Observables on $\mathbb{H}^A \otimes \mathbb{H}^B$

Special case: $M_A : \mathbb{H}^A \to \mathbb{H}^A$ self-adj.,

$M_B : \mathbb{H}^B \to \mathbb{H}^B$ " ,

$M_A \otimes M_B : \mathbb{H}^A \otimes \mathbb{H}^B \to \mathbb{H}^A \otimes \mathbb{H}^B$

$|\varphi\rangle \otimes |\psi\rangle \mapsto |M_A \varphi\rangle \otimes |M_B \psi\rangle$ (exc. is well def.)

is self-adj.: $|\eta\rangle \in \mathbb{H}^A$, $|\xi\rangle \in \mathbb{H}^B$,

$$\langle \eta \otimes \xi | M_A \otimes M_B \varphi \otimes \psi \rangle^{\mathbb{H}^A \otimes \mathbb{H}^B} = \langle \eta | M_A \varphi \rangle^{\mathbb{H}^A} \langle \xi | M_B \psi \rangle^{\mathbb{H}^B} =$$
$$= \langle M_A \eta | \varphi \rangle^{\mathbb{H}^A} \langle M_B \xi | \psi \rangle^{\mathbb{H}^B} = \langle (M_A \eta) \otimes (M_B \xi) | \varphi \otimes \psi \rangle^{\mathbb{H}^A \otimes \mathbb{H}^B} =$$
$$= \langle M_A \otimes M_B \, \eta \otimes \xi | \varphi \otimes \psi \rangle^{\mathbb{H}^A \otimes \mathbb{H}^B}.$$

$M_A \otimes M_B$ is represented by a matrix in $\mathbb{C}^{(m_A m_B) \times (m_A m_B)}$, the Kronecker product between the matrices $(\underbrace{\langle e_i | M_A e_k \rangle}_{M^A_{ik}})_{i,k=1,\dots,m_A}$ and $(\underbrace{\langle f_j | M_B f_l \rangle}_{M^B_{jl}})_{j,l=1,\dots,m_B}$, that is

$$\left(\begin{array}{c|c|c|c} M^A_{11} M^B & M^A_{12} M^B & \cdots & M^A_{1m_A} M^B \\ \hline M^A_{21} M^B & M^A_{22} M^B & \cdots & M^A_{2m_A} M^B \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M^A_{m_A 1} M^B & M^A_{m_A 2} M^B & \cdots & M^A_{m_A m_A} M^B \end{array}\right).$$

Ex.: $\sigma_x \otimes \sigma_x \rightsquigarrow \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$;

$\sigma_x \otimes \sigma_z \rightsquigarrow \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$

In the example we can check that the two observables $\sigma_x \otimes \sigma_x$ and $\sigma_z \otimes \sigma_z$ commute:

$$(\sigma_x \otimes \sigma_x)(\sigma_z \otimes \sigma_z) |\varphi\rangle \otimes |\psi\rangle = |(\sigma_x \sigma_z \varphi) \otimes (\sigma_x \sigma_z \psi)\rangle,$$
$$(\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x) |\varphi\rangle \otimes |\psi\rangle = |(\sigma_z \sigma_x \varphi) \otimes (\sigma_z \sigma_x \psi)\rangle,$$
$$\sigma_x \sigma_z = -i \sigma_y, \quad \sigma_z \sigma_x = i \sigma_y.$$

Bell's basis consists of eigenvectors both for $\sigma_x \otimes \sigma_x$ and $\sigma_z \otimes \sigma_z$.

By the postulates, the expectation of $M_A \otimes M_B$ on the (mixed) state $\rho \in D(\mathbb{H}^A \otimes \mathbb{H}^B)$ is $\mathrm{Tr}((M_A \otimes M_B)\rho)$. If we represent ρ with density matrix w.r.t. ONB $\{|e_i \otimes f_j\rangle\}_{i=1,\dots,m_A;\ j=1,\dots,m_B}$

$$\mathrm{Tr}((M_A \otimes M_B)\rho) = \sum_{\substack{i=1,\dots,m_A,\\ j=1,\dots,m_B}} \langle e_i \otimes f_j | (M_A \otimes M_B) \rho \, e_i \otimes f_j \rangle =$$
$$= \sum_{i,j} \langle M_A e_i \otimes M_B f_j | \sum_{k,l} | e_k \otimes f_l \rangle \rho_{(k,l)(i,j)} \rangle =$$
$$= \sum_{i,j,k,l} \langle M_A e_i \otimes M_B f_j | e_k \otimes f_l \rangle \rho_{(k,l)(i,j)} =$$
$$= \sum_{i,j,k,l} \langle M_A e_i | e_k \rangle \langle M_B f_j | f_l \rangle \rho_{(k,l)(i,j)}.$$

If $M_B = \mathbb{1}$ we get $\mathrm{Tr}((M_A \otimes \mathbb{1}_B)\rho) =$

$$= \sum_{i,k=1}^{m_A} \langle M_A e_i | e_k \rangle \underbrace{\left(\sum_{j=1}^{m_B} \rho_{(k,j)(i,j)} \right)}_{\rho^A_{ik}}.$$

Notice that $\mathrm{Tr}(\rho) = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \rho_{(i,j)(i,j)}$.

Instead we define $(\mathrm{Tr}^B(\rho))_{i,k} = \sum_{j=1}^{m_B} \rho_{(i,j)(k,j)}$,

$$(\mathrm{Tr}^A(\rho))_{j,l} = \sum_{i=1}^{m_A} \rho_{(i,j)(i,l)}.$$

Def.: let $M : \mathbb{H}^A \otimes \mathbb{H}^B \to \mathbb{H}^A \otimes \mathbb{H}^B$ be linear; then $\mathrm{Tr}^B(M) : \mathbb{H}^A \to \mathbb{H}^A$ is the (*) only linear operator $L^A : \mathbb{H}^A \to \mathbb{H}^A$ s.t. $\mathrm{Tr}(K L^A) = \mathrm{Tr}((K \otimes \mathbb{1}_B) M)$ $\forall K : \mathbb{H}^A \to \mathbb{H}^A$.

Similar for $\mathrm{Tr}^A(M)$.

Explicit formula: pick any ONB $\{|e_i \otimes f_j\rangle\}$. Then

$$\mathrm{Tr}^B(M) = \sum_{i,k=1}^{m_A} |e_i\rangle\langle e_k| \left(\sum_{j=1}^{m_B} \langle e_i \otimes f_j | M \, e_k \otimes f_j \rangle \right).$$

Similar for $\mathrm{Tr}^A(M)$.

(*) If L^A and $\tilde{L}^A$ are like that, then $\forall K : \mathbb{H}^A \to \mathbb{H}^A$ $\mathrm{Tr}(K(L^A - \tilde{L}^A)) = 0$, pick $K = (L^A - \tilde{L}^A)^*$.

Properties of partial trace Tr^A (exc.):

1) linearity ok;

1') more generally, $\mathrm{Tr}^A((\mathbb{1}_A \otimes K) M) = K \, \mathrm{Tr}^A(M)$, $K : \mathbb{H}^B \to \mathbb{H}^B$;

2) if M is self-adj. and non-negative, then $\mathrm{Tr}^A(M)$ is also " " " ;

3) $\mathrm{Tr}(\mathrm{Tr}^A(M)) = \mathrm{Tr}(M)$.

In particular, if $\rho \in D(\mathbb{H}^A \otimes \mathbb{H}^B)$, $\mathrm{Tr}^A(\rho) = \rho^B \in D(\mathbb{H}^B)$, $\mathrm{Tr}^B(\rho) = \rho^A \in D(\mathbb{H}^A)$ are <u>reduced</u> density operators (from ρ).

Exc.: 1) if $M : \mathbb{H}^A \otimes \mathbb{H}^B \otimes \mathbb{H}^C \to \mathbb{H}^A \otimes \mathbb{H}^B \otimes \mathbb{H}^C$, then $\mathrm{Tr}^{AB}(M) = \mathrm{Tr}^B(\mathrm{Tr}^A(M))$;

2) if $U_B : \mathbb{H}^B \to \mathbb{H}^B$ is unitary, then $\mathrm{Tr}^B(M) = \mathrm{Tr}^B((\mathbb{1}_A \otimes U_B) M (\mathbb{1}_A \otimes U_B^*))$;

3) $U : \mathbb{H}^A \otimes \mathbb{H}^B \to \mathbb{H}^A \otimes \mathbb{H}^B$ unitary $\Rightarrow \mathrm{Tr}^A(U)$ unitary?

4) Compute $\mathrm{Tr}^B(|\Phi^+\rangle\langle\Phi^+|)$.