Composite systems

$$\{0,1\} \rightsquigarrow \{0,1\}^m = \{(x_{m-1}, x_{m-2}, \dots, x_1, x_0) \mid x_i \in \{0,1\}\}$$

$$\big\downarrow$$

$$\mathbb{H} = \mathbb{C}^2 \dashrightarrow ?$$

- Tensor product space: $\mathbb{H}^{AB} = \mathbb{H}^A \otimes \mathbb{H}^B$.
- How to compute "marginals", reduced states $\rho^A \in \mathcal{D}(\mathbb{H}^A)$, $\rho^B \in \mathcal{D}(\mathbb{H}^B)$ given $\rho \in \mathcal{D}(\mathbb{H}^{AB})$? Partial trace operator.

The tensor product of (two) Hilbert spaces $\mathbb{H}^A$ with scalar product $\langle\,|\,\rangle^{\mathbb{H}^A}$, $\mathbb{H}^B$ with $\langle\,|\,\rangle^{\mathbb{H}^B}$:

define for $|\varphi\rangle \in \mathbb{H}^A$, $|\psi\rangle \in \mathbb{H}^B$ the functional

$$|\varphi\rangle \otimes |\psi\rangle : \mathbb{H}^A \times \mathbb{H}^B \longrightarrow \mathbb{C}.$$

$$(\xi, \eta) \longmapsto \langle \xi | \varphi \rangle^{\mathbb{H}^A} \cdot \langle \eta | \psi \rangle^{\mathbb{H}^B}$$

It is (bi-)antilinear.

Notation: $|\varphi\rangle \otimes |\psi\rangle = |\varphi \otimes \psi\rangle = |\varphi, \psi\rangle = |\varphi\rangle|\psi\rangle = |\varphi\psi\rangle$.

Def.: $\mathbb{H}^A \otimes \mathbb{H}^B = \{\Psi : \mathbb{H}^A \times \mathbb{H}^B \to \mathbb{C} \text{ bi-antilinear}\}$.

Remarks: i) $\mathbb{H}^A \otimes \mathbb{H}^B$ is a $\mathbb{C}$-v.s.;

ii) if $\{|e_i\rangle\}_{i=1,\dots,m_A} \subseteq \mathbb{H}^A$, $\{|f_j\rangle\}_{j=1,\dots,m_B} \subseteq \mathbb{H}^B$ are ONB, then $\{|e_i\rangle \otimes |f_j\rangle\}_{\substack{i=1,\dots,m_A \\ j=1,\dots,m_B}}$ is a basis and

$$\forall \Psi \in \mathbb{H}^A \otimes \mathbb{H}^B \qquad \Psi = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \Psi_{ij} \, |e_i\rangle \otimes |f_j\rangle$$

with $\Psi_{ij} = \Psi(e_i, f_j)$;

iii) $\dim(\mathbb{H}^A \otimes \mathbb{H}^B) = \dim(\mathbb{H}^A) \cdot \dim(\mathbb{H}^B)$.

Def.: a scalar product on $\mathbb{H}^A \otimes \mathbb{H}^B$ is defined $\forall \varphi_1, \varphi_2 \in \mathbb{H}^A$, $\psi_1, \psi_2 \in \mathbb{H}^B$ as $\langle \varphi_1 \otimes \psi_1 | \varphi_2 \otimes \psi_2 \rangle^{\mathbb{H}^A \otimes \mathbb{H}^B} = \langle \varphi_1 | \varphi_2 \rangle^{\mathbb{H}^A} \cdot \langle \psi_1 | \psi_2 \rangle^{\mathbb{H}^B}$ and extend by linearity.

Exc.: $\langle\,|\,\rangle^{\mathbb{H}^A \otimes \mathbb{H}^B}$ is well defined.

Remark: if $\underset{\mathbb{H}^A}{\{e_i\}} \,\cap$, $\underset{\mathbb{H}^B}{\{f_j\}} \,\cap$ are ONB, then $\{|e_i \otimes f_j\rangle\} \subseteq \mathbb{H}^A \otimes \mathbb{H}^B$ is ONB and $\langle \Psi, \Phi \rangle = \sum_{i,j} \overline{\Psi_{ij}} \Phi_{ij} \overset{\text{exc.}}{=} \operatorname{tr}(M_\Psi^* M_\Phi)$,

$$M_\Psi = (\Psi_{ij})_{\substack{i=1,\dots,m_A \\ j=1,\dots,m_B}} \in \mathbb{C}^{m_A \times m_B}.$$

$$\mathbb{C}^{m_A \times m_B} \simeq \mathbb{C}^{m_A m_B}$$

$$\begin{pmatrix} \underline{\quad r_1 \quad} \\ \underline{\quad r_2 \quad} \\ \vdots \\ r_{m_A} \end{pmatrix} \longmapsto (r_1, \dots, r_{m_A})^T.$$

Remark: $\quad \| |\varphi\rangle \otimes |\psi\rangle \|_{\mathbb{H}^A \otimes \mathbb{H}^B} = \|\varphi\|_{\mathbb{H}^A} \|\psi\|_{\mathbb{H}^B}$.

One can generalize the tensor product to $3, 4, \dots, m$ factors. Notice that $(\mathbb{H}^A \otimes \mathbb{H}^B) \otimes \mathbb{H}^C \neq \mathbb{H}^A \otimes \mathbb{H}^B \otimes \mathbb{H}^C$, but are isomorphic as Hilbert spaces. The same for $\mathbb{H}^A \otimes \mathbb{H}^B$ and $\mathbb{H}^B \otimes \mathbb{H}^A$, but for us the order is important.

Ex.: $\mathbb{H}^A = \mathbb{H}^B = \mathbb{C}^2$, $\{|0\rangle^{\mathbb{H}^A}, |1\rangle^{\mathbb{H}^A}\}$, $\{|0\rangle^{\mathbb{H}^B}, |1\rangle^{\mathbb{H}^B}\}$.

$\mathbb{H}^A \otimes \mathbb{H}^B \; (\cong \mathbb{C}^4)$ with ONB $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Computational basis of m-fold tensor product of qubit systems:

$$\mathbb{H}^m = \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{m \text{ factors}}, \quad \dim \mathbb{H}^m = 2^m.$$

Recursively: $\mathbb{H}^1 = \mathbb{C}^2$, $\mathbb{H}^{m+1} = \mathbb{C}^2 \otimes \mathbb{H}^m$.

The computational basis of $\mathbb{H}^m$ is defined as

$$\{|s\rangle\}_{s \in \{0,1\}^m}, \quad s = (x_{m-1}, x_{m-2}, \dots, x_1, x_0), \; x_i \in \{0,1\},$$

$$|s\rangle = |x_{m-1}\rangle \otimes |x_{m-2}\rangle \otimes \dots \otimes |x_1\rangle \otimes |x_0\rangle.$$

The computational basis is of ONB of $\mathbb{H}^m$ (by induction).

Another notation: given s as above, we let $s_{\text{base } 2} = \sum_{i=0}^{m-1} x_i \cdot 2^i \in \{0, 1, \dots, 2^m - 1\}$. Given $x \in \{0, 1, \dots, 2^m - 1\}$ we let $\mathbb{H}^m \ni |x\rangle^m = |s\rangle$ where $x = s_{\text{base } 2}$.

Another ON basis in $\mathbb{H}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2$ is the BELL basis:

$$|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}.$$

States and observables for composite systems

Postulate: given systems $\mathbb{H}^A$, $\mathbb{H}^B$, the composite system of the two is represented by $\mathbb{H}^A \otimes \mathbb{H}^B$; (pure) states are represented by $|\psi\rangle \in \mathbb{H}^A \otimes \mathbb{H}^B$ with $\|\psi\| = 1$ and mixed states are $\rho \in \mathcal{D}(\mathbb{H}^A \otimes \mathbb{H}^B)$.