Def.: $f*g(x) = \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy$. È ben definito in x se $|f|*|g|(x) < +\infty$.

Teorema: $r, p_1, p_2 \in [1,+\infty]$, $1+\frac{1}{r} = \frac{1}{p_1}+\frac{1}{p_2}$, $f_i \in L^{p_i} \Rightarrow f_1 * f_2$ è ben definito q.o. e $\|f_1 * f_2\|_r \leq \|f_1\|_{p_1} \|f_2\|_{p_2}$.

Dim.: $r<+\infty$ ($r=+\infty$ dopo) WLOG $f_1, f_2 \geq 0$
(se $\| |f_1| * |f_2| \|_r < +\infty$, $|f_1|*|f_2| < +\infty$ q.o. $\Rightarrow f_1 * f_2$ ben def. q.o.).

Caso $0 < \|f_1 * f_2\|_r < +\infty$:

$$\|f_1 * f_2\|_r^r = \int (\underbrace{f_1 * f_2(x)}_{h})^r\,dx = \int f_1 * f_2(x)\, h^{r-1}(x)\,dx =$$

$$= \int\int f_1(x-y) f_2(y)\, h^{r-1}(x)\,dy\,dx \;=\; \qquad \beta_1, \beta_2 \geq 0,\ \beta_1+\beta_2 = r-1,\ \alpha_1, \alpha_2 \in [0,1]$$

$$= \iint (f_1^{\alpha_1} h^{\beta_1})(f_2^{\alpha_2} h^{\beta_2})(f_1^{1-\alpha_1} f_2^{1-\alpha_2}) \leq$$

$$\underset{\substack{\text{Hölder} \\ \text{con } \frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{1-\gamma_1-\gamma_2}}}{\leq} \left(\iint f_1^{\alpha_1/\gamma_1} h^{\beta_1/\gamma_1}\right)^{\gamma_1} \left(\iint f_2^{\alpha_2/\gamma_2} h^{\beta_2/\gamma_2}\right)^{\gamma_2} \left(\iint f_1^{\frac{1-\alpha_1}{1-\gamma_1-\gamma_2}} f_2^{\frac{1-\alpha_2}{1-\gamma_1-\gamma_2}}\right)^{1-\gamma_1-\gamma_2} =$$

Imponiamo $\frac{\alpha_i}{\gamma_i} = \frac{1-\alpha_i}{1-\gamma_1-\gamma_2} = p_i$, $\frac{\beta_i}{\gamma_i} = r \Rightarrow$

$$\Rightarrow \gamma_i = \frac{p_i - 1}{p_i}$$

$$= \left(\iint f_1^{p_1} h^r\right)^{\frac{p_2-1}{p_2}} \left(\iint f_2^{p_2} h^r\right)^{\frac{p_1-1}{p_1}} \left(\iint f_1^{p_1} f_2^{p_2}\right)^{1-\left(\frac{p_1-1}{p_1}+\frac{p_2-1}{p_2}\right)} =$$

$$= \|f_1\|_{p_1} \|f_2\|_{p_2} \|h\|_r^{r-1}.$$

Se $\|f_1 * f_2\|_r = +\infty$, siano $f_{1,n}, f_{2,n}$ limitate a supporto cpt t.c. $f_{1,n} \uparrow f_1$, $f_{2,n} \uparrow f_2 \Rightarrow f_{1,n} * f_{2,n} \uparrow f_1 * f_2$. La tesi segue dunque per convergenza monotona (la disuguaglianza è vera per $f, g \geq 0$ lim. a supp cpt perché in questo caso pure $f*g$ lo è). $\square$

Lemma: τ_h è continua su L^p per $p<+\infty$ (controesempio per $p=+\infty$: $f = \mathbb{1}_{[0,+\infty)}$).

Dim.: caso $f \in C_c$, $\|\tau_h f - f\|_p^p = \int |f(x-h) - f(x)|^p\,dx$.
Per h sufficientemente piccolo, $\mathrm{supp}(|f(x-h) - f(x)|) \subset \mathrm{supp}(f) + B(0,1)$ e $|f(x-h) - f(x)| \leq 2\|f\|_\infty < +\infty$, inoltre f continua $\Rightarrow f(x-h) \to f(x)$.
Allora per convergenza dominata ho la tesi.

Caso f generica: sia g C_c t.c. $\|f-g\|_p \leq \varepsilon$.

$$\|\tau_h f - f\|_p \leq \underbrace{\|\tau_h f - \tau_h g\|_p}_{= \|f-g\|_p} + \|\tau_h g - g\|_p + \|g - f\|_p \leq$$

$$\leq 2\varepsilon + \|\tau_h g - g\|_p \xrightarrow[h\to 0]{} 2\varepsilon$$, concludo per arbitrarietà di ε. $\square$

Prop.: $\frac{1}{p_1}+\frac{1}{p_2} = 1$, $f_i \in L^{p_i} \Rightarrow \|f_1 * f_2\|_\infty \leq \|f_1\|_{p_1} \|f_2\|_{p_2}$, $f_1 * f_2$ è uniformemente continua. $1<p_1, p_2 < +\infty \Rightarrow f_1 * f_2 \in C_0$.

Dim.: $|f_1 * f_2(x)| \leq \int |f_1|(x-y)|f_2|(y)\,dy \underset{\text{Hölder}}{\leq} \| |f_1|(x-\cdot)\|_{p_1} \|f_2\|_{p_2} = \|f_1\|_{p_1}\|f_2\|_{p_2}$.
WLOG $p_1 < +\infty$,

$$|f_1 * f_2(x-h) - f_1 * f_2(x)| \leq \int |f_1(x-h-y) - f_1(x-y)|\,|f_2(y)|\,dy \leq$$

$$\leq \|\tau_h f_1(x-\cdot) - f_1(x-\cdot)\|_{p_1} \|f_2\|_{p_2} = \|\tau_h f_1 - f_1\|_{p_1} \|f_2\|_{p_2} \xrightarrow{h\to 0} 0.$$

Approssimo f_1, f_2 con $f_{1,n}, f_{2,n} \in C_c\ (\subset C_0)$. Se dimostro che $f_{1,n} * f_{2,n} \to f_1 * f_2$ uniformemente, ho concluso.

$$\|f_1 * f_2 - f_{1,n} * f_{2,n}\|_\infty \leq \|f_1 * f_2 - f_1 * f_{2,n}\|_\infty + \|f_1 * f_{2,n} - f_{1,n} * f_{2,n}\|_\infty \leq$$

$$\leq \underbrace{\|f_1\|_{p_1}}_{<+\infty} \underbrace{\|f_2 - f_{2,n}\|_{p_2}}_{\to 0} + \underbrace{\|f_1 - f_{1,n}\|_\infty}_{\to 0} \underbrace{\|f_{2,n}\|_{p_2}}_{\to \|f_2\|_{p_2} < +\infty}. \ \square$$

Prop.: $f_1 \in L^{p_1}$ continua e derivabile con continuità lungo h, $\frac{\partial f_1}{\partial h} \in L^{p_1}$, $f_2 \in L^{p_2} \Rightarrow f_1 * f_2$ continua e derivabile con continuità lungo h, $\left(\frac{1}{p_1}+\frac{1}{p_2}=1\right)$

$$\frac{\partial}{\partial h}(f_1 * f_2) = \frac{\partial f_1}{\partial h} * f_2.$$

Dim. (caso $d=1$): $f_1' * f_2$ è continua (caso $r=+\infty$ del teorema).

$$\int_a^b f_1' * f_2(x)\,dx = \int_a^b \int_{\mathbb{R}} f_1'(x-y) f_2(y)\,dy\,dx \underset{\text{Fubini}}{=}$$

$$= \int_{\mathbb{R}} \int_a^b f_1'(x-y) f_2(y)\,dx\,dy = \int_{\mathbb{R}} (f_1(b-y) - f_1(a-y)) f_2(y)\,dy =$$

$= f_1 * f_2(b) - f_1 * f_2(a)$. La tesi segue dal TFCI. $\square$

Prop.: $f \in L^p$, $g \in L^1 \Rightarrow f * \sigma_\delta g \to \left(\int g\right) f$ in L^p.

Dim.:

$$\|\underbrace{f*g - \left(\textstyle\int g\right) f}_{h}\|_p^p = \int \left|f*g(x) - \left(\textstyle\int g\right) f(x)\right| |h(x)|^{p-1}\,dx \leq$$

$$\leq \int |h(x)|^{p-1} \left(\int |f(x-y) - f(x)|\,|g(y)|\,dy\right) dx \overset{\text{Fubini}}{=}$$

$$= \int |g(y)| \left(\int |(\tau_y f - f)(x)|\,|h(x)|^{p-1}\,dx\right) dy \overset{\text{Hölder},\ q = \frac{p}{p-1}}{\leq}$$

$$\leq \int |g(y)|\, \|\tau_y f - f\|_p \underbrace{\| |h|^{p-1} \|_q}_{= \|h\|_p^{p-1}}\,dy. \quad \int \sigma_\delta g = \int g \Rightarrow$$

$$\Rightarrow \left\|f * \sigma_\delta g - \left(\textstyle\int g\right) f\right\|_p \leq \int \left|\tfrac{1}{\delta} g\left(\tfrac{y}{\delta}\right)\right| \|\tau_y f - f\|_p\,dy =$$

$$\underset{y=\delta z}{=} \int |g(z)|\, \|\tau_{\delta z} f - f\|_p\,dz \xrightarrow{\delta\to 0} 0$$ per convergenza dominata. $\square$