Postulates of QM

States/Observables: a quantum system is described with a H. space $\mathbb{H}$.

- An observable is any self-adjoint $A:\mathbb{H}\to\mathbb{H}$.
- A (pure) state is any $|\psi\rangle\in\mathbb{H}$ s.t. $\|\psi\|=1$.

We say that the physical system is in state $|\psi\rangle$ if for every observable A the quantum expected value of A is $\langle A\rangle_{|\psi\rangle}=\langle\psi|A\psi\rangle$.
The physical system S will be often identified with $\mathbb{H}^S$.
$|\langle A\rangle_{|\psi\rangle}|\le\|A\|$.

By spectral theorem, given A we can diagonalize it wit ONB $(|e_{j,\alpha}\rangle)_{j,\alpha}$ so $A=\sum_{j,\alpha}\lambda_j|e_{j,\alpha}\rangle\langle e_{j,\alpha}|$ and $\forall\,|\psi\rangle\in\mathbb{H}$
$\langle A\rangle_{|\psi\rangle}=\sum_{j,\alpha}\lambda_j|\langle e_{j,\alpha}|\psi\rangle|^2$.
If $A=\mathrm{Id}=1$ we get $\langle 1\rangle_{|\psi\rangle}=\sum_\alpha 1\,|\langle e_\alpha|\psi\rangle|^2=\|\psi\|^2=1$.
Notice also that if A,B are observables and $\lambda\in\mathbb{R}$,

$$\langle A+\lambda B\rangle_{|\psi\rangle}=\langle A\rangle_{|\psi\rangle}+\lambda\langle B\rangle_{|\psi\rangle}.$$

Probabilities: assume that the system is in the state $|\psi\rangle\in\mathbb{H}$ and let A be an observable. The (possible) outcomes of measuring A are $\sigma(A)$. For every $\lambda\in\sigma(A)$ the probability of measuring λ is $\mathbb{P}_\psi(\lambda)=\|P_\lambda\psi\|^2$, where P_λ is the (orthogonal) projection on the corresponding eigenspace.
Consider P_λ as an observable, so $\mathbb{P}_\psi(\lambda)=\|P_\lambda\psi\|^2=\langle P_\lambda\psi|P_\lambda\psi\rangle=$
$=\langle\psi|P_\lambda^2\psi\rangle=\langle\psi|P_\lambda\psi\rangle=\langle P_\lambda\rangle_{|\psi\rangle}$.
Question: when $|\psi\rangle$ and $|\varphi\rangle$ state vectors satisfy $\langle A\rangle_{|\varphi\rangle}=\langle A\rangle_{|\psi\rangle}$ $\forall A$ observables?
Answer: iff $\exists\,\alpha\in\mathbb{R}$ s.t. $\underbrace{e^{i\alpha}}_{\text{phase}}|\varphi\rangle=|\psi\rangle$. ($\Leftarrow$) is trivial.
($\Rightarrow$) exc..

Def.: a ray is $R_\psi=\{e^{i\alpha}|\psi\rangle \mid \alpha\in\mathbb{R}\}$.
Let $|\psi\rangle$ and $|\varphi\rangle$ be states and $a,b\in\mathbb{C}$ s.t. $a|\psi\rangle+b|\varphi\rangle$ is a state vector, i.e. $\|a|\psi\rangle+b|\varphi\rangle\|=1$. Any such combination is called a quantum superposition (of $|\psi\rangle$ and $|\varphi\rangle$).
Ex.: if $\langle\psi|\varphi\rangle=0$, then $\|a|\psi\rangle+b|\varphi\rangle\|^2=\underbrace{|a|^2}_{p}+\underbrace{|b|^2}_{1-p}\overset{?}{=}1$, $p\in[0,1]$.
Let A be an observable,

$$\langle A\rangle_{\sqrt{p}|\psi\rangle+\sqrt{1-p}|\varphi\rangle}=p\langle A\rangle_{|\psi\rangle}+(1-p)\langle A\rangle_{|\varphi\rangle}+\underbrace{2\sqrt{p(1-p)}\,\mathrm{Re}\langle\psi|A\varphi\rangle}_{\text{interference term}}.$$

Exc.: compute $\langle A\rangle_{\sqrt{p}|\varphi\rangle+e^{i\alpha}\sqrt{1-p}|\psi\rangle}$.

Prop.: if $|\psi\rangle,|\varphi\rangle\in\mathbb{H}$ are state vectors, then the probability of observing $|\varphi\rangle$ if the system is in state $|\psi\rangle$ is given by "$\mathbb{P}_\psi(|\varphi\rangle)$" $=|\langle\varphi|\psi\rangle|^2$.
Proof.: introduce the observable $A=|\varphi\rangle\langle\varphi|$.
$\mathbb{P}_\psi(1)=\langle A\rangle_\psi=|\langle\varphi|\psi\rangle|^2$. □

Def.: the uncertainty of an observable A in a state vector $|\psi\rangle$ is defined as $\Delta_\psi(A)=\sqrt{\langle(A-\langle A\rangle_\psi\mathrm{Id})^2\rangle_\psi}=$
$=\sqrt{\langle\psi|(A-\langle A\rangle_\psi\mathrm{Id})^2\psi\rangle}=\sqrt{\langle(A-\langle A\rangle_\psi\mathrm{Id})\psi|(A-\langle A\rangle_\psi\mathrm{Id})\psi\rangle}=\|(A-\langle A\rangle_\psi\mathrm{Id})\psi\|$.
Def.: A is sharp in the state $|\psi\rangle$ if $\Delta_\psi(A)=0$.
Exc.: $(\Delta_\psi(A))^2=\langle A^2\rangle_\psi-(\langle A\rangle_\psi)^2$, $\Delta_\psi(\lambda A)=|\lambda|\Delta_\psi(A)$ $\forall\lambda\in\mathbb{R}$,
$\Delta_\psi(A-\lambda\mathrm{Id})=\Delta_\psi(A)$ $\forall\lambda\in\mathbb{R}$.
Prop.: $\Delta_\psi(A)=0\iff|\psi\rangle$ is an eigenvector of A (with eigenvalue $\langle A\rangle_\psi$).
Proof: WLOG $\langle A\rangle_\psi=0$. Then $0=\Delta_\psi(A)=\|A\psi\|\iff A\psi=0$. □
Def.: A,B observables are compatible if $[A,B]=0$.
Compatible observables can be "measured" at the same time.
Th.: if $[A,B]=0$, then $\exists\,(e_j)_j$ ONB s.t. both A,B are diagonal.
If A,B are not compatible, then $\forall\psi$ state
$\frac{1}{2}\langle i[A,B]\rangle_{|\psi\rangle}\le\Delta_\psi(A)\Delta_\psi(B)$ (Heisenberg uncertainty inequality).
Proof: let $K=A+iB$, $K^*=A-iB$,

$$0\le\langle KK^*\rangle_\psi=\langle A^2\rangle_\psi+\langle B^2\rangle_\psi-\langle i[A,B]\rangle_\psi\Rightarrow$$

$$\Rightarrow\langle i[A,B]\rangle_\psi\le\langle A^2\rangle_\psi+\langle B^2\rangle_\psi.$$

Taking $A\rightsquigarrow A-\langle A\rangle_\psi\mathrm{Id}$, $B\rightsquigarrow B-\langle B\rangle_\psi\mathrm{Id}$ we get

$$\langle i[A,B]\rangle_\psi=\langle i[A-\langle A\rangle_\psi\mathrm{Id},B-\langle B\rangle_\psi\mathrm{Id}]\rangle_\psi\le\Delta_\psi^2(A)+\Delta_\psi^2(B).$$

Taking $A\rightsquigarrow\lambda A$, $B\rightsquigarrow\frac{1}{\lambda}B$ ($\lambda\in\mathbb{R}$) we get

$$\langle i[A,B]\rangle_\psi=\langle i[\lambda A,\tfrac{1}{\lambda}B]\rangle_\psi\le\lambda^2\Delta_\psi^2(A)+\frac{1}{\lambda^2}\Delta_\psi^2(B).$$

Choose $\lambda=-\sqrt{\frac{\Delta_\psi(B)}{\Delta_\psi(A)}}$. □