Qubit: $\mathbb{H}$ (2 dim.), $|0\rangle, |1\rangle$, σ_z.

$|\psi\rangle \in \mathbb{H}$ has the form $|\psi\rangle = a|0\rangle + b|1\rangle$ with $|a|^2+|b|^2=1$, $a, b \in \mathbb{C}$.

$|\psi\rangle = e^{-i\varphi/2}\cos(\theta/2)|0\rangle + e^{i\varphi/2}\sin(\theta/2)|1\rangle$ for suitable angles $\theta, \varphi \rightsquigarrow$ Bloch sphere representation.

What about mixed states? A mixed qubit state is given by ρ density operator, i.e., $\rho^* = \rho$, $\operatorname{tr}\rho = 1$, $\rho \geq 0$.

$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$; $a, d \in \mathbb{R}$, $b = \bar{c}$, $a + d = 1$.

Introduce real parameters $x_1, x_2, x_3 \in \mathbb{R}$ s.t.

$a = \frac{1+x_3}{2}$, $d = \frac{1-x_3}{2}$, $b = \frac{x_1 - i x_2}{2}$, $c = \frac{x_1 + i x_2}{2}$.

$\rho = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1 - i x_2 \\ x_1 + i x_2 & 1 - x_3 \end{pmatrix}$. Eigenvalues of ρ:

$$(1+x_3-2\lambda)(1-x_3-2\lambda) - (x_1 - i x_2)(x_1 + i x_2) = 0$$
$$(1-2\lambda)^2 - x_3^2 = x_1^2 + x_2^2$$
$$(1-2\lambda)^2 = \|x\|_2^2, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3.$$
$$\lambda = \frac{1 \pm \|x\|_2}{2} \geq 0 \Rightarrow \|x\|_2 \leq 1.$$

Bloch sphere representation: $\rho \mapsto x \in \underbrace{B}_{\text{unit ball in } \mathbb{R}^3}$.

Remark 1: ρ is pure $\Longleftrightarrow$ it is represented by a point on the unit sphere (that is, $\|x\|_2 = 1$).

Remark 2: $\frac{1}{2}(\mathbb{1} + x_1\sigma_x + x_2\sigma_y + x_3\sigma_z) = \frac{1}{2}(\mathbb{1} + x\cdot\sigma) = \rho$.

Proof of remark 1: ρ is pure iff $\rho^2 = \rho$;

$\rho^2 = \frac{1}{4}(\mathbb{1} + (x\cdot\sigma)(x\cdot\sigma) + 2x\cdot\sigma) = \frac{1}{4}(\mathbb{1} + \|x\|_2^2\mathbb{1} + 2x\cdot\sigma)$,

$= \rho \Longleftrightarrow \|x\|_2^2 = 1$.

Remark 3: for $j = 1, 2, 3$, $\operatorname{tr}(\rho_x\sigma_j) = x_j$:

$\operatorname{tr}\left(\frac{1}{2}(\mathbb{1} + x\cdot\sigma)\sigma_j\right) = \frac{1}{2}\operatorname{tr}(\sigma_j + (x\cdot\sigma)\sigma_j) =$

$= \frac{1}{2}\left(\underbrace{\operatorname{tr}(\sigma_j)}_{=0} + \sum_{k=1}^{3} x_k \operatorname{tr}(\sigma_k\sigma_j)\right) =$

$= \frac{1}{2}x_j\operatorname{tr}(\mathbb{1}) = x_j$.

Operators on qubits: we want to study unitary operators on $\mathbb{H}$.

Exc.: let A be an operator on $\mathbb{H}$; $e^A := \sum_{k=0}^{+\infty}\frac{1}{k!}A^k$. Suppose that $A^2 = \mathbb{1}$; then $\forall \alpha \in \mathbb{R}$

$$e^{i\alpha A} = \mathbb{1} + i\alpha A - \frac{1}{2}\alpha^2\mathbb{1} - \frac{i}{3!}\alpha^3 A + \frac{1}{4!}\alpha^4\mathbb{1} + \frac{i}{5!}\alpha^5 A + \dots$$
$$= \cos(\alpha)\mathbb{1} + i\sin(\alpha)A.$$

Def. (rotation): let $\hat{m}$ be a unit vector in $\mathbb{R}^3$, $\alpha \in \mathbb{R}$ and define the operator $\overset{\hat{m}}{S^2}$ $D_{\hat{m}}(\alpha) = e^{-i\alpha/2\,\hat{m}\cdot\sigma}$ (spin operator).

Properties:

(1) $(m\cdot\sigma)^2 = \mathbb{1} \Rightarrow D_{\hat{m}}(\alpha) = \cos(\alpha/2)\mathbb{1} - i\sin(\alpha/2)\,\hat{m}\cdot\sigma$;

(2) $D_{\hat{m}}(\alpha)^* = D_{\hat{m}}(-\alpha)$ (exc.);

(3) $D_{\hat{m}}(\alpha)D_{\hat{m}}(\alpha)^* = \mathbb{1}$, so rotation is unitary;

(4) $D_{\hat{m}}(\alpha)D_{\hat{m}}(\beta) = D_{\hat{m}}(\alpha+\beta)$ (exc.).

Lemma: let U be a unitary operator on $\mathbb{H}$. Then $\exists\, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ s.t. the matrix of U w.r.t. $\{|0\rangle, |1\rangle\}$ is

$$U = e^{i\alpha}\begin{pmatrix} e^{-i\frac{\beta+\delta}{2}}\cos(\gamma/2) & -e^{i\frac{\delta-\beta}{2}}\sin(\gamma/2) \\ e^{i\frac{\beta-\delta}{2}}\sin(\gamma/2) & e^{i\frac{\beta+\delta}{2}}\cos(\gamma/2) \end{pmatrix}.$$

Idea of proof: $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$; U unitary $\Rightarrow$

$\Rightarrow UU^* = \mathbb{1}$ so $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, that is,

$|a|^2 + |b|^2 = 1$, $a\bar{c} + b\bar{d} = 0$, $|c|^2 + |d|^2 = 1$.

Details in the book, Section 2.5, Lemma 2.32. □

Lemma: let U be a unitary operator on $\mathbb{H}$; then $\exists\, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ s.t. $U = e^{i\alpha}D_{\hat{z}}(\beta)D_{\hat{y}}(\gamma)D_{\hat{z}}(\delta)$.

Proof: do the calculations and use previous lemma. □

Lemma: U unitary on $\mathbb{H}$. Then there exist operator A, B, C on $\mathbb{H}$, $\alpha \in \mathbb{R}$ s.t. • $ABC = \mathbb{1}$; • $U = e^{i\alpha}A\sigma_x B\sigma_x C$.

Idea of proof: we know $U = e^{i\alpha}D_{\hat{z}}(\beta)D_{\hat{y}}(\gamma)D_{\hat{z}}(\delta)$; take $A = D_{\hat{z}}(\beta)D_{\hat{y}}(\gamma/2)$, $B = D_{\hat{y}}(-\gamma/2)D_{\hat{z}}(-\frac{\beta+\delta}{2})$, $C = D_{\hat{z}}\left(\frac{\delta-\beta}{2}\right)$. □

Lemma: U unitary on $\mathbb{H}$. Then there exist $\alpha, \xi \in \mathbb{R}$ (angles), $\hat{m}$ unitary vector s.t. $U = e^{i\alpha}D_{\hat{m}}(\xi)$.

Idea of proof: use representation $U = e^{i\alpha}\begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$, split matrix as $c_0\mathbb{1} + c_1\sigma_x + c_2\sigma_y + c_3\sigma_z$ and from the c_j deduce coordinates of $\hat{m}$ and angle ξ. See Lemma 2.35 in the book. □

Exc.: let A be an operator on $\mathbb{H}$. Then $\exists\, z_0, z_1, z_2, z_3 \in \mathbb{C}$ s.t. $A = z_0\mathbb{1} + z\cdot\sigma$, $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{C}^3$. If A is unitary then $|z_0|^2 + \|z\|_2^2 = 1$. Hint: $A = e^{i\alpha}D_{\hat{m}}(\xi)$.

Hadamard operator: $H := \frac{\sigma_x + \sigma_z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ w.r.t. $\{|0\rangle, |1\rangle\}$.

$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$. $H^2 = \mathbb{1}$.

$H = e^{i\frac{3\pi}{2}}D_{\hat{z}}(0)D_{\hat{y}}\left(\frac{\pi}{2}\right)D_{\hat{z}}(-\pi)$.