Entanglement

$\mathbb{H}^A$ system A
$\mathbb{H}^B$ system B $\longrightarrow$ composite system $\mathbb{H}^A \otimes \mathbb{H}^B$

$|\varphi\rangle \in \mathbb{H}^A$, $|\psi\rangle \in \mathbb{H}^B$ states, then $|\varphi\rangle \otimes |\psi\rangle \in \mathbb{H}^A \otimes \mathbb{H}^B$ state.

Is the converse true? No.

Ex.: Bell states in a two qubit systems. $|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$,

PA $|\Phi^+\rangle = |\varphi\rangle \otimes |\psi\rangle$, $|\varphi\rangle = \varphi_0|0\rangle + \varphi_1|1\rangle$, $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle \Rightarrow$

$\Rightarrow |\varphi\rangle \otimes |\psi\rangle = \varphi_0\psi_0|00\rangle + \varphi_0\psi_1|01\rangle + \varphi_1\psi_0|10\rangle + \varphi_1\psi_1|11\rangle \Rightarrow$

$\Rightarrow \varphi_0\psi_1 = 0 \Rightarrow \varphi_0\psi_0 = 0 \circ \varphi_1\psi_1 = 0$, contradiction.

Def.: let $|\Psi\rangle \in \mathbb{H}^A \otimes \mathbb{H}^B$. If $\exists\, |\varphi\rangle \in \mathbb{H}^A$, $|\psi\rangle \in \mathbb{H}^B$ s.t. $|\Psi\rangle = |\varphi\rangle \otimes |\psi\rangle$, then $|\Psi\rangle$ is <u>separable</u>, otherwise it is <u>entangled</u>.

Remark: if $|\Psi\rangle = |\varphi\rangle \otimes |\psi\rangle$ then $\rho_\Psi = |\Psi\rangle\langle\Psi| = (|\varphi\rangle\langle\varphi|) \otimes (|\psi\rangle\langle\psi|)$.

Def.: let $\rho \in D(\mathbb{H}^A \otimes \mathbb{H}^B)$ (mixed state on composite system). ρ is called <u>separable</u> if $\rho = \sum_{j \in I} p_j \rho_j^{(A)} \otimes \rho_j^{(B)}$, $I \subseteq \mathbb{N}$, $\sum_j p_j = 1$, $p_j \geq 0$, with $\rho_j^{(A)} \in D(\mathbb{H}^A)$, $\rho_j^{(B)} \in D(\mathbb{H}^B)$; otherwise it is <u>entangled</u>.

Theorem: the two definition are consistent.

Proof: see the book. $\square$

Ex. (on two qubit sistems):
- Bell states are entangled;
- $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ is separable.

Theorem: let $|\Psi\rangle$ be a pure state on composite system $\mathbb{H}^A \otimes \mathbb{H}^B$. Then $|\Psi\rangle$ is separable $\Leftrightarrow$ ($\rho^B(\rho) = tr^A(\rho)$, $\rho^A(\rho) = tr^B(\rho)$) $\rho^A(|\Psi\rangle)$ and $\rho^B(|\Psi\rangle)$ are pure states.

Proof: see the book, but maybe we'll do it later. $\square$

Ex.: $tr^B(|\Phi^+\rangle\langle\Phi^+|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.

Entanglement swapping

$\longrightarrow$ A, B, C, D qubits

Our composite system is $\mathbb{H}^{ABCD} = \mathbb{H}^A \otimes \mathbb{H}^B \otimes \mathbb{H}^C \otimes \mathbb{H}^D$.

Prepare state $|\Phi\rangle \in \mathbb{H}^{ABCD}$ as $|\Phi\rangle := |\Psi^-\rangle^{AB} \otimes |\Psi^-\rangle^{CD}$.

Exc.: show that $|\Phi\rangle = \frac{1}{2}\Big(|0101\rangle - |1001\rangle - |0110\rangle + |1010\rangle\Big) =$

$= \frac{1}{2}\Big(|\Psi^+\rangle^{AD} \otimes |\Psi^+\rangle^{BC} - |\Psi^-\rangle^{AD} \otimes |\Psi^-\rangle^{BC} + - |\Phi^+\rangle^{AD} \otimes |\Phi^+\rangle^{BC} + |\Phi^-\rangle^{AD} \otimes |\Phi^-\rangle^{BC}\Big)$.

Define observables $\Sigma_z = \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1}$, $\Sigma_x = \mathbb{1} \otimes \sigma_x \otimes \sigma_x \otimes \mathbb{1}$.

These commute because $\sigma_z \otimes \sigma_z$ and $\sigma_x \otimes \sigma_x$ commute.

Recall that measuring $\sigma_x \otimes \sigma_x$ and $\sigma_z \otimes \sigma_z$ allow us to prepare the four Bell states.

Let's measure Σ_x, Σ_z on $\mathbb{H}^B \otimes \mathbb{H}^C$:

Σ_z^{BC}	Σ_x^{BC}	composite system	$\mathbb{H}^A \otimes \mathbb{H}^D$
+1	+1	$\lvert\Phi^+\rangle^{AD} \otimes \lvert\Phi^+\rangle^{BC}$	$\lvert\Phi^+\rangle^{AD}$
+1	−1	$\lvert\Phi^-\rangle^{AD} \otimes \lvert\Phi^-\rangle^{BC}$	$\lvert\Phi^-\rangle^{AD}$
−1	+1	$\lvert\Psi^+\rangle^{AD} \otimes \lvert\Psi^+\rangle^{BC}$	$\lvert\Psi^+\rangle^{BC}$
−1	−1	$\lvert\Psi^-\rangle^{AD} \otimes \lvert\Psi^-\rangle^{BC}$	$\lvert\Psi^-\rangle^{BC}$

Quantum copier

A quantum copier is an operator $K: \mathbb{H} \otimes \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ s.t. for a fixed state $|w\rangle \in \mathbb{H}$ $|\psi\rangle \otimes |w\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$ $\forall |\psi\rangle \in \mathbb{H}$.

Theorem (no cloning): there is no quantum copier.

Proof: on a two qubit system $\mathbb{H} \cong \mathbb{C}^2$. Fix $|w\rangle \in \mathbb{H}$ and suppose we have a q-copier K.

$K(|0\rangle \otimes |w\rangle) = |00\rangle$, $K(|1\rangle \otimes |w\rangle) = |11\rangle$.

$$K\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |w\rangle\right) = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\overset{\Vert}{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)} \quad \neq \text{ contradiction. } \square$$

EPR states and Bell telephone

$$\underset{\text{Alice's qubit}}{\mathbb{H}^A} \otimes \underset{\text{Bob's qubit}}{\mathbb{H}^B} \ni |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow_{\hat z}\rangle \otimes |\uparrow_{\hat z}\rangle + |\downarrow_{\hat z}\rangle \otimes |\downarrow_{\hat z}\rangle) \overset{\text{exc.}}{=}$$

$$= \frac{1}{\sqrt{2}}(|\uparrow_{\hat x}\rangle \otimes |\uparrow_{\hat x}\rangle + |\downarrow_{\hat x}\rangle \otimes |\downarrow_{\hat x}\rangle).$$

Alice measures σ_z on her qubit and gets $1 \Rightarrow$ her qubit is in state $|\uparrow_{\hat z}\rangle$ and Bob's qubit must be in state $|\uparrow_{\hat z}\rangle$ as well.

Encode a classical bit: $0 \rightsquigarrow \sigma_z^A$, $1 \rightsquigarrow \sigma_x^A$. But Bob also has to measure, so he needs many measurement (try to see why).