Def.: a Hilbert space $\mathbb{H}$ is a vector space over $\mathbb{C}$ endowed with a (positive definite) scalar product $\mathbb{H}\times\mathbb{H}\to\mathbb{C}$
$(\varphi,\psi)\mapsto\langle\varphi|\psi\rangle$

- $\langle\varphi|\psi\rangle=\overline{\langle\psi|\varphi\rangle}\ \forall\varphi,\psi\in\mathbb{H}$
- $\langle\psi|\psi\rangle\geqslant 0\ \forall\psi\in\mathbb{H}$
- $\langle\psi|\psi\rangle=0\iff\psi=0$
- $\langle\psi|a\varphi_1+b\varphi_2\rangle=a\langle\psi|\varphi_1\rangle+b\langle\psi|\varphi_2\rangle\ \forall\psi,\varphi_1,\varphi_2\in\mathbb{H},a,b\in\mathbb{C}\Rightarrow$
$\Rightarrow\langle a\varphi_1+b\varphi_2|\psi\rangle=\bar a\langle\varphi_1|\psi\rangle+\bar b\langle\varphi_2|\psi\rangle$ (anti-linear w.r.t. the first argument)

Remark: $\langle\cdot|\cdot\rangle$ induces a norm on $\mathbb{H}$: $\|\psi\|=\sqrt{\langle\psi|\psi\rangle}\ \forall\psi\in\mathbb{H}$
- $\mathbb{H}$ is complete w.r.t. the induced norm.

Remarks: (1) $\forall a\in\mathbb{C},\forall\psi,\varphi\in\mathbb{H}\ \langle a\varphi|\psi\rangle=\bar a\langle\varphi|\psi\rangle$, $\|a\varphi\|=|a|\cdot\|\varphi\|$
(2) let $\psi\in\mathbb{H}$, $\langle\psi|\varphi\rangle=0\ \forall\varphi\in\mathbb{H}\iff\psi=0$
(3) complex parallelogram identity: $\forall\varphi,\psi\in\mathbb{H}$ it holds
$\langle\psi|\varphi\rangle=\frac{1}{4}(\|\psi+\varphi\|^2-\|\psi-\varphi\|^2+i\|\psi-i\varphi\|^2-i\|\psi+i\varphi\|^2)$ (ex.)

Def.:
- a vector $\psi\in\mathbb{H}$ is a <u>unit</u> (normed) vector if $\|\psi\|=1$
- $\varphi,\psi\in\mathbb{H}$ are <u>orthogonal</u> if $\langle\varphi|\psi\rangle=0$
- given $\psi\in\mathbb{H}$, the <u>orthogonal subspace</u> to ψ is $\mathbb{H}_{\psi\perp}=\{\varphi\in\mathbb{H}\mid\langle\varphi|\psi\rangle=0\}$

Let $0\neq\psi\in\mathbb{H}$ fixed, $\varphi\in\mathbb{H}$ any vector; then $\varphi-\underbrace{\frac{\langle\varphi|\psi\rangle}{\|\psi\|^2}\psi}_{\text{projection of }\varphi\text{ along }\psi}\in\mathbb{H}_{\psi\perp}$

Def.: let $\mathbb{H}$ be a Hilbert space, I an index set;
- vectors $\{\varphi_j\}_{j\in I}\subset\mathbb{H}$ are linearly independent if ... (you know the rest)
- $\mathbb{H}$ is finite-dimensional if ... (yktr)
- an orthonormal base (ONB) is ... (yktr)

Remark: we only work with separable H. spaces (i.e. basis has a countable number of elements)

Remark: $\{e_j\}_{j\in I}$ ONB, $\psi=\sum_j a_j e_j$; a_j is uniquely defined by $a_j=\langle e_j|\psi\rangle$

Let $\psi,\varphi\in\mathbb{H}$, $\{e_j\}_{j\in I}$ ONB, $\psi=\sum_j\psi_j e_j$, $\varphi=\sum_j\varphi_j e_j\Rightarrow\langle\varphi|\psi\rangle=\sum_j\bar\varphi_j\psi_j$.
$\|\psi\|^2=\langle\psi|\psi\rangle=\sum_j\psi_j^2$.
If $\langle\varphi|\psi\rangle=0$ then $\|\varphi+\psi\|^2=\|\varphi\|^2+\|\psi\|^2$.
In general, $\|\varphi+\psi\|\leqslant\|\varphi\|+\|\psi\|$ (using $|\langle\varphi|\psi\rangle|\leqslant\|\varphi\|\cdot\|\psi\|$, CS).
For $\psi\in\mathbb{H}$ fixed, consider the linear map $\mathbb{H}\to\mathbb{C}$, $\varphi\mapsto\langle\psi|\varphi\rangle$; this is also continuous. Conversely (Riesz representation theorem), $\forall$ linear and continuous map $f:\mathbb{H}\to\mathbb{C}$ $\exists!\ \psi\in\mathbb{H}$ s.t. $f(\varphi)=\langle\psi|\varphi\rangle\ \forall\varphi\in\mathbb{H}$.
$\mathbb{H}^*=\{f:\mathbb{H}\to\mathbb{C}$ linear and continuous$\}$ is the <u>dual space</u> of $\mathbb{H}$.
There is a bijection between $\mathbb{H}$ and $\mathbb{H}^*$.
Dirac notation: $\psi\in\mathbb{H}$, <u>bra vectors</u> are $\langle\psi|\in\mathbb{H}^*$
<u>ket vectors</u> are $|\psi\rangle\in\mathbb{H}$

Let $A:\mathbb{H}\to\mathbb{H}$ linear, $|\psi\rangle\in\mathbb{H}$, $A|\psi\rangle=|A\psi\rangle$. $\{e_j\}$ ONB on $\mathbb{H}$,
$|\psi\rangle=\sum_j|e_j\rangle\langle e_j|\psi\rangle$, $|A\psi\rangle=\sum_j|e_j\rangle\langle e_j|A\psi\rangle=$
$=\sum_j|e_j\rangle\langle e_j|A\sum_k|e_k\rangle\langle e_k|\psi\rangle\rangle$, $A=\sum_{j,k}|e_j\rangle\underbrace{\langle e_j|Ae_k\rangle}_{A_{jk}}\langle e_k|$.

Finite dimensional case: $\mathbb{H}\cong\mathbb{C}^n$,
$|e_j\rangle\longleftrightarrow\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\begin{matrix}\\ \\ \\ \leftarrow j\\ \\ \\ \\ \end{matrix}$, $j=1,\dots,n$. $\langle e_j|\longleftrightarrow(0,\dots,0,\overset{\overset{j}{\uparrow}}{1},0,\dots,0)$, $j=1,\dots,n$.

$|\psi\rangle\in\mathbb{H}$, $|\psi\rangle=\sum_j\psi_j|e_j\rangle\longleftrightarrow\begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}\in\mathbb{C}^n$,
$\langle\varphi|=\sum_j\langle e_j|\bar\varphi_j\longleftrightarrow(\bar\varphi_1,\dots,\bar\varphi_n)$,
$|\psi\rangle\langle\varphi|\longleftrightarrow\begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}(\bar\varphi_1\dots\bar\varphi_n)=\begin{pmatrix}\psi_1\bar\varphi_1 & \cdots & \psi_1\bar\varphi_n\\ \vdots & & \vdots\\ \psi_n\bar\varphi_1 & \cdots & \psi_n\bar\varphi_n\end{pmatrix}$