Exc.: spin of the electron: vector in $\mathbb{R}^3$.

We are interested in observables S_x, S_y, S_z.

Hilbert space of our system is 2-dim.: $\mathbb{H} \cong \mathbb{C}^2$.

From now on, $\mathbb{H}$ denotes always a 2-dim. space.

ONB is chosen as $\underset{\text{up}}{|\uparrow_{\hat z}\rangle}, \underset{\text{down}}{|\downarrow_{\hat z}\rangle}$.

We also denote $|0\rangle := |\uparrow_{\hat z}\rangle, |1\rangle := |\downarrow_{\hat z}\rangle$ (⚠ $|0\rangle \neq$ zero vector).

$$\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{C}^2 \ni \begin{pmatrix}0\\1\end{pmatrix}$$

As matrices: $S_x = \frac{1}{2}\sigma_x$ and cyc, $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$: Pauli matrices. Also: $\sigma_x = \sigma_1$ and cyc.

$S_z|\uparrow_{\hat z}\rangle = \frac{1}{2}|\uparrow_{\hat z}\rangle$, $S_z|\downarrow_{\hat z}\rangle = -\frac{1}{2}|\downarrow_{\hat z}\rangle$.

Properties of Pauli matrices:

(1) $\sigma_j\sigma_k = \delta_{jk}\mathbb{1} + i\varepsilon_{jkl}\sigma_l$;

(2) $[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l$;

(3) $\{\sigma_j, \sigma_k\} = 2\delta_{jk}\mathbb{1}$ ($\{A,B\} := AB + BA$);

(4) σ_j is unitary and self-adjoint, $j = 1,2,3$.

Suppose system is in state $|0\rangle$. What happens if we measure σ_z ($2\cdot z$-spin)? From the postulates I expect that:

- measurement is sharp;
- expected value is 1;
- system stays in state $|0\rangle$ after measurement.

$\langle\sigma_z\rangle_{|0\rangle} = 1$, $\Delta_{|0\rangle}(\sigma_z) = 0$. Try with $|1\rangle$.

Suppose system is in state $|0\rangle$. What happens if we measure σ_x ($2\cdot x$-spin)?

$\langle\sigma_x\rangle_{|0\rangle} = 0$, $\Delta_{|0\rangle}(\sigma_x) = 1$, not sharp.

Possible outcomes: ± 1. What are the eigenstates of σ_x?

$\frac{1}{\sqrt2}|0\rangle + \frac{1}{\sqrt2}|1\rangle = |\uparrow_{\hat x}\rangle$, $\frac{1}{\sqrt2}|0\rangle - \frac{1}{\sqrt2}|1\rangle = |\downarrow_{\hat x}\rangle$.

- With probability 1/2 we find 1 and after that the system is in state $|\uparrow_{\hat x}\rangle$;
- with probability 1/2 we find -1 and after that the system is in state $|\downarrow_{\hat x}\rangle$.

Suppose I have measured σ_x and found 1. The system is now in state $|\uparrow_{\hat x}\rangle$. What happens if I measure σ_z again? Exc..

Remark: these two observables do not commute.

Qubits

Classical bit: two possible states, 0 and 1.

Quantum bit: quantum system defined by a 2-dim. Hilbert space $\mathbb{H}$ (qubit space) with ONB $\{|0\rangle, |1\rangle\}$ and an observable σ_z with eigenstates $\{|0\rangle, |1\rangle\}$ and eigenvalues ±1.

$|0\rangle$ and $|1\rangle$ correspond to classical bits 0, 1, but in $\mathbb{H}$ we also have all the states of the form $|\psi\rangle = a|0\rangle + b|1\rangle$, $|a|^2 + |b|^2 = 1$, $a, b \in \mathbb{C}$.

Measuring σ_z on a qubit yields 1 or -1; after that the qubit will be in state $|0\rangle$ or $|1\rangle$, respectively.

Parametrization of a qubit state

$|a|^2 + |b|^2 = 1 \Rightarrow \exists$ angles α, β, θ s.t. $a = e^{i\alpha}\cos(\theta/2)$, $b = e^{i\beta}\sin(\theta/2)$.

So, up to a global phase, I can write

$$|\psi\rangle = e^{-i\varphi/2}\cos(\theta/2)\,|0\rangle + e^{i\varphi/2}\sin(\theta/2)\,|1\rangle,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi.$$

Bloch sphere representation: associate $|\psi\rangle_{\varphi,\theta}$ with point $\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}$ on the sphere $S^2 \subseteq \mathbb{R}^3$.

Fix a qubit state $|\psi\rangle$. How can we build an observable that has $|\psi\rangle$ as an eigenstate?

Notation: let $a = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \in \mathbb{R}^3$ and define $a\cdot\sigma = \sum_{k=1}^{3} a_k\sigma_k$.

Exc.: $(a\cdot\sigma)(b\cdot\sigma) = (a\cdot b)\mathbb{1} + i(a\times b)\cdot\sigma$.

Define $\hat n_{|\psi\rangle} = \begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix} \in \mathbb{R}^3$, $\hat n_{|\psi\rangle}\cdot\sigma = \begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix}$

(operator for spin in direction $\hat n$). It is the operator we are looking for. Let $|\psi\rangle = |\uparrow_{\hat n_{|\psi\rangle}}\rangle$; analogously,

$$|\downarrow_{\hat n_{|\psi\rangle}}\rangle = \begin{pmatrix}-e^{-i\varphi/2}\sin(\theta/2)\\ e^{i\varphi/2}\cos(\theta/2)\end{pmatrix}.$$