Homework 3

Istituzioni di Algebra

Due date: December 9, 2024

1 Proving stuff

Exercise P1. Let R be a commutative unitary ring. For every R-module M, denote by $M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ the Pontryagin dual. Recall that $M \mapsto M^*$ is a contravariant functor.

1. Show that $A \to B \to C$ is exact if and only if $C^* \to B^* \to A^*$ is exact.
2. Given R-modules A, M, construct a natural map $A^* \otimes_R M \to \mathrm{Hom}_R(M, A)^*$. Show that it is an isomorphism whenever M is finitely presented.
3. Let M be a flat, finitely presented module. Prove that M is projective.

Hint for 2. Do first the case $M = R^m$ and then use the (appropriate) exactness of $\otimes$, Hom and $N \mapsto N^*$.

Hint for 3. Copy the proof of "B flat $\iff$ B^* injective" that we saw in class.

Exercise P2. Let G, G' be groups, N be a normal abelian subgroup of G, and π be the natural projection $\pi : G \to G/N$.

1. Show that there exists a class $u \in H^2(G/N, N)$ such that the following holds: given a group homomorphism $\overline{\varphi} : G' \to G/N$, there exists a group homomorphism $\varphi : G' \to G$ such that $\overline{\varphi} = \pi \circ \varphi$ if and only if $\varphi^*(u) \in H^2(G', N)$ is trivial.
2. Let $\overline{\phi} : G' \to G/N$ be a group homomorphism such that there exists a homomorphism $\varphi : G' \to G$ with $\overline{\varphi} = \varphi \circ \pi$. Show that the set $\{\psi : G' \to G \mid \pi \circ \psi = \overline{\varphi}\}$ is in bijection with $Z^1(G', N)$, the set of 1-cocycles of G' with values in N.

Exercise P3. Let R be a commutative unitary ring and let M be an R-module. Let

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$$

be a resolution of M by *flat* modules (that is: the above sequence is exact and each F_i is a flat R-module). Let N be any R-module. Show that the i-th homology of the complex

$$\cdots \to F_3 \otimes N \to F_2 \otimes N \to F_1 \otimes N \to F_0 \otimes N \to 0$$

coincides with $\mathrm{Tor}_i(M, N)$, for all $i \geq 0$. In other words, one can use flat objects instead of projective objects to compute Tor.

Hint. Dimension shifting.

Exercise P4. Let $(A, \mathfrak{m})$ be a Noetherian local ring and set $k = A/\mathfrak{m}$.

1. Show that, for any A-module N, one has
$$\mathfrak{m} \cdot \mathrm{Tor}_i(k, N) = (0)$$
for all $i \geq 0$. Deduce in particular that $\mathrm{Tor}_i(k, N)$ has a natural structure of k-vector space.
2. Similarly, prove that for all $i \geq 0$ and all A-modules N we have $\mathfrak{m} \cdot \mathrm{Ext}^i(N, k) = (0)$ and $\mathfrak{m} \cdot \mathrm{Ext}^i(k, N) = (0)$, so that the Ext groups $\mathrm{Ext}^i(N, k)$ and $\mathrm{Ext}^i(k, N)$ have a structure of k-vector space.
3. Show that $\mathrm{Ext}^i(k, k)$ and $\mathrm{Tor}_i(k, k)$ are finite-dimensional over k and that for all $i \geq 0$ we have
$$\dim_k \mathrm{Ext}^i(k, k) = \dim_k \mathrm{Tor}_i(k, k).$$

2 Computing stuff

Exercise C1. Determine the number of isomorphism classes of (not necessarily commutative) groups G such that there exists a normal subgroup $N \triangleleft G$ with $N \cong G/N \cong \mathbb{Z}/4\mathbb{Z}$.

Note. Don't forget to prove that the groups you find are not isomorphic!

Exercise C2. Consider the group $G := \mathbb{Z}$, acting trivially on $A := \mathbb{Z}$. Compute $H^n(G, A)$ for all $n \geq 0$.

Exercise C3.

1. Let R be the ring $\mathbb{Z}[\sqrt{-5}]$ and let $I = (7, 3+\sqrt{-5})$, $J = (7, 3-\sqrt{-5})$. Describe $\operatorname{Ext}^1_R(R/I, J)$, both as an abelian group and as an R-module.
2. Compute the cardinality of $\# \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/7\mathbb{Z})$. How many (isomorphism classes of) abelian groups G are there that fit in an exact sequence

$$1 \to \mathbb{Z}/7\mathbb{Z} \to G \to \mathbb{Z}/7\mathbb{Z} \to 1?$$

   Explain this discrepancy.

Exercise C4.

1. Prove that $\operatorname{Ext}^1(\mathbb{Q}, \mathbb{Z})$ has a structure of $\mathbb{Q}$-vector space.
2. Show that $\# \operatorname{Ext}^1(\mathbb{Q}, \mathbb{Z}) = 2^{\aleph_0}$ if and only if $\operatorname{Ext}^1(\mathbb{Q}, \mathbb{Z}) \cong (\mathbb{R}, +)$.

   (The statement is that every $\mathbb{Q}$-vector space which is set-theoretically in bijection with $\mathbb{R}$ is also isomorphic to $\mathbb{R}$ as a $\mathbb{Q}$-vector space)
3. Comparing $\operatorname{Ext}^1(\mathbb{Q}, \mathbb{Z})$ with $\operatorname{Ext}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$, deduce that $\operatorname{Ext}^1(\mathbb{Q}, \mathbb{Z}) \cong (\mathbb{R}, +)$.