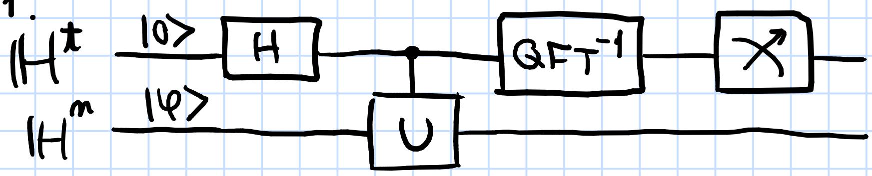
Quantum phase estimation

Problem: assume that we have an eigenstate $|\varphi\rangle \in \mathbb{H}^n$ for a unitary U, i.e. $U|\varphi\rangle = e^{2\pi i\theta}|\varphi\rangle$; can we estimate θ using a quantum circuit?

The general solution requires to implement a controlled U and apply it several times to t additional qubits in order to approximate $\theta \in [0,1]$ up to t binary digits, i.e. we find with high probability $\lfloor 2^t \theta \rfloor$.

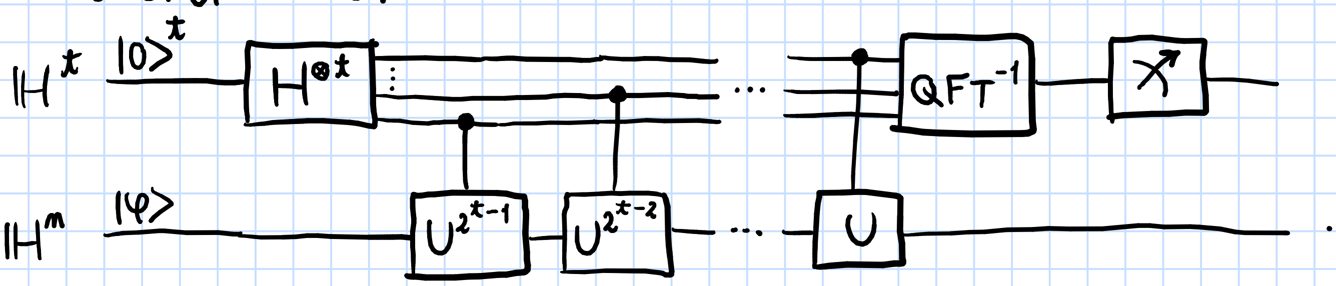
Ex.: $t=1$.



Notice that $t=1 \Rightarrow QFT^{-1}=H$.

$P(0)=\cos^2(\pi\theta)$, $P(1)=\sin^2(\pi\theta)$. $P(0)>P(1) \iff$
$\iff \theta \in [0, 1/4) \cup (3/4, 1]$.

General case:



$$P(|x\rangle^t) = \frac{1}{2^{2t}}\left|\sum_{\xi=0}^{2^t-1} e^{2\pi i \frac{\xi}{2^t}(2^t\theta - x)}\right|^2, \ \gtrsim c \text{ if } \left|\frac{x}{2^t} - \theta\right| \leq \frac{1}{2\cdot 2^t}.$$

Consider $|\varphi\rangle = \frac{1}{\sqrt{r}}\sum_{k=0}^{r-1}|f(k)\rangle \in \mathbb{H}^K$ (notation of Shor's alg.),

then if $U|f(k)\rangle = |f(k+1)\rangle$, $U|\varphi\rangle = |\varphi\rangle$
$(f(n)=b^n: \ U|x\rangle = |x\cdot b\rangle)$.

Pick $s \in \{0,1,\dots,r-1\}$ and consider

$$|\varphi_s\rangle = \frac{1}{\sqrt{r}}\sum_{k=0}^{r-1} e^{-2\pi i \frac{sk}{r}}|f(k)\rangle, \ U|\varphi_s\rangle = e^{2\pi i s/r}|\varphi_s\rangle,$$

estimate $\theta = s/r \rightsquigarrow$ find $r = s/\theta$.

$\frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}|\varphi_s\rangle = |f(0)\rangle$. We can use it to find r, but there are some things to keep in mind.