Exc.: try the proof of uncertainty principle with $K=A+B$.

Heisenberg inequality on $\mathbb{H}=L^2(\mathbb{R})$

Q = position operator, $(Q\psi)(x)=x\cdot\psi(x)$

P = momentum operator, $(P\psi)(x)=-i\frac{d}{dx}\psi(x)$

One can prove: if ψ is $C^1_c(\mathbb{R})$, then

$$[Q,P]\psi(x)=-ix\frac{d}{dx}\psi(x)+i\frac{d}{dx}(x\psi(x))=i\psi(x)=i\,\mathrm{Id}\,\psi(x).$$

These are called canonical commutation relations.

Heisenberg inequality reads as $\frac{1}{2}\leq\Delta_\psi(Q)\Delta_\psi(P)$.

Two (last) postulates

- If $|\psi\rangle$ is the state of your quantum system and the observable A is measured with outcome $\lambda\in\sigma(A)$, the state after the measurement is described by $|P_\lambda\psi\rangle/\|P_\lambda\psi\|$ with $P_\lambda:\mathbb{H}\to\mathbb{H}$ the orthogonal proj. on $\mathrm{Eig}(\lambda,A)$.
- If instead the system is closed (isolated) the evolution of a state $|\psi_t\rangle$ from time t_0 to time t_1 is described by a unitary $U(t_0,t_1)$ s.t. $|\psi_{t_1}\rangle=U(t_0,t_1)|\psi_{t_0}\rangle$.

An observable H is the hamiltonian of a system if $U(t_0,t_1)=e^{-i(t_1-t_0)H}$.

Assuming that H does not depend on t, one can write $|\psi_t\rangle=e^{-itH}|\psi_0\rangle$.

$\partial_t|\psi_t\rangle=-iH|\psi_t\rangle\Rightarrow i\partial_t|\psi_t\rangle=H|\psi_t\rangle$, Schrödinger's equation.

Mixed states

Remember: $|\psi\rangle=a|\varphi_0\rangle+b|\varphi_1\rangle$ is a superposition.

Idea: $(p_i)_{i=1}^N$ classical probability, $(|\psi_i\rangle)_{i=1}^N$ state vectors.

Density operator: $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$ if $\langle\psi_i|\psi_j\rangle=\delta_{ij}$.

Def.: a mixed state on a system $\mathbb{H}$ is described by any density operator $\rho:\mathbb{H}\to\mathbb{H}$ s.t.

- is self-adjoint; • is non-negative; • has unit trace.

We write $\mathcal{D}(\mathbb{H})=\{\rho:\mathbb{H}\to\mathbb{H}\mid\rho^*=\rho,\ \rho\geq0,\ \mathrm{Tr}(\rho)=1\}$.

Remark:
- $\mathcal{D}(\mathbb{H})$ is convex;
- if $|\psi\rangle\in\mathbb{H}$ is a (pure) state vector, $\rho=|\psi\rangle\langle\psi|\in\mathcal{D}(\mathbb{H})$.
- if $|\varphi\rangle\in R_{|\psi\rangle}$ ($\exists\,\alpha\in\mathbb{R}$ s.t. $e^{i\alpha}|\varphi\rangle=|\psi\rangle$), $|\psi\rangle\langle\psi|=|\varphi\rangle\langle\varphi|$;
- if $U:\mathbb{H}\to\mathbb{H}$ is unitary, then $\forall\rho\in\mathcal{D}(\mathbb{H})$ $U\rho U^*\in\mathcal{D}(\mathbb{H})$.

Theorem: if $\rho\in\mathcal{D}(\mathbb{H})$, $\dim(\mathbb{H})<+\infty$, then
- $\exists\,(p_i)_i$, $p_i\in[0,1]$, $\sum_i p_i=1$, $(|\psi_i\rangle)_i$ ONB of $\mathbb{H}$ s.t. $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$;
- $\rho^2\leq\rho$ as quadratic forms, with $=$ iff ρ is pure;
- $\|\rho\|\leq1$ " " " " " " (exc.).

Proof: apply spectral theorem to ρ: $(|\varphi_{j,\alpha}\rangle)_{j,\alpha}$ ONB so that $\rho=\sum_j\lambda_j\sum_\alpha|\varphi_{j\alpha}\rangle\langle\varphi_{j\alpha}|$, $\{\lambda_j\}_j=\sigma(\rho)$. Up to renumbering, $|\varphi_{j\alpha}\rangle\rightsquigarrow|\psi_i\rangle$, $\lambda_j\rightsquigarrow p_i$.

$\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|\Rightarrow\rho^2=\sum_i p_i^2|\psi_i\rangle\langle\psi_i|\leq\sum_i p_i|\psi_i\rangle\langle\psi_i|=\rho$ because $p_i\in[0,1]\Rightarrow p_i^2\leq p_i$; $=\Leftrightarrow p_i\in\{0,1\}$ $\forall i$, WLOG $p_1=1\Rightarrow\rho=|\psi_1\rangle\langle\psi_1|$. $\square$

Postulates for mixed states:
- given A observable, its expectation over a mixed state ρ is $\mathrm{Tr}(A\rho)=\sum_i p_i\langle A\rangle_{\psi_i}$ where $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$, $|\psi_i\rangle$ ONB;
- the probability of observing $\lambda\in\sigma(A)$ if system is in the state ρ is $\mathbb{P}_\rho(\lambda)=\mathrm{Tr}(P_\lambda\rho)=\sum_i p_i\,\mathbb{P}_{\psi_i}(\lambda)$;
- after measuring $\lambda\in\sigma(A)$, the state ρ is updated to $\frac{P_\lambda\rho P_\lambda}{\mathbb{P}_\rho(\lambda)}$.

If A is measured but the outcome is not given the state ρ is updated to $\tilde{\rho}=\sum_{\lambda\in\sigma(A)}\mathbb{P}_\rho(\lambda)\cdot\left(\frac{P_\lambda\rho P_\lambda}{\mathbb{P}_\rho(\lambda)}\right)$.

- If the system is closed it evolves from time t_0 to t_1 according to a unitary $U(t_0,t_1)$: $\rho_{t_1}=U(t_0,t_1)\rho_{t_0}U^*(t_0,t_1)$.

The Schrödinger's eq. associated to H is
$$i\partial_t\rho_t=[H,\rho_t].$$

- Uncertainty of A over ρ is $\Delta_\rho(A)=(\langle(A-\langle A\rangle_\rho\mathrm{Id})^2\rangle_\rho)^{1/2}$.

Exc.: Heisenberg inequality for mixed states?

If $\rho\in\mathcal{D}(\mathbb{H})$ is represented as $\rho=\sum_{i=1}^m q_i|\varphi_i\rangle\langle\varphi_i|$, $|\varphi_i\rangle$ state vectors, $q_i>0$, but also $\rho=\sum_{j=1}^n p_j|\psi_j\rangle\langle\psi_j|$, $p_j>0$, $(|\psi_j\rangle)_j$ ONB.

Then $m\geq n$ and there exists $U\in\mathbb{C}^{m\times m}$ s.t. $UU^*=\mathrm{Id}$ with $\sqrt{q_i}|\varphi_i\rangle=\sum_{j=1}^n U_{ij}\sqrt{p_j}|\psi_j\rangle$.

Hint: $U_{ij}=\sqrt{\frac{q_i}{p_j}}\langle\psi_j|\varphi_i\rangle$ for $i=1,\dots,m$, $j=1,\dots,n$.

Exc.: if $\rho,\rho'\in\mathcal{D}(\mathbb{H})$ s.t. $\langle A\rangle_\rho=\langle A\rangle_{\rho'}$ $\forall A$ obs. then $\rho=\rho'$.

Hint: $\mathrm{Tr}(A\rho)=\mathrm{Tr}(A\rho')\Rightarrow\mathrm{Tr}(A(\rho-\rho'))=0$.