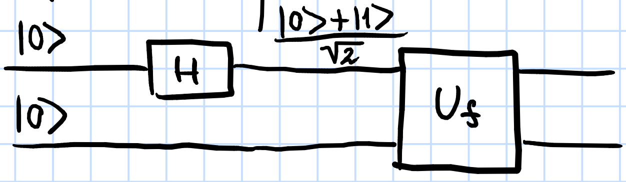
Deutsch's problem

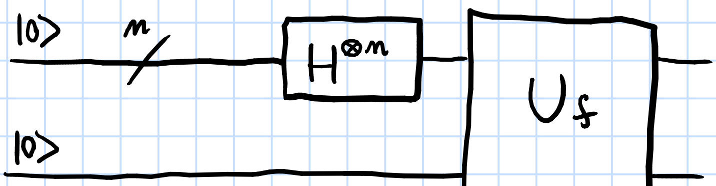
$f:\{0,1\}^n \to \{0,1\}$ either constant or balanced ($\#f^{-1}(0)=\#f^{-1}(1)$).
We can perform evaluations of f.
Determine (with certainty) wheter f is constant or balanced using the smallest possible number of evaluation.
Classical answer: $2^{n-1}+1$.
Quantum answer: 1 (Deutsch-Jozsa algorithm, ~1992).

Quantum parallelism: $f:\{0,1\}\to\{0,1\}$



$\frac{|0,f(0)\rangle+|1,f(1)\rangle}{\sqrt{2}}$.

$f:\{0,1\}^n \to \{0,1\}$

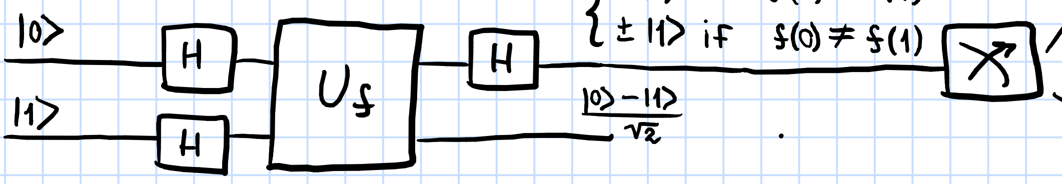


$\frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}|x\rangle\otimes|f(x)\rangle$.

Exc.: prove that if $x\in\{0,1\}^n$, $H^{\otimes n}|x\rangle=\frac{1}{2^{n/2}}\sum_{y\in\{0,1\}^n}(-1)^{\overbrace{x\odot y}^{\sum_{j=0}^{n-1}x_j\cdot y_j}}|y\rangle$.
Hint: induction.

Deutsch-Jozsa algorithm

Case $n=1$:

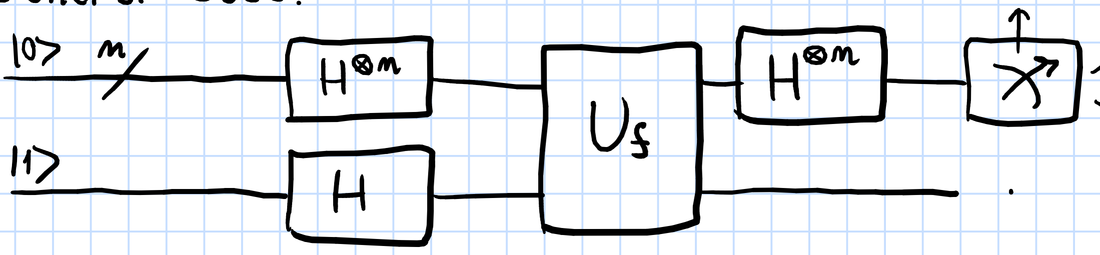


$\begin{cases}\pm|0\rangle & \text{if } f(0)=f(1)\\ \pm|1\rangle & \text{if } f(0)\neq f(1)\end{cases}$

$\frac{|0\rangle-|1\rangle}{\sqrt{2}}$.

0 f constant
1 f balanced

General case:



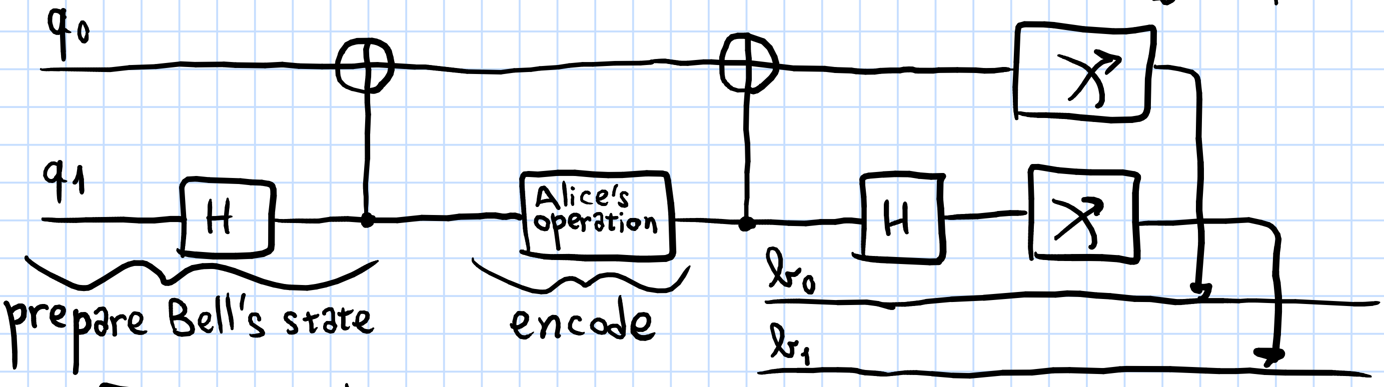
measure $|0\rangle^n$ or not

1 if f constant
0 if f balanced

(Super)dense coding

Two qubit system in state $\Phi^+=\frac{|00\rangle+|11\rangle}{\sqrt{2}}$.
Give the first qubit to Alice and the second to Bob.
Alice performs an operation on her qubit (encoding 2 classical bits) and sends it to Bob.

Classical bits	Alice's operation	Global state	After CNOT	After H
0 0	Id	$\frac{\vert 00\rangle+\vert 11\rangle}{\sqrt{2}}$	$\frac{\vert 00\rangle+\vert 10\rangle}{\sqrt{2}}$	$\vert 00\rangle$
0 1	Z	$\frac{\vert 00\rangle-\vert 11\rangle}{\sqrt{2}}$	$\frac{\vert 00\rangle-\vert 10\rangle}{\sqrt{2}}$	$\vert 10\rangle$
1 0	X	$\frac{\vert 10\rangle+\vert 01\rangle}{\sqrt{2}}$	$\frac{\vert 11\rangle+\vert 01\rangle}{\sqrt{2}}$	$\vert 01\rangle$
1 1	ZX	$\frac{-\vert 10\rangle+\vert 01\rangle}{\sqrt{2}}$	$\frac{-\vert 11\rangle+\vert 01\rangle}{\sqrt{2}}$	$\vert 11\rangle$



Exc.: what is the reduced density matrix associated to Alice's qubit?

Teleportation

Alice has a qubit in state $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$.
We also assume that Alice and Bob have each one qubit of an entangled pair (say, in state Φ^+).