD) e inoltre $w|_{\partial B(x_0, r)} = 0$

$$\Rightarrow w \equiv 0 \text{ su } B(x_0, r)$$

□



Metodo di Perron

Consideriamo

$$\begin{cases} \Delta u = 0 & \text{su } \Omega \\ u|_{\partial\Omega} = \varphi \in C(\partial\Omega) \end{cases} \quad \Omega \text{ connesso limitato}$$

Def. $u \in C(\Omega)$ si dice *subarmonica* se
 $u(x) \leq \int_{S_{r,x}} u \, ds \quad \forall 0 < r < r_0(x)$

Sia $Y_\varphi = \{u: \Omega \rightarrow \mathbb{R} \mid u \in C(\Omega) \cap \text{Sub}(\Omega) \text{ e } u(y) \leq \varphi(y) \, \forall y \in \partial\Omega\}$

$$u(y) = \sup_{u \in Y_\varphi} u(y)$$

$$Y_\varphi \neq \emptyset \text{ perché } u = \inf \varphi \in Y_\varphi$$

Mostreremo che: $\bullet \Delta u = 0 \, \forall \Omega$

$\bullet u|_{\partial\Omega} = \varphi$ sotto opportune ipotesi su Ω

Proposizione (reticolo) Se $u, v \in \text{Sub}(\Omega)$, allora $\max\{u, v\} = u \vee v \in \text{Sub}(\Omega)$.

DIMOSTRAZIONE

$$\begin{aligned} u(x) &\leq \int_{S_{r,x}} u \, ds \quad \forall 0 < r < r_0(x), \quad v(x) \leq \int_{S_{r,x}} v \, ds \quad \forall 0 < r < r_1(x) \\ \Rightarrow \max\{u(x), v(x)\} &\leq \max\left\{ \int_{S_{r,x}} u \, ds, \int_{S_{r,x}} v \, ds \right\} \quad \forall 0 < r < \min\{r_0(x), r_1(x)\} \\ &\leq \int_{S_{r,x}} \max\{u, v\} \, ds \end{aligned}$$

□

Proposizione (principio del massimo) Se $u \in \text{Sub}(\Omega) \cap C(\bar{\Omega})$, allora $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

DIMOSTRAZIONE

Vale l'alternativa: (i) u costante

(ii) u non ha max interni

considerando $\{u(x) = \max_{\bar{\Omega}} u\}$ aperto e chiuso.

□

Lemma di incollamento

Sia $u \in \text{Sub}(\Omega)$, $B(x_0, r) \subset \Omega$.

Allora se $\begin{cases} \Delta v = 0 & \text{su } B(x_0, r) \\ v = u & \text{su } \partial B(x_0, r) \end{cases}$,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{se } x \notin B(x_0, r) \\ v(x) & \text{se } x \in B(x_0, r) \end{cases} \quad \text{è } \text{Sub}(\Omega)$$

e inoltre $\tilde{u}(x) \geq u(x)$.

DIMOSTRAZIONE

Abbiamo 3 casi.

$\bullet x \in B^c(x_0, r)$: uso il fatto che $u \in \text{Sub}(\Omega)$

$\bullet x \in B(x_0, r)$: uso che v armonica e quindi vale (HF)

$\bullet x \in \partial B(x_0, r)$: $\int_{S_{r,x}} \tilde{u} \, ds = \frac{1}{|S_r|} \left(\int_{S_{r,x} \cap B^c(x_0, r)} \tilde{u} \, ds + \int_{S_{r,x} \cap B(x_0, r)} \tilde{u} \, ds \right)$

Dico che $u(x) \leq \tilde{u}(x)$ su $B(x_0, r)$: $u - v$ è subarmonica su $B(x_0, r)$

quindi $u - v = 0$ su $\partial B(x_0, r) \Rightarrow u(x) \leq v(x)$ su $B(x_0, r)$

$$\geq \frac{1}{|S_r|} \left(\int_{S_{r,x} \cap B^c(x_0, r)} u \, ds + \int_{S_{r,x} \cap B(x_0, r)} u \, ds \right) \geq u(x) = \tilde{u}(x)$$

□

Sia (u_n) t.c. $\Delta u_n = 0$ su Ω , $\sup \|u_n\|_{L^\infty(\Omega)} < +\infty$, $u_n(x) \geq u(x) \forall x \in \Omega$
 Allora $\Delta u = 0$ su Ω

Consideriamo $g \in C(\partial\Omega)$

$$Y_g = \{u \in \text{Sub}(\Omega) \cap C(\bar{\Omega}) : u \leq g \text{ su } \partial\Omega\}$$

Oss $u \in Y_g \implies \max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} g$

$$u(x) = \sup_{u \in Y_g} u(x)$$

teorema $\Delta u = 0$ su Ω .

DIMOSTRAZIONE

Fissiamo $x_0 \in \Omega$ e $B = B(x_0, r) \subset \Omega$

Per definizione, $\exists (u_n) \subset Y_g$ t.c. $u_n(x_0) \xrightarrow{n \rightarrow \infty} u(x_0)$

Posso supporre che $u_n \uparrow$ e $\Delta u_n = 0$ su B :

da u_1 , considero $\bar{u}_1 \in Y_g$ per incollamento

da u_2 , considero $\bar{u}_1 \vee u_2 \in Y_g, \dots$ poi itero

Quindi $u_n \rightarrow \bar{u}$ t.c. $\bar{u} \in Y_g$, $\bar{u}(x_0) = u(x_0)$, $\Delta \bar{u} = 0$ su B

Dico che $u(y) = \bar{u}(y) \forall y \in B$

Per def, $\exists (u_n) \subset Y_g$ t.c. $u_n(y) \xrightarrow{n \rightarrow \infty} u(y)$

Posso supporre che $u_n(x_0) \xrightarrow{n \rightarrow \infty} u(x_0)$:

Infatti posso sempre considerare $u_n \vee u_n$

Inoltre posso supporre per reticolo e incollamento che $u_n \uparrow$ e $\Delta u_n = 0$ su B

Quindi abbiamo che $u_n \rightarrow \bar{u}$ t.c. $\Delta \bar{u} = 0$ su B , $u_n(x_0) \rightarrow \bar{u}(x_0) = u(x_0)$

e $u_n \geq u \implies \bar{u}(x) \geq u(x) \forall x \in B$

Ora $\Delta(\bar{u} - u) = 0$ su B , $\bar{u}(x_0) - u(x_0) = 0$, $\bar{u} - u \leq 0$ su B

Per il massimo forte, $\bar{u} - u = 0 \implies u(y) = \bar{u}(y) = \bar{u}(y)$. □

Domanda: $u(y) = g(y)$ su $\partial\Omega$?

DEF $x_0 \in \partial\Omega$ è regolare se $\exists \phi_{x_0} \in \text{Sub}(\Omega) \cap C(\bar{\Omega})$ t.c.
 $\phi_{x_0}(x_0) = 0$ e $\phi_{x_0}(x) < 0 \forall x \in \partial\Omega \setminus \{x_0\}$

teorema Se esiste $B(y_0, r)$ t.c. $\overline{B(y_0, r)} \cap \partial\Omega = \{x_0\}$, $B(y_0, r) \cap \Omega \neq \emptyset$
 allora x_0 è regolare.

DIMOSTRAZIONE

Basta scegliere $\phi_{x_0}(x) = \frac{1}{|x - y_0|^{n-2}} - \frac{1}{r^{n-2}}$ □

teorema Sia $v(x)$ la funzione di Perron su Ω con dato g .
Se $x_0 \in \partial\Omega$ è regolare, allora $\lim_{x \rightarrow x_0} v(x) = g(x_0)$.

DIMOSTRAZIONE

• $\liminf_{x \rightarrow x_0} v(x) \geq g(x_0)$

Dico che $\forall \varepsilon > 0 \exists s_\varepsilon(x) \in Y_g$ t.c. $s_\varepsilon(x_0) = g(x_0) - \varepsilon$

$\Rightarrow \exists B(x_0, \delta)$ t.c. $s_\varepsilon(x) \geq g(x_0) - 2\varepsilon \quad \forall x \in B(x_0, \delta) \cap \Omega$

$\Rightarrow \forall x \in B(x_0, \delta) \cap \Omega \quad v(x) \geq s_\varepsilon(x) \geq g(x_0) - 2\varepsilon$

Ora costruiamo $s_\varepsilon(x) := M\phi_{x_0}(x) + g(x_0) - \varepsilon \in \text{Sub}(\Omega) \cap C(\bar{\Omega})$:

$\exists M \gg 1$ t.c. $M\phi_{x_0}(x) + g(x_0) - \varepsilon \leq g(x) \quad \forall x \in \partial\Omega$

$M\phi_{x_0}(x) + g(x_0) - \varepsilon \leq g(x_0) - \varepsilon \Rightarrow$ poiché g è continua in x_0 , $g(x_0) - \frac{\varepsilon}{2} \leq g(x) \quad \forall x \in \partial\Omega \cap B(x_0, \delta)$

Se mi trovo su $\partial\Omega \cap B(x_0, \delta)$, allora

$$M\phi_{x_0}(x) + g(x_0) - \varepsilon \leq M \underbrace{\sup_{\partial\Omega \cap B(x_0, \delta)} \phi}_{< 0} + g(x_0) - \varepsilon \leq \inf_{\partial\Omega} g$$

• $\limsup_{x \rightarrow x_0} v(x) \leq g(x_0)$

$\exists s_\varepsilon \in Y_g$ t.c. $s_\varepsilon(x_0) = -g(x_0) - \varepsilon$

$\forall w \in Y_g : s_\varepsilon(x) + w(x) \leq -g(x) + g(x) = 0$ su $\partial\Omega$

Per il principio del massimo, $s_\varepsilon + w \leq 0$ su $\Omega \quad \forall w \in Y_g$

$\Rightarrow w(x) \leq -s_\varepsilon(x) \quad \forall w \in Y_g \quad \forall x \in \Omega$

Inoltre sappiamo che $-s_\varepsilon(x_0) = g(x_0) + \varepsilon$

$\Rightarrow -s_\varepsilon(x) \leq g(x_0) + 2\varepsilon \quad \text{se } x \in B(x_0, \delta) \cap \Omega$

$\Rightarrow w(x) \leq g(x_0) + 2\varepsilon \quad \forall w \in Y_g \quad \forall x \in B(x_0, \delta) \cap \Omega$

□

proposizione $\Delta u_n = 0$ su Ω , $\sup_n \|u_n\|_{C(\bar{\Omega})} < +\infty$, $u_n \nearrow$.
Allora $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ è armonica su Ω .

DIMOSTRAZIONE

Verifico che $u \in C(\bar{\Omega})$ e vale $f_{\text{sern}} u = u(x) \quad \forall x(x) \in \Omega$

Per ipotesi, $u_n(x) = f_{\text{sern}} u_n dx$

$$\downarrow \quad \downarrow$$

$$u(x) = f_{\text{sern}} u dx \quad \text{per CondDom (o ConMon)}$$

Ora $u_n(x) = f_{\text{sern}} u_n dx$

$$\downarrow \quad \downarrow$$

$$u(x) = f_{\text{sern}} u dx$$

Dico che se $u \in L^\infty$, $x \mapsto \int_{\text{sern}} u dx$ è C^0

$$\int_{\text{sern}} u dx = \int_{\Omega} \chi_{\text{sern}} u dx \xrightarrow{\text{se } x_n \rightarrow x, \chi_{\text{sern}} \rightarrow \chi_{\text{sern}}} \int_{\Omega} \chi_{\text{sern}} u dx \quad \text{per CondDom}$$

□

esempio Se $x_n \rightarrow 0$, $x_n \in \mathbb{R} \setminus \mathbb{Q}$
 $\chi_{\mathbb{Q} \cup x_n} \not\rightarrow \chi_{\mathbb{Q}}$

teorema Sia $\Delta u_n = 0$ su Ω connesso, $u_n \nearrow$, $\exists x_0 \in \Omega$ t.c. $\sup \|u_n(x_0)\| < +\infty$.
Allora $u_n \xrightarrow{\text{lim}(\Omega)} u$ e quindi $\Delta u = 0$.

Disuguaglianza di Harnack

Teorema: disuguaglianza di Harnack

Sia $\Omega \subseteq \mathbb{R}^n$ aperto connesso, $K \subset \Omega$ compatto.
Allora $\exists C = C(\Omega, K) > 0$ t.c. $\forall u$ con $\Delta u = 0$ su Ω , $u \geq 0$
vale $\max u \leq C \min u$.

Oss. Non si può estendere il risultato a tutto Ω

- la connessione è fondamentale

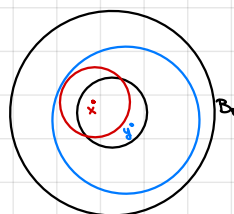
Lemma $\Delta u = 0$ su B_r , $u \geq 0$ Allora $\forall x, y \in B_{r/4}$ $u(x) \leq 3^n u(y)$

DIMOSTRAZIONE

$$B(x, \frac{r}{4}) \subseteq B(y, \frac{3}{4}r) \subseteq B_r$$

$$\Rightarrow u(x) = \int_{B(x, \frac{r}{4})} u dz^n \leq \frac{\int_{B(y, \frac{3}{4}r)} u dz^n}{|B(x, \frac{r}{4})|} =$$

$$= \int_{B(y, \frac{3}{4}r)} u dz^n \frac{|B(y, \frac{3}{4}r)|}{|B(x, \frac{r}{4})|} = u(y) 3^n$$



□

DIMOSTRAZIONE

$$K \subset \Omega, r = \text{dist}(K, \Omega^c) > 0$$

Per compattezza $\exists x_1, \dots, x_N \in K$ t.c. $K \subseteq \bigcup_{i=1}^N B(x_i, \frac{r}{2})$

Assumiamo K compatto e connesso (infatti, se Ω è connesso,

ogni $K \subset \Omega$ compatto è contenuto in un compatto connesso, congiungendo i centri).

Se $N=2$, per il Lemma

$$u(x) \leq 3^n u(z) \leq 3^{2n} u(y)$$

Ora $\forall i, j \in \{1, \dots, N\} \exists i_1, \dots, i_k = j$ t.c. $B(x_{i_1}, \frac{r}{2}) \cap B(x_{i_2}, \frac{r}{2}) \neq \emptyset$.

$$I_1 = \{i_1\}, I_2 = \{i_2, \dots, i_k\} \text{ t.c. } B(x_{i_1}, \frac{r}{2}) \cap B(x_{i_2}, \frac{r}{2}) \neq \emptyset, \dots$$

Iterando, $I_m = \{i_m, \dots, i_N\} \text{ t.c. } B(x_{i_m}, \frac{r}{2}) \cap \bigcup_{k=i_m+1}^N B(x_{i_k}, \frac{r}{2}) \neq \emptyset$ saturiamo gli indici $\{1, \dots, N\}$.

Quindi $u(x) \leq 3^{nN} u(y) \forall x, y \in K$.



□

Teorema $\Delta u_n = 0$ su Ω connesso, $u_n \uparrow$, $|u_n(x_0)| < C$
Allora $\forall K$ compatto in Ω $u_n \xrightarrow{L^\infty(K)} u$ e inoltre $\Delta u = 0$.

DIMOSTRAZIONE

$$\forall \ell, m \in \mathbb{N}, \ell > m \quad \Delta(u_\ell - u_m) = 0 \text{ e } u_\ell - u_m \geq 0$$

Per Harnack, $\exists C = C(\Omega, \tilde{K})$ t.c. se $\tilde{K} = K \cup \{x_0\}$, allora

$$0 \leq u_\ell(x) - u_m(x) \leq C(u_\ell(x_0) - u_m(x_0))$$

Siccome $u_\ell(x_0)$ converge a $\sup u_n(x_0)$, $(u_n(x_0))$ è di Cauchy:

$$\forall \varepsilon > 0 \quad C(u_\ell(x_0) - u_m(x_0)) < C\varepsilon \text{ se } \ell, m > 1$$

$$u_\ell(x) - u_m(x) \geq 0$$

□

teorema Sia (u_n) f.c. $\Delta u_n = 0$ su Ω , $\sup \|u_n\|_{L^\infty(\Omega)} < +\infty$
 Allora $\exists u_n$ f.c. $u_n \xrightarrow{L^\infty(K)} u \quad \forall K$ compatto

DIMOSTRAZIONE

Basta dimostrarlo per un singolo compatto, poi applico un processo diagonale.

Sia $K \subset \Omega$ compatto. Restringiamo che (u_n) è equicontinua su K .

Usiamo le stime del gradiente

Sia $r = \text{dist}(K, \partial\Omega)$, $K \subset \bigcup_{i=1}^N B(x_i, \frac{r}{2})$

$\sup \|\nabla u_n\|_{L^\infty(B(x_i, \frac{r}{2}))} < +\infty$:

$$|\partial_{x_j} u(x)| = \left| \int_{B(x, \frac{r}{2})} \partial_{x_j} u \, dz^n \right| = \frac{1}{|B(x, \frac{r}{2})|} \left| \int_{B(x, \frac{r}{2})} \partial_{x_j} u \, dz^n \right| = \quad \text{con } x \in B(x_i, \frac{r}{2})$$

$$= \frac{C}{r^n} \left| \int_{B(x, \frac{r}{2})} u \, \nu_j \, d\sigma \right| \leq \frac{C}{r^n} \int_{B(x, \frac{r}{2})} |u| \, d\sigma \leq \frac{C}{r}$$

Possiamo quindi applicare Ascoli-Arzelà su $B(x_i, \frac{r}{2})$

Poiché è ricoperto da un numero finito di palle, vale su K . □

**teorema di
Liouville (forte)**

Sia u con $\Delta u = 0$ su $\mathbb{R}^n$, $\inf u > -\infty$.
 Allora $u \equiv \text{cost}$.

DIMOSTRAZIONE (1)

Possiamo supporre $u \geq 0$.

$$\partial_{x_j} u(x) = \int_{B(x, r)} \partial_{x_j} u \, dz^n = \frac{\int_{B(x, r)} u \, \nu_j \, d\sigma}{|B(x, r)|}$$

$$\Rightarrow |\partial_{x_j} u(x)| \leq \frac{C}{r^n} \int_{B(x, r)} |u| \, d\sigma = \frac{C}{r^n} r^{n-1} u(x) = \frac{C}{r} \xrightarrow{r \rightarrow +\infty} 0$$

DIMOSTRAZIONE (2)

$u_e(x) := u(x)$, $\Delta u_e = 0$, $u_e \geq 0$.

Applico Harnack con $\Omega = \mathbb{R}^n$, $K = \overline{B(0, 1)}$:

$\forall x \in K \quad u(x) = u_e(x) \leq C u_e(0) = C u(0)$

$$\Rightarrow \sup_{\text{each}} u \leq C u(0) \quad \forall e \Rightarrow \sup_{\mathbb{R}^n} u \leq C u(0)$$

Siccome $u \geq 0$, $u \in L^\infty(\mathbb{R}^n)$. □

EQUAZIONE DEL CALORE

Consideriamo

$$\begin{cases} \partial_t u - \Delta_x u = 0 & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0, x) = f(x) \end{cases}$$

In generale

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(0, x) = 0 \end{cases} \Rightarrow u \equiv 0$$

quindi non può esserci unicità.

Esistenza: $e^{t\Delta} f(x)$ semigrupp del calore
(in analogia con: $u' = Au \Rightarrow u(t) = e^{tA}$)

Trasformata di Fourier

$$\begin{aligned} \mathcal{F}f(\xi) &= \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx \\ \mathcal{F}'g(x) &= \check{g}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} g(\xi) d\xi \end{aligned}$$

Quando ha senso, $\mathcal{F}' \circ \mathcal{F} f = f$

Inoltre $\mathcal{F}(\Delta f) = |\xi|^2 \hat{f}(\xi)$, $\mathcal{F}'(fg) = \mathcal{F}'(f) * \mathcal{F}'(g)$

Se $\partial_t u - \Delta_x u = 0$, allora $\mathcal{F}_{x \rightarrow \xi}(\partial_t u - \Delta_x u) = 0$, cioè

Ora $\int e^{-ix \cdot \xi} \partial_t u(t, x) dx = \frac{d}{dt} \int e^{-ix \cdot \xi} u(t, x) dx = \frac{d}{dt} \hat{u}(t, \xi)$, quindi

$$\begin{cases} \frac{d}{dt} \hat{u}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) = 0 \\ \hat{u}(0, \xi) = \hat{f}(\xi) \end{cases}$$

$$\Rightarrow \hat{u}(t, \xi) = e^{-t|\xi|^2} \hat{f}(\xi)$$

$$\Rightarrow u(t, x) = \mathcal{F}' \hat{u}(t, \xi) = \mathcal{F}'(e^{-t|\xi|^2} \hat{f}(\xi)) = \mathcal{F}'(e^{-t|\xi|^2}) * f(x)$$

Calcolo di $\mathcal{F}'(e^{-t|\xi|^2})$

$$\mathcal{F}'(e^{-t|\xi|^2})(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-t|\xi|^2} d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \prod_{j=1}^n e^{ix_j \xi_j - t \xi_j^2} d\xi_1 \dots d\xi_n = \prod_{j=1}^n \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix_j \xi_j - t \xi_j^2} d\xi_j = \prod_{j=1}^n \frac{1}{\sqrt{t}} g_t(x_j)$$

Basta allora calcolare $g_t(x) = \int_{\mathbb{R}} e^{ix\xi - t\xi^2} d\xi$

$$\int_{\mathbb{R}} e^{ix\xi - t\xi^2} d\xi = \int_{\mathbb{R}} e^{-(t\xi^2 - ix\xi)} d\xi = \int_{\mathbb{R}} e^{-(t(\xi - \frac{ix}{2t})^2 + (-\frac{ix}{2t})^2)} d\xi = e^{-\frac{x^2}{4t}} \int_{\mathbb{R}} e^{-(t\xi - \frac{ix}{2})^2} d\xi = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}} \int_{\mathbb{R}} e^{-(\eta - \frac{ix}{2\sqrt{t}})^2} d\eta$$

Lemma Per ogni $y \in \mathbb{R}$, $\int_{\mathbb{R}} e^{-(\eta + iy)^2} d\eta = \int_{\mathbb{R}} e^{-\eta^2} d\eta$.

DIMOSTRAZIONE

Consideriamo $z \mapsto e^{-z^2}$ olomorfa su $\mathbb{C}$

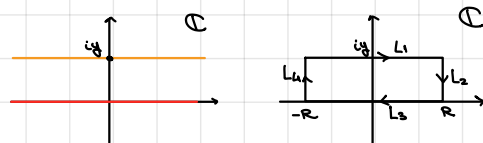
Per olomorfia

$$\int_{L_1} e^{-z^2} dz + \int_{L_2} e^{-z^2} dz + \int_{L_3} e^{-z^2} dz + \int_{L_4} e^{-z^2} dz = 0$$

$$\begin{array}{cccc} \int_{L_1} e^{-z^2} dz & \int_{L_2} e^{-z^2} dz & \int_{L_3} e^{-z^2} dz & \int_{L_4} e^{-z^2} dz \\ \downarrow R \rightarrow +\infty & \downarrow & \downarrow R \rightarrow +\infty & \downarrow \\ \int_{\mathbb{R}} e^{-(x+iy)^2} dx & 0 & -\int_{\mathbb{R}} e^{-x^2} dx & 0 \end{array}$$

$$|\int_{L_1} e^{-z^2} dz| = |\int_0^y e^{-(R+in)^2} i d\eta| = |\int_0^y e^{-R^2 + \eta^2 + 2i\eta R} d\eta| \leq \int_0^y e^{-R^2 + \eta^2} d\eta = o(y) e^{-R^2} \xrightarrow{R \rightarrow +\infty} 0$$

□



$$\text{Quindi } g_t(x) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}} \int_{\mathbb{R}} e^{-\eta^2} d\eta = \sqrt{\pi} \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}$$

$$\text{Allora } \mathcal{F}^{-1}(e^{-t\xi^2})(x) = \prod_{j=1}^n \frac{\sqrt{\pi}}{2\pi t} e^{-\frac{x_j^2}{4t}} = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} =: K_t(x)$$

**Teorema di
esistenza**

$$\text{Sia } K_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad \int_{\mathbb{R}^n} K_t(x) dx = 1$$

Sia $f \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Allora $u(t, x) := K_t * f(x) \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}^n$ soddisfa:

$$(1) \quad u(t, x) \in C^\infty((0, +\infty) \times \mathbb{R}^n)$$

$$(2) \quad \partial_t u - \Delta_x u = 0 \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}^n$$

$$(3) \quad \forall x_0 \in \mathbb{R}^n \quad \lim_{(t, x) \rightarrow (0, x_0)} u(t, x) = f(x_0)$$

DIMOSTRAZIONE

$$(1) \quad \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{n/2}} f(y) dy : \text{ si può derivare sotto il segno di integrale}$$

$$(2) \quad \text{Basta verificare che } \partial_t K_t(x) - \Delta_x K_t(x) = 0$$

$$(4\pi)^{\frac{n}{2}} \partial_t K_t(x) = \frac{|x|^2}{4t^2} e^{-\frac{|x|^2}{4t}} - \frac{n}{2} e^{-\frac{|x|^2}{4t}} t^{-\frac{n}{2}-1}$$

$$\begin{aligned} (4\pi)^{\frac{n}{2}} \Delta_x K_t(x) &= \frac{1}{t^{\frac{n}{2}}} \Delta_x (e^{-\frac{|x|^2}{4t}}) = \frac{1}{t^{\frac{n}{2}}} \left(\partial_r^2 e^{-\frac{r^2}{4t}} + \frac{n-1}{r} \partial_r e^{-\frac{r^2}{4t}} \right) = \\ &= \frac{1}{t^{\frac{n}{2}}} \left(\partial_r \left(-\frac{r}{2t} e^{-\frac{r^2}{4t}} \right) + \frac{n-1}{r} \left(-\frac{r}{2t} e^{-\frac{r^2}{4t}} \right) \right) = \frac{1}{t^{\frac{n}{2}}} \left(-\frac{1}{2t} e^{-\frac{r^2}{4t}} - \frac{r}{2t} \left(-\frac{r}{2t} e^{-\frac{r^2}{4t}} \right) - \frac{n-1}{2t} e^{-\frac{r^2}{4t}} \right) = \\ &= -\frac{n}{2t^{\frac{n}{2}+1}} e^{-\frac{r^2}{4t}} + \frac{r^2}{4t^{\frac{n}{2}+2}} e^{-\frac{r^2}{4t}} \end{aligned}$$

$$(3) \quad \lim_{t \rightarrow 0^+, x \rightarrow x_0} u(t, x) = f(x_0)$$

$$\begin{aligned} |u(t, x) - f(x_0)| &= \left| \int_{\mathbb{R}^n} K_t(x-y) f(y) dy - f(x_0) \right| = \left| \int_{\mathbb{R}^n} K_t(x-y) f(y) dy - \int_{\mathbb{R}^n} K_t(y) f(x_0) dy \right| = \\ &= \left| \int_{\mathbb{R}^n} K_t(y) f(x+y) dy - \int_{\mathbb{R}^n} K_t(y) f(x_0) dy \right| = \left| \int_{\mathbb{R}^n} K_t(y) (f(x+y) - f(x_0)) dy \right| \end{aligned}$$

Per la continuità di f , $\forall \varepsilon > 0 \exists \delta > 0$ t.c. $|f(x) - f(x_0)| < \varepsilon$ se $|x - x_0| < \delta$

$$\left| \int_{\mathbb{R}^n} K_t(y) (f(x+y) - f(x_0)) dy \right| \leq \underbrace{\int_{|x+y-x_0| < \delta} K_t(y) |f(x+y) - f(x_0)| dy}_I + \underbrace{\int_{|x+y-x_0| > \delta} K_t(y) |f(x+y) - f(x_0)| dy}_{II}$$

$$I \leq \varepsilon \int_{\mathbb{R}^n} K_t(y) dy = \varepsilon$$

$$II \leq 2\|f\|_{L^\infty} \int_{|x+y-x_0| > \delta} K_t(y) dy$$

Considero ora $|x - x_0| < \frac{\delta}{2}$: allora la condizione $|x+y-x_0| > \delta$

$$\text{implica } |y| > |y+x-x_0| - |x-x_0| > \frac{\delta}{2}$$

$$\Rightarrow II \leq 2\|f\|_{L^\infty} \int_{|y| > \frac{\delta}{2}} K_t(y) dy \quad \text{se } |x-x_0| < \frac{\delta}{2}$$

Dico che $\exists t_\varepsilon > 0$ t.c. $\int_{|y| > \frac{\delta}{2}} K_t(y) dy < \varepsilon$ se $0 < t < t_\varepsilon$

Se ciò è vero, trovo $|u(t, x) - f(x_0)| < \varepsilon + 2\|f\|_{L^\infty} \varepsilon$ se $0 < t < t_\varepsilon$, $|x - x_0| < \frac{\delta}{2}$

Mostriamo quindi che $\forall \delta > 0 \int_{|y| > \frac{\delta}{2}} K_t(y) dy \xrightarrow{t \rightarrow 0^+} 0$.

$$\int_{|y| > \frac{\delta}{2}} \frac{e^{-\frac{|y|^2}{4t}}}{(4\pi t)^{n/2}} dy \stackrel{z = \frac{y}{\sqrt{t}}}{=} \int_{|z| > \frac{\delta}{2\sqrt{t}}} \frac{e^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}} dz \xrightarrow{t \rightarrow 0^+} 0 \quad \text{per Lebesgue}$$

Oss Se $f \in UC(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, allora

$$e^{t\Delta} f \xrightarrow{t \rightarrow 0} f$$

Se $f \in L^p$, $e^{t\Delta} f \xrightarrow{t \rightarrow 0} f$?

Oss Siano $T_t \in \mathcal{L}(X)$, $\sup_t \|T_t\|_{\mathcal{L}(X)} < C$. Se $T_t \varphi \xrightarrow{t \rightarrow 0} \varphi \quad \forall \varphi \in D$ denso in X , allora $T_t x \xrightarrow{t \rightarrow 0} x \quad \forall x \in X$

Nel nostro caso: $X = L^p(\mathbb{R}^n)$, $T_t f = e^{t\Delta} f$

Oss $\|K_t * f\|_{L^p} \leq \frac{\|K_t\|_{L^1}}{1} \|f\|_{L^p}$ per Young $\forall p \in [1, +\infty]$

teorema Se $\varphi \in L^p$, $p \in [1, +\infty)$, allora $e^{t\Delta} \varphi \xrightarrow{t \rightarrow 0} \varphi$.

DIMOSTRAZIONE

Consideriamo $D = C_c^\infty(\mathbb{R}^n)$: se $\varphi \in C_c^\infty(\mathbb{R}^n)$, allora $e^{t\Delta} \varphi \xrightarrow{t \rightarrow 0} \varphi$?

Se $p \in [1, +\infty)$:

$$\|e^{t\Delta} \varphi - \varphi\|_{L^p} \leq \frac{\|e^{t\Delta} \varphi - \varphi\|_{L^1}^{\frac{1}{p}} \|e^{t\Delta} \varphi - \varphi\|_{L^\infty}^{1-\frac{1}{p}} \xrightarrow{t \rightarrow 0} 0$$

$\leq C \quad \xrightarrow{t \rightarrow 0} 0$ per sopra

Valde in realtà per $p \in [1, +\infty)$: se provo $e^{t\Delta} \varphi \xrightarrow{t \rightarrow 0} \varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n)$ concludo

Basta il caso $p=1$

Sia $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\text{supp } \varphi \subseteq B_R(0)$: vogliamo $\|e^{t\Delta} \varphi - \varphi\|_{L^1(\mathbb{R}^n)} \xrightarrow{t \rightarrow 0} 0$

$$\|e^{t\Delta} \varphi - \varphi\|_{L^\infty(B_{2R}(0))} \xrightarrow{t \rightarrow 0} 0 \implies \|e^{t\Delta} \varphi - \varphi\|_{L^1(B_{2R}(0))} \xrightarrow{t \rightarrow 0} 0$$

Rimane $\|e^{t\Delta} \varphi - \varphi\|_{L^1(B_{2R}^c(0))} = \|e^{t\Delta} \varphi\|_{L^1(B_{2R}^c(0))}$

$$\int_{|x|>2R} \left| \int_{|y|<R} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{n/2}} \varphi(y) dy \right| dx \leq \|\varphi\|_{L^\infty} \int_{|x|>2R} \int_{|y|<R} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{n/2}} dy dx \leq$$

$$\leq \|\varphi\|_{L^\infty} \int_{|x|>2R} \int_{|y|<R} \frac{e^{-\frac{|x|^2}{16t}}}{(4\pi t)^{n/2}} dx dy \leq CR^n \int_{|x|>2R} \frac{e^{-\frac{|x|^2}{16t}}}{(4\pi t)^{n/2}} dx \leq CR^n \int_{|z|>\frac{2R}{\sqrt{t}}} e^{-\frac{|z|^2}{4}} dz \xrightarrow{t \rightarrow 0} 0$$

□

Oss Si può dimostrare che $e^{t\Delta} \varphi \xrightarrow{t \rightarrow 0} \varphi$ q.o. $\forall \varphi \in L^1$

Non unicità

teorema

Esiste una soluzione $u \in C^\infty([0, +\infty) \times \mathbb{R}) \cap C^0([0, +\infty) \times \mathbb{R})$, $u \neq 0$, di

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & (t, x) \in (0, +\infty) \times \mathbb{R} \\ u(0, x) = 0 \end{cases}$$

Dimostrazione

Cerchiamo $u(t, x) = \sum_{k=0}^{\infty} f_{2k}(t) x^{2k}$

$$\partial_t u = \sum_{k=0}^{\infty} f'_{2k}(t) x^{2k}$$

$$\partial_x^2 u = \sum_{k=0}^{\infty} (2k+2)(2k+1) f_{2k+2}(t) x^{2k}$$

Se $\partial_t u - \partial_x^2 u = 0$, allora $f'_{2k} = f_{2k+2} (2k+2)(2k+1)$

Dico che $f_{2k}(t) = \frac{f_0^{(k)}}{(2k)!}$: per induzione

$$f_{2k+2}(t) = \frac{f'_{2k}}{(2k+2)(2k+1)} = \frac{f_0^{(k+1)}}{(2k+2)(2k+1)(2k)!} = \frac{f_0^{(k+1)}}{(2k+2)!}$$

Quindi $u(t, x) = \sum_{k=0}^{\infty} \frac{f_0^{(k)}}{(2k)!} x^{2k}$ con f_0 da definire t.c. la serie converga e $u(0, x) = 0$
 $\Rightarrow f_0(t) = e^{-\frac{1}{2}t}$

Dimostriamo che $\sum_{k=0}^{\infty} \frac{(e^{-\frac{1}{2}t})^{(k)}}{(2k)!} x^{2k}$ è una serie assolutamente convergente su $[0, +\infty) \times [-R, R]$:

$$\sum_{k=0}^{\infty} \sup_{[0, +\infty) \times [-R, R]} \frac{|(e^{-\frac{1}{2}t})^{(k)}|}{(2k)!} x^{2k} \leq \sum_{k=0}^{\infty} \frac{R^{2k}}{(2k)!} \sup_{[0, +\infty)} |(e^{-\frac{1}{2}t})^{(k)}|$$

Lemma Esiste $a > 0$ t.c. $(e^{-\frac{1}{2}t})^{(k)} \leq \frac{2^k k!}{t^k} e^{-\frac{a}{2}}$.

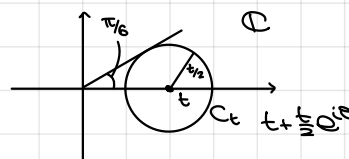
Dimostrazione

Se $f(z) \in \mathcal{H}(\Omega)$, $\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \oint_{C_z} \frac{f(z)}{(z-z_0)^{k+1}} dz$

Scegliamo $z_0 = t \in \mathbb{R}$, $\Omega = \mathbb{C} \setminus \{0\}$, $f(z) = e^{-\frac{1}{2}z}$

$$|(e^{-\frac{1}{2}t})^{(k)}| = \left| \frac{k!}{2\pi i} \oint_{C_t} \frac{e^{-\frac{1}{2}z}}{(z-t)^{k+1}} dz \right| \leq$$

$$\leq \frac{k!}{2\pi} \int_0^{2\pi} \frac{\sup |e^{-\frac{1}{2}z}|}{(t/2)^{k+1}} \frac{1}{2} d\theta = \frac{k! 2^k}{t^k} \max_{C_t} |e^{-\frac{1}{2}z}|$$



In polari $z = \rho e^{i\theta} \in C_t \Rightarrow \rho \in [\frac{1}{2}t, \frac{3}{2}t]$, $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$

$$|e^{-\frac{1}{2}z}| = |e^{-\frac{1}{2}\rho e^{i\theta}}| = |e^{-\frac{1}{2}\rho(\cos\theta - i\sin\theta)}| = e^{-\frac{\rho}{2}\cos\theta} \leq e^{-\frac{\rho}{2}\frac{\sqrt{3}}{2}}$$

$$\frac{\rho\sqrt{3}}{2} \geq \frac{\rho\sqrt{3}}{2} \frac{1}{2} = \frac{\rho}{2}$$

□

$$\sup_{[0, +\infty)} |(e^{-\frac{1}{2}t})^{(k)}| \leq 2^k k! \sup_{[0, +\infty)} \frac{|e^{-\frac{a}{2}}|}{t^k} = 2^k k! \sup_{s=\frac{1}{2}} |e^{-as} s^{\frac{k}{2}}| = 2^k k! e^{-\frac{1}{2}} \left(\frac{k}{2a}\right)^{\frac{k}{2}}$$

Derivandola, troviamo $-ae^{-as} s^{\frac{k}{2}} + \frac{k}{2} e^{-as} s^{\frac{k}{2}-1} = 0 \Rightarrow s = \frac{k}{2a}$

$$\sum_{k=0}^{\infty} \frac{R^{2k}}{(2k)!} \sup_{[0, +\infty)} |(e^{-\frac{1}{2}t})^{(k)}| \leq \sum_{k=0}^{\infty} \frac{R^{2k}}{(2k)!} 2^k k! e^{-\frac{1}{2}} \left(\frac{k}{2a}\right)^{\frac{k}{2}} = \sum_{k=0}^{\infty} M^k \frac{k!}{(2k)!} k^{\frac{k}{2}} \leq \sum_{k=0}^{\infty} M^k \frac{1}{k!} k^{\frac{k}{2}}$$

Applichiamo il criterio della radice: $\frac{M\sqrt{k}}{\sqrt{k!}} \stackrel{\text{Stirling}}{\sim} \frac{M\sqrt{k}}{((\frac{k}{e})^k \sqrt{2\pi k})^{\frac{1}{k}}} \sim \frac{M\sqrt{k}}{k(\frac{\sqrt{2\pi k}}{e})^{\frac{1}{k}}} \rightarrow 0$
 Allo stesso modo si fa per le derivate.

□

$$\begin{cases} \partial_t u - \Delta_x u = 0 & (t, x) \in (0, +\infty) \times \mathbb{R}^n \\ u(0, x) = 0 \end{cases}$$

$\Rightarrow u \equiv 0$ sotto quali ipotesi?

Se $|u(t, x)| \leq C e^{c|x|^2}$, allora $u \equiv 0$

o anche $|u(t, x)| \leq C(t) e^{c(t)|x|^2}$ con $C \in L^\infty_{loc}$

Se $u \geq 0$, allora $u \equiv 0$

def. Dato $\Omega \subseteq \mathbb{R}^n, T > 0$, si definisce la frontiera parabolica di $\Omega \times [0, T]$ l'insieme:

$$\partial^p(\Omega \times [0, T]) = \partial(\Omega \times [0, T]) \setminus \Omega \times \{T\} = \bar{\Omega} \times \{0\} \cup \partial\Omega \times [0, T]$$

teorema:
principio del massimo

Sia $u \in C(\bar{\Omega} \times [0, T]) \cap C'_t(\Omega \times (0, T)) \cap C^2_x(\Omega \times (0, T))$
soluzione di $\partial_t u - \Delta_x u = 0$ su $\Omega \times (0, T)$.
Se Ω è limitato, allora

$$\max_{\bar{\Omega} \times [0, T]} u = \max_{\partial^p(\Omega \times [0, T])} u \quad \text{e} \quad \min_{\bar{\Omega} \times [0, T]} u = \min_{\partial^p(\Omega \times [0, T])} u$$

DIMOSTRAZIONE

Se $\partial_t u - \Delta_x u > 0$, allora $\min_{\bar{\Omega} \times [0, T]} u = \min_{\partial^p(\Omega \times [0, T])} u$.

Infatti, se $(t_0, x_0) \in \Omega \times (0, T)$ fosse di minimo, allora $\partial_t u(t_0, x_0) - \Delta_x u(t_0, x_0) \leq 0$ $\nlessgtr$ ≤ 0
se $(t_0, x_0) \in \Omega \times \{T\}$ fosse di minimo, allora $\partial_t u(t_0, x_0) - \Delta_x u(t_0, x_0) \leq 0$ $\nlessgtr$ ≤ 0

Consideriamo ora $u_\varepsilon := u - \varepsilon|x|^2$ Allora

$$\partial_t u_\varepsilon - \Delta_x u_\varepsilon > 0 \Rightarrow \begin{array}{ccc} \min_{\bar{\Omega} \times [0, T]} u_\varepsilon & = & \min_{\partial^p(\Omega \times [0, T])} u_\varepsilon \\ \downarrow \varepsilon \rightarrow 0 & & \downarrow \varepsilon \rightarrow 0 \\ \min_{\bar{\Omega} \times [0, T]} u & = & \min_{\partial^p(\Omega \times [0, T])} u \end{array}$$

Analogamente per il massimo. □

corollario Dato Ω limitato, il problema

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{su } \Omega \times (0, +\infty) \\ u(0, x) = f(x) \\ u(t, y) = 0 & \forall y \in \partial\Omega \end{cases}$$

ammette un'unica soluzione.

DIMOSTRAZIONE

Se u_1 e u_2 sono soluzioni, allora

$$\begin{cases} \partial_t (u_1 - u_2) - \Delta (u_1 - u_2) = 0 \\ (u_1 - u_2)(0, x) = 0 \\ (u_1 - u_2)(t, y) = 0 & \forall y \in \partial\Omega \end{cases}$$

$$\Rightarrow u_1 - u_2 \equiv 0$$

□

**teorema di
unicità
condizionata**

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{su } \mathbb{R}^n \times (0, +\infty) \\ u(0, x) = 0 \end{cases}$$

Se $|u(t, x)| \leq C e^{A|x|^2}$ su $(0, T) \times \mathbb{R}^n$, allora
 $u \equiv 0$ su $[0, T] \times \mathbb{R}^n$.

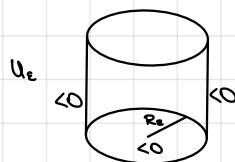
DIMOSTRAZIONE

Claim $\exists w(t, x) \geq 0$, $\partial_t w - \Delta w = 0$, $|w(t, x)| \geq C e^{(A+\delta)|x|^2}$ su $(0, T) \times \mathbb{R}^n$.

Se tale w esiste per $\delta_0 > 0$, allora introduciamo $u_\varepsilon := u - \varepsilon w$

Si ha che $\exists R_\varepsilon > 0$ t.c.

$$\begin{cases} u_\varepsilon < 0 & \text{su } B_{R_\varepsilon}^c(t) \times [0, T] \\ u_\varepsilon(0, x) < 0 \end{cases}$$



quindi per il principio del massimo

$$u_\varepsilon(t, x) < 0 \text{ su } B_{R_\varepsilon}(t) \times [0, T]$$

Se ora fissiamo $\bar{R} > 0$, allora abbiamo che se $\varepsilon \rightarrow 0$,

$$\text{allora } R_\varepsilon > \bar{R} \Rightarrow u_\varepsilon(t, x) < 0 \text{ su } B_{\bar{R}}(t) \times [0, T]$$

$$\text{cioè } u(t, x) < \varepsilon w(t, x) \text{ su } B_{\bar{R}}(t) \times [0, T]$$

$$\text{Mandando } \varepsilon \rightarrow 0, u(t, x) \leq 0 \text{ su } B_{\bar{R}}(t) \times [0, T]$$

$$\text{Essendo } \bar{R} \text{ arbitrario, si ha } u \leq 0 \text{ su } \mathbb{R}^n \times [0, T]$$

$$\text{Applicando lo stesso ragionamento a } -u, \text{ si ha } -u \leq 0 \text{ su } \mathbb{R}^n \times [0, T]$$

$$\Rightarrow u \equiv 0 \text{ su } \mathbb{R}^n \times [0, T].$$

Per dimostrare il claim, basta scegliere $w(t, x)$ come la gaussiana troncata

$$w(t, x) = \frac{e^{-\frac{|x|^2}{4(T+\delta-t)}}}{(T+\delta-t)^{n/2}} \text{ con } \delta > 0, \delta < 1.$$

□

Consideriamo l'equazione del calore non lineare con condizioni di Dirichlet:

$$\begin{cases} \partial_t u - \Delta u = f(u) & (t, x) \in (0, +\infty) \times \mathbb{R}^n \\ u(0, x) = \varphi(x) \end{cases} \text{ con } f(u) = \pm u|u|^{p-1} \text{ con } p > 1$$

L'equazione integrale associata è:

$$u(t, x) = \underbrace{e^{t\Delta} \varphi(x) + \int_0^t e^{(t-s)\Delta} F(u(s)) ds}_{T_\varphi u}$$

Abbiamo quindi un problema di punto fisso: $T_\varphi u = u$

Poniamo $\forall t > 0 \quad X_t = L^p([0, t]; L_x^\infty)$

Lemma $\sup \|e^{t\Delta} \psi\|_{L_x^\infty} \leq \|\psi\|_{L_x^\infty}$
 $\sup_{t \in [0, \tau]} \left\| \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_{L_x^\infty} \leq \|F\|_{L^1([0, \tau]; L_x^\infty)}$

DIMOSTRAZIONE

In fatti $e^{t\Delta} \psi = G_t * \psi$ e $\|G_t * \psi\| \leq \|G_t\|_1 \|\psi\|_{L_x^\infty} = \|\psi\|_{L_x^\infty}$
 e $\left\| \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_{L_x^\infty} \leq \int_0^t \|e^{(t-s)\Delta} F(s)\|_{L_x^\infty} ds \leq \int_0^t \|F(s)\|_{L_x^\infty} ds$ $\square$

teorema Per ogni $\varphi \in L_x^\infty$, $p > 1$, $\exists \tau = \tau(\|\varphi\|_{L_x^\infty}) > 0$ t.c.
 (i) $T_\varphi : B_{2\|\varphi\|_{L_x^\infty}}(X_\tau) \hookrightarrow$
 (ii) $T_\varphi : B_{2\|\varphi\|_{L_x^\infty}}(X_\tau)$ è una contrazione

corollario Se $T_{\max}(\varphi)$ è il tempo massimale di esistenza,
 allora abbiamo due alternative:

- (i) $T_{\max}(\varphi) = +\infty$
- (ii) $T_{\max}(\varphi) < +\infty$ e $\limsup_{t \rightarrow T_{\max}} \|u(t, x)\|_{L_x^\infty} = +\infty$

Oss Se fosse $\tau = \tau(\varphi)$, non si può dedurre il corollario [casi critici]

DIMOSTRAZIONE

$$(i) \quad \|T_\varphi u\|_{X_\tau} \leq \|e^{t\Delta} \varphi\|_{X_\tau} + \left\| \int_0^t e^{(t-s)\Delta} F(u(s)) ds \right\|_{X_\tau} \leq \|\varphi\|_{L_x^\infty} + \|F(u)\|_{L^1([0, \tau]; L_x^\infty)} \leq$$

$$\leq \|\varphi\|_{L_x^\infty} + \int_0^\tau \|u(s)\|_{L_x^\infty}^{p+1} ds \leq \|\varphi\|_{L_x^\infty} + \tau \|u\|_{X_\tau}^{p+1}$$

Dico che $\exists \tau = \tau(\|\varphi\|_{L_x^\infty})$ t.c. $\|T_\varphi u\|_{X_\tau} \leq 2\|\varphi\|_{L_x^\infty}$ se $\|u\|_{X_\tau} \leq 2\|\varphi\|_{L_x^\infty}$

$$\|T_\varphi u\|_{X_\tau} \leq \|\varphi\|_{L_x^\infty} + \tau \|u\|_{X_\tau}^{p+1} \leq \|\varphi\|_{L_x^\infty} + \tau (2\|\varphi\|_{L_x^\infty})^{p+1} \leq 2\|\varphi\|_{L_x^\infty}$$

che è vero se $\tau (2\|\varphi\|_{L_x^\infty})^p \leq \|\varphi\|_{L_x^\infty}$, cioè $\tau \leq \frac{1}{2^p \|\varphi\|_{L_x^\infty}^p}$

$$(ii) \quad \|T_\varphi u_1 - T_\varphi u_2\|_{X_\tau} \leq \frac{1}{2} \|u_1 - u_2\|_{X_\tau} \quad \forall u_1, u_2 \in B_{2\|\varphi\|_{L_x^\infty}}(X_\tau)$$

$$\|T_\varphi u_1 - T_\varphi u_2\|_{X_\tau} = \left\| \int_0^t e^{(t-s)\Delta} (F(u_1(s)) - F(u_2(s))) ds \right\|_{X_\tau} \leq \int_0^t \|F(u_1(s)) - F(u_2(s))\|_{X_\tau} ds \leq$$

$$\leq \tau \|F(u_1) - F(u_2)\|_{L^p([0, \tau]; L_x^\infty)} \leq \tau \|u_1 - u_2\|_{X_\tau}^{(p+1)} (2\|\varphi\|_{L_x^\infty})^p \leq \frac{1}{2} \|u_1 - u_2\|_{X_\tau}$$

$$\leq \|u_1(s) - u_2(s)\|^{(p+1)} \max\{|u_1(s)|, |u_2(s)|\}^p$$

che è vero se $\tau^{(p+1)} (2\|\varphi\|_{L_x^\infty})^p \leq \frac{1}{2}$, cioè $\tau \leq \frac{1}{(p+1) 2^{p+1} \|\varphi\|_{L_x^\infty}^p}$

Quindi basta prendere $\tau = \frac{1}{(p+1) 2^{p+1} \|\varphi\|_{L_x^\infty}^p}$ $\square$

Esistenza globale per dati piccoli

$$\begin{cases} \partial_t u - \partial_x^2 u = f(u) & (t,x) \in (0,+\infty) \times \mathbb{R} \\ u(0,x) = \varphi(x) \end{cases}$$

Se riesco a stimare $\|u(t,x)\|_{L_x^2}$ su $[0,T]$ $\forall T$, posso applicare il corollario.
Qua serve il segno - per f , e non è sempre detto si possa fare :
quindi non seguiremo questa strada.

def $Y = \{ u: (0,+\infty) \times \mathbb{R} \rightarrow \mathbb{R} \mid \sup_t \langle t \rangle^{\frac{n}{2}} \|u(t,x)\|_{L_x^2} + \sup_t \|u(t,x)\|_{L_x^4} < +\infty \}$
cioè $\underbrace{(\| \langle t \rangle^{\frac{n}{2}} u(t,x) \|_{L_{t,x}^2} + \|u(t,x)\|_{L_x^4 L_t^2})}_{\|u\|_Y} < +\infty$ dove $\langle t \rangle = \sqrt{1+t^2}$

Lemma $\|e^{t\Delta} \varphi\|_{L_t^2 L_x^4} \leq \|\varphi\|_{L_x^4}$
 $\|e^{t\Delta} \varphi\|_{L_x^2} \leq \frac{C}{\langle t \rangle^{\frac{n}{2}}} (\|\varphi\|_{L_x^4} + \|\varphi\|_{L_x^2})$
 cioè $\| \langle t \rangle^{\frac{n}{2}} \|e^{t\Delta} \varphi\|_{L_t^2 L_x^2} \leq C(\|\varphi\|_{L_x^4} + \|\varphi\|_{L_x^2})$

DIMOSTRAZIONE

In fatti $\|G_t * \varphi\|_{L_x^4} \leq \|G_t\|_{L_x^4} \|\varphi\|_{L_x^4} = \|\varphi\|_{L_x^4}$
 $\|e^{t\Delta} \varphi\|_{L_x^2} \leq \|\varphi\|_{L_x^2}$
 $\|e^{t\Delta} \varphi\|_{L_x^2} = \|G_t * \varphi\|_{L_x^2} \leq \|G_t\|_{L_x^2} \|\varphi\|_{L_x^4} \leq \frac{C}{t^{\frac{n}{2}}} \|\varphi\|_{L_x^4}$
 $\Rightarrow \|e^{t\Delta} \varphi\|_{L_x^2} \leq \min \{ 1, \frac{C}{t^{\frac{n}{2}}} \} (\|\varphi\|_{L_x^4} + \|\varphi\|_{L_x^2})$
 $\approx \frac{C}{\langle t \rangle^{\frac{n}{2}}}$ □

Quindi $\|e^{t\Delta} \varphi\|_Y \leq C \|\varphi\|$

Lemma $\left\| \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_Y \leq \|F\|_{L_t^2 L_x^4} + \left\| \langle t \rangle^{\frac{n}{2}} \int_0^t \frac{\|F(s)\|_{L_x^4} + \|F(s)\|_{L_x^2}}{\langle t-s \rangle^{\frac{n}{2}}} ds \right\|_{L_t^2}$

DIMOSTRAZIONE

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_Y &= \left\| \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_{L_t^2 L_x^4} + \left\| \langle t \rangle^{\frac{n}{2}} \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_{L_t^2 L_x^2} \\ &\leq \sup_t \int_0^t \|e^{(t-s)\Delta} F(s)\|_{L_x^4} ds + \sup_t \langle t \rangle^{\frac{n}{2}} \left(\int_0^t \|e^{(t-s)\Delta} F(s)\|_{L_x^2} ds \right) \leq \\ &\leq \sup_t \int_0^t \|F(s)\|_{L_x^4} ds + \sup_t \langle t \rangle^{\frac{n}{2}} \int_0^t \frac{\|F(s)\|_{L_x^4} + \|F(s)\|_{L_x^2}}{\langle t-s \rangle^{\frac{n}{2}}} ds \\ &\quad \|F\|_{L_t^2 L_x^4} \end{aligned}$$
 □

teorema

Sia $p > 2$, $\varphi \in L^1_x \cap L^\infty_x$.

Esiste $\varepsilon_0 > 0$ t.c. se $\|\varphi\| = \|\varphi\|_{L^1_x} + \|\varphi\|_{L^\infty_x} < \varepsilon_0$, allora

(i) $T_\varphi : B_{2\|\varphi\|}(Y) \hookrightarrow$

(ii) $T_\varphi : B_{2\|\varphi\|}(Y) \hookrightarrow$ è una contrazione

DIMOSTRAZIONE

$$\|T_\varphi u\|_Y \leq \|e^{t\Delta} \varphi\|_Y + \left\| \int_0^t e^{(t-s)\Delta} u u^p ds \right\|_Y \leq C \|\varphi\| + C \|u\|_Y^p$$

$$\left\| \int_0^t e^{(t-s)\Delta} u u^p ds \right\|_Y \leq C \|u u^p\|_{L^1_x L^p_x} + C \sup_t \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{\|u u^p\|_{L^1_x} + \|u u^p\|_{L^\infty_x}}{\langle t-s \rangle^{\frac{p}{2}}} ds$$

$$\leq C \int_0^{+\infty} \|u u^p\|_{L^1_x} ds + C \sup_t \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{\|u(s)\|_{L^1_x} \|u(s)\|_{L^\infty_x}^p + \|u(s)\|_{L^\infty_x}^{p+1}}{\langle t-s \rangle^{\frac{p}{2}}} ds$$

$$\leq \underbrace{C \int_0^{+\infty} \|u(s)\|_{L^1_x} \|u(s)\|_{L^\infty_x}^p ds}_I + \underbrace{C \sup_t \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{\|u(s)\|_{L^1_x} \|u(s)\|_{L^\infty_x}^p}{\langle t-s \rangle^{\frac{p}{2}}} ds}_{II} + \underbrace{C \sup_t \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{\|u(s)\|_{L^\infty_x}^{p+1}}{\langle t-s \rangle^{\frac{p}{2}}} ds}_{III}$$

$$I \leq C \sup_t \|u(s)\|_{L^1_x} \int_0^{+\infty} \|u(s)\|_{L^\infty_x}^p ds \leq C \|u\|_Y \int_0^{+\infty} \frac{\|u(s)\|_{L^\infty_x}^p}{\langle s \rangle^{\frac{p}{2}}} \langle s \rangle^{\frac{p}{2}} ds \leq C \|u\|_Y \|u\|_Y^p \int_0^{+\infty} \frac{1}{\langle s \rangle^{\frac{p}{2}}} ds$$

$$\langle +\infty \iff \frac{np}{2} > 1$$

$$II \leq \sup_t \langle t \rangle^{\frac{p}{2}} \|u\|_Y \int_0^t \frac{\|u(s)\|_{L^\infty_x}^p}{\langle t-s \rangle^{\frac{p}{2}}} ds = \sup_t \|u\|_Y \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{\langle s \rangle^{\frac{p}{2}} \|u\|_{L^\infty_x}^p}{\langle s \rangle^{\frac{p}{2}} \langle t-s \rangle^{\frac{p}{2}}} ds \leq \|u\|_Y^p \sup_t \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{1}{\langle s \rangle^{\frac{p}{2}} \langle t-s \rangle^{\frac{p}{2}}} ds$$

$$\langle t \rangle^{\frac{p}{2}} \int_0^t \frac{1}{\langle s \rangle^{\frac{p}{2}} \langle t-s \rangle^{\frac{p}{2}}} ds \simeq \langle t \rangle^{\frac{p}{2}} \int_0^{t/2} \frac{1}{\langle s \rangle^{\frac{p}{2}} \langle t-s \rangle^{\frac{p}{2}}} ds + \langle t \rangle^{\frac{p}{2}} \int_{t/2}^t \frac{1}{\langle s \rangle^{\frac{p}{2}} \langle t-s \rangle^{\frac{p}{2}}} ds$$

$$\simeq \underbrace{\frac{\langle t \rangle^{\frac{p}{2}}}{\langle t \rangle^{\frac{p}{2}}} \int_0^{t/2} \frac{1}{\langle s \rangle^{\frac{p}{2}}} ds}_{\langle t \rangle^{\frac{p}{2}}} + \underbrace{\frac{\langle t \rangle^{\frac{p}{2}}}{\langle t \rangle^{\frac{p}{2}}} \int_{t/2}^t \frac{1}{\langle t-s \rangle^{\frac{p}{2}}} ds}_{\simeq \frac{\langle t \rangle^{\frac{p}{2}}}{\langle t \rangle^{\frac{p}{2}}} \langle t \rangle^{-\frac{p}{2}+1} = \langle t \rangle^{1-\frac{np}{2}}}$$

$$1 - \frac{np}{2} < 0$$

$$III \leq \sup_t \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{\|u(s)\|_{L^\infty_x}^{p+1}}{\langle t-s \rangle^{\frac{p}{2}} \langle s \rangle^{\frac{p}{2}+1}} ds \simeq \|u\|_Y^p \sup_t \langle t \rangle^{\frac{p}{2}} \int_0^t \frac{1}{\langle t-s \rangle^{\frac{p}{2}} \langle s \rangle^{\frac{p}{2}+1}} ds$$

$$\simeq \underbrace{\langle t \rangle^{\frac{p}{2}} \int_0^{t/2} \frac{1}{\langle t-s \rangle^{\frac{p}{2}} \langle s \rangle^{\frac{p}{2}+1}} ds}_{\simeq \int_0^{t/2} \frac{1}{\langle s \rangle^{\frac{p}{2}+1}} ds} + \underbrace{\langle t \rangle^{\frac{p}{2}} \int_{t/2}^t \frac{1}{\langle t-s \rangle^{\frac{p}{2}} \langle s \rangle^{\frac{p}{2}+1}} ds}_{\simeq \frac{\langle t \rangle^{\frac{p}{2}}}{\langle t \rangle^{\frac{p}{2}+1}} \int_{t/2}^t \frac{1}{\langle t-s \rangle^{\frac{p}{2}}} ds \simeq \frac{1}{\langle t \rangle^{\frac{p}{2}}} \langle t \rangle^{-\frac{p}{2}+1} \simeq \langle t \rangle^{-\frac{p}{2}+1-\frac{np}{2}}}$$

$$\simeq \int_0^{t/2} \frac{1}{\langle s \rangle^{\frac{p}{2}+1}} ds$$

$$\simeq \frac{\langle t \rangle^{\frac{p}{2}}}{\langle t \rangle^{\frac{p}{2}+1}} \int_{t/2}^t \frac{1}{\langle t-s \rangle^{\frac{p}{2}}} ds \simeq \frac{1}{\langle t \rangle^{\frac{p}{2}}} \langle t \rangle^{-\frac{p}{2}+1} \simeq \langle t \rangle^{-\frac{p}{2}+1-\frac{np}{2}}$$

$$-(\frac{p}{2}+1)\frac{np}{2} + 1 < 0$$

Quindi $\|T_\varphi u\|_Y \leq C \|\varphi\| + C \|u\|_Y^p \leq 2C \|\varphi\|$ se $\|u\|_Y \leq 2C \|\varphi\|$ e $\|\varphi\| < 1$

infatti: $\|T_\varphi u\|_Y \leq C \|\varphi\| + C (2C \|\varphi\|)^p \leq 2C \|\varphi\|$ per $\|\varphi\|^p < \frac{1}{(2C)^p}$



teorema

$$\begin{cases} \partial_t u - \Delta u = u|u|^{p-1} & (t,x) \in (0,+\infty) \times \mathbb{R}^n, \text{ con } 0 < p < \frac{2}{n} \\ u(0,x) = \phi & \\ \text{Allora } u \equiv 0 & \end{cases} \quad u \in C^\infty([0,+\infty) \times \mathbb{R}^n) \cap C^0([0,+\infty) \times \mathbb{R}^n) \text{ e } u \geq 0$$

Oss $T_\phi u = e^{t\Delta} \phi + \int_0^t e^{(t-s)\Delta} (u|u|^{p-1}) ds$

$Y \rightsquigarrow Y_+ = Y \vee 0$ è chiuso e T_ϕ -invariante, quindi basta $u \geq 0$

DIMOSTRAZIONE

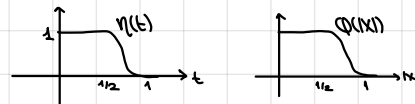
Funzioni test

Sia $\phi(t,x) \in C_c^\infty([0,+\infty) \times \mathbb{R}^n)$, $\phi \geq 0$

$$\begin{aligned} \iint |u|^{p+1} \phi \, dx \, dt &= \int_0^\infty \int_{\mathbb{R}^n} (\partial_t u - \Delta u) \phi(t,x) \, dx \, dt = \\ &= \int_0^\infty \int_{\mathbb{R}^n} \partial_t (u\phi) - u \Delta \phi - \partial_t \phi u \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^n} u(-\Delta \phi - \partial_t \phi) - \int_{\mathbb{R}^n} u(0) \phi(0) \, dx \\ &\leq \iint u(-\Delta \phi - \partial_t \phi) \end{aligned}$$

Sceglia delle ϕ :

$$\phi_R(t,x) = \left(\eta\left(\frac{t}{R^2}\right) \varphi\left(\frac{x}{R}\right) \right)^m \quad \text{dove } m = 2\left(\frac{p+1}{p}\right)$$



Claim $|\Delta \phi_R - \partial_t \phi_R| \leq \frac{C}{R^2} (\phi_R(t,x))^{\frac{1}{p+1}}$

Quindi

$$\iint |u|^{p+1} \phi_R \, dx \, dt \leq \int_0^\infty \int_{B(0,R)} \frac{C}{R^2} u (\phi_R(t,x))^{\frac{1}{p+1}} \, dx \, dt \stackrel{\text{Hölder}}{\leq} \frac{C}{R^2} \left(\iint |u|^{p+1} \phi_R \, dx \, dt \right)^{\frac{1}{p+1}} R^{(2+n)\frac{p}{p+1}}$$

$$\Rightarrow \left(\iint |u|^{p+1} \phi_R \right)^{1 - \frac{1}{p+1}} \leq C R^{(2+n)\frac{p}{p+1} - 2}$$

Mandando $R \rightarrow +\infty$, basta $(2+n)\frac{p}{p+1} - 2 < 0$, cioè $np < 2$

Oss Se $p = \frac{n}{2}$, trovo $u \in L^{p+1}$, ma in realtà gli integrali sono su corone o su intervalli lontani dall'origine, quindi resta integrando contro caratteristiche che periodicamente tendono a 0.

Mostriamo il claim:

$$|\Delta \phi_R(t,x)| = \left| \eta\left(\frac{t}{R^2}\right)^m |\Delta \varphi^m\left(\frac{x}{R}\right)| \right| \leq \frac{C}{R^2} \left| \eta\left(\frac{t}{R^2}\right)^m \left[\Delta \varphi\left(\frac{x}{R}\right) \varphi^{m-2}\left(\frac{x}{R}\right) + \varphi^{m-2}\left(\frac{x}{R}\right) |\nabla \varphi\left(\frac{x}{R}\right)|^2 \right] \right|$$

$$\begin{aligned} \Delta \varphi^m\left(\frac{x}{R}\right) &= \operatorname{div}(\nabla \varphi^m\left(\frac{x}{R}\right)) = m \operatorname{div}\left(\frac{1}{R} \nabla \varphi\left(\frac{x}{R}\right) \varphi^{m-1}\left(\frac{x}{R}\right)\right) = \\ &= \frac{m}{R} \left(\frac{1}{R} \Delta \varphi\left(\frac{x}{R}\right) \varphi^{m-1}\left(\frac{x}{R}\right) + \frac{1}{R} \nabla \varphi\left(\frac{x}{R}\right) \cdot (m-1) \varphi^{m-2}\left(\frac{x}{R}\right) \nabla \varphi\left(\frac{x}{R}\right) \right) \end{aligned}$$

$$\leq \frac{C}{R^2} \left| \eta\left(\frac{t}{R^2}\right)^m \left[\varphi\left(\frac{x}{R}\right)^{m-1} + \varphi\left(\frac{x}{R}\right)^{m-2} \right] \right| \leq \frac{C}{R^2} \left| \eta\left(\frac{t}{R^2}\right)^m \varphi\left(\frac{x}{R}\right)^{m-1} + \frac{C}{R^2} \left| \eta\left(\frac{t}{R^2}\right)^m \varphi\left(\frac{x}{R}\right)^{m-2} \right| \right| \leq \frac{C}{R^2} \left| \eta\left(\frac{t}{R^2}\right)^m \varphi\left(\frac{x}{R}\right)^{\frac{m}{p+1}} \right|$$

Basta imporre $\begin{cases} m \geq \frac{n}{p+1} \\ m-1 \geq \frac{n}{p+1} \\ m-2 \geq \frac{n}{p+1} \end{cases} \Rightarrow m-2 \geq \frac{n}{p+1} \Rightarrow m \geq 2\frac{p+1}{p}$

$$|\partial_t \phi_R(t,x)| = \frac{m}{R^2} \left| \eta\left(\frac{t}{R^2}\right) \eta^{m-1}\left(\frac{t}{R^2}\right) \varphi^m\left(\frac{x}{R}\right) \right| \leq \frac{C}{R^2} \left| \eta^{m-1}\left(\frac{t}{R^2}\right) \varphi^m\left(\frac{x}{R}\right) \right| \leq \frac{C}{R^2} \left| \eta\left(\frac{t}{R^2}\right)^{\frac{m}{p+1}} \varphi\left(\frac{x}{R}\right)^{\frac{m}{p+1}} \right|$$

$\text{se } \begin{cases} m-1 \geq \frac{n}{p+1} \\ m \geq \frac{n}{p+1} \end{cases}$



EQUAZIONE DI SCHRÖDINGER

Consideriamo

$$\begin{cases} i\partial_t u + \Delta u = 0 & (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0,x) = \phi(x) \end{cases}$$

Come per l'eq del calore:

$$\begin{cases} i\partial_t \hat{u} - |\xi|^2 \hat{u} = 0 \\ \hat{u}(0, \xi) = \hat{\phi}(\xi) \end{cases} \Rightarrow \hat{u}(t, \xi) = e^{-i|\xi|^2 t} \hat{\phi}(\xi) \\ \Rightarrow u(t, x) = \mathcal{F}^{-1}(e^{-i|\xi|^2 t}) * \phi$$

Gruppo di Schrödinger

$$\mathcal{S}'(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : x^\alpha \partial^\beta \phi \in L^\infty \forall \alpha, \beta\} \text{ con seminorma } p_{\alpha, \beta}(\phi) = \|x^\alpha \partial^\beta \phi\|_{L^\infty}$$

$\mathcal{F}: \mathcal{S} \rightarrow \mathcal{S}$ è invertibile

$\mathcal{S}'(\mathbb{R}^n)$ è il duale topologico di $\mathcal{S}(\mathbb{R}^n)$

$T: \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathbb{C}$ lineare e continua

Se $f \in L^\infty(\mathbb{R}^n)$, posso definire

$$T_f: \phi \in \mathcal{S} \mapsto \int_{\mathbb{R}^n} f \phi, T_f \in \mathcal{S}'$$

$\delta_0: \phi \mapsto \phi(0)$ non si può rappresentare in tal modo

Ma $f \in L^\infty \Rightarrow f \in \mathcal{D}'$, non necessariamente $f \in \mathcal{S}'$

$$\mathcal{F}: \mathcal{S}' \rightarrow \mathcal{S}'$$

$$\langle \mathcal{F}T, \phi \rangle = \langle T, \mathcal{F}\phi \rangle$$

esempio $\langle \mathcal{F}\delta_0, \phi \rangle = \langle \delta_0, \hat{\phi} \rangle = \hat{\phi}(0) = \int_{\mathbb{R}^n} \phi(x) dx \Rightarrow \hat{\delta}_0 = T_1$

$$\langle \mathcal{F}(e^{-i|\xi|^2 t}), \phi \rangle = \langle e^{-i|\xi|^2 t}, \hat{\phi} \rangle = \int e^{-i|\xi|^2 t} \hat{\phi} d\xi$$

$$\phi \in \mathcal{S} \Rightarrow \mathcal{F}^{-1}(\phi(x)) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \phi(\xi) d\xi$$

$$\langle \mathcal{F}^{-1}T, \phi \rangle = \langle T, \mathcal{F}^{-1}\phi \rangle, \mathcal{F}^{-1}: \mathcal{S}' \rightarrow \mathcal{S}'$$

$$T_n \xrightarrow{\mathcal{F}^{-1}} T \text{ se } \langle T_n, \phi \rangle \rightarrow \langle T, \phi \rangle \forall \phi \in \mathcal{S} \\ \Rightarrow \hat{T}_n \xrightarrow{\mathcal{F}} \hat{T}$$

Consideriamo $f_\varepsilon(\xi) := e^{-\varepsilon|\xi|^2 - i|\xi|^2 t} \in L^1, T_{f_\varepsilon} \in \mathcal{S}'$

$$T_{f_\varepsilon} \xrightarrow{\mathcal{F}^{-1}} e^{-i|\xi|^2 t} \Rightarrow \hat{T}_{f_\varepsilon} \xrightarrow{\mathcal{F}} e^{-i|\xi|^2 t}$$

Se $f \in L^1, \hat{T}_f = T_{\hat{f}}$

$$\mathcal{F}^{-1}f_\varepsilon = \mathcal{F}^{-1}(e^{-\varepsilon|\xi|^2 - i|\xi|^2 t}) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi - \varepsilon|\xi|^2 - i|\xi|^2 t} d\xi = \frac{1}{(2\pi)^n} \int e^{-\varepsilon|\xi|^2} e^{ix \cdot \xi - i|\xi|^2 t} d\xi =: K(t, \varepsilon, x)$$

$$F(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-|\xi|^2 z} e^{ix \cdot \xi} d\xi, z = \varepsilon + it$$



F è olomorfa su $\{\operatorname{Re} z > 0\}$

$$F(\alpha + i0) = \frac{1}{(2\pi)^n} \int e^{-\alpha|\xi|^2} e^{ix \cdot \xi} d\xi = \frac{1}{(4\pi\alpha)^{n/2}} e^{-\frac{x^2}{4\alpha}}$$

ha estensione olomorfa su $\{\operatorname{Re} z > 0\}$

$$\frac{1}{(4\pi 2)^{n/2}} e^{-\frac{x^2}{4z}} \text{ (dove la radice ha } \operatorname{Re} > 0 \text{)}$$

Poiché sono olomorfe su $\{\operatorname{Re} z > 0\}$ e coincidono sulla semiretta reale positiva

$$\frac{1}{(2\pi)^n} \int e^{-|\xi|^2 z} e^{ix \cdot \xi} d\xi = \frac{1}{(4\pi 2)^{n/2}} e^{-\frac{x^2}{4z}} \text{ su } \operatorname{Re} z > 0, \operatorname{Re}(iz) > 0$$

Ora per $z = \varepsilon + it$

$$\frac{1}{(2\pi)^n} \int e^{-|\xi|^2 (\varepsilon + it)} e^{ix \cdot \xi} d\xi = \frac{e^{-\frac{x^2}{4(\varepsilon + it)}}}{(4\pi(\varepsilon + it))^{n/2}} \xrightarrow{z \rightarrow 0} \frac{e^{-\frac{x^2}{4it}}}{(4\pi it)^{n/2}}$$

puntualmente, e anche debolmente in $\mathcal{S}'$ per Carleman

$$\frac{e^{-\frac{x^2}{4it}}}{(4\pi it)^{n/2}} \text{ è il nucleo di Schrödinger}$$