Remarks on QPE:

$|\varphi\rangle$ eigenvector of U unitary, $U|\varphi\rangle = e^{2\pi i\theta}|\varphi\rangle$, remember the QPE algorithm.

Question: what if $|\varphi\rangle$ is unknown?

Ex.: if $|\varphi\rangle = \alpha_0|\varphi_0\rangle + \alpha_1|\varphi_1\rangle$ with $|\alpha_0|^2+|\alpha_1|^2=1$ and $U|\varphi_0\rangle = e^{2\pi i\theta_0}|\varphi_0\rangle$, $U|\varphi_1\rangle = e^{2\pi i\theta_1}|\varphi_1\rangle$, $\langle\varphi_0|\varphi_1\rangle = 0$.

When we apply the algorithm we get

$$\frac{1}{2^t}\sum_{l=0}^{2^t-1}\sum_{k=0}^{2^t-1}\left(e^{2\pi i(\theta_0-\frac{l}{2^t})k}\alpha_0|l\rangle\otimes|\varphi_0\rangle + e^{2\pi i(\theta_1-\frac{l}{2^t})k}\alpha_1|l\rangle\otimes|\varphi_1\rangle\right).$$

When we measure I/O register, we obtain $l\in\{0,1,\dots,2^t-1\}$ with probability $P(l) = \left\|\frac{1}{2^t}\sum_{k=0}^{2^t-1}\left(e^{2\pi i(\theta_0-\frac{l}{2^t})k}\alpha_0|\varphi_0\rangle + e^{2\pi i(\theta_1-\frac{l}{2^t})k}\alpha_1|\varphi_1\rangle\right)\right\|^2 =$

$$= \frac{1}{2^{2t}}|\alpha_0|^2\left|\sum_{k=0}^{2^t-1}e^{2\pi i(\theta_0-\frac{l}{2^t})k}\right|^2 + \frac{1}{2^{2t}}|\alpha_1|^2\left|\sum_{k=0}^{2^t-1}e^{2\pi i(\theta_1-\frac{l}{2^t})k}\right|^2.$$

This situation arises in Quantum Counting (when we want to find m in Grover's algorithm).

Harrow-Hassidim-Lloyd (HHL) algorithm (2008)

Solving linear systems of equations using quantum computers

Problem: given $A\in\mathbb{C}^{N\times N}$, $b\in\mathbb{C}^N$, find $x\in\mathbb{C}^N$ s.t. $Ax=b$.

Gaussian elimination: $O(N^3)$.

Recall the conditioning number of A is $k(A)=\|A\|\cdot\|A^{-1}\|$; if A is positive and hermitian, $k(A)=\frac{\lambda_{max}}{\lambda_{min}}$.

A is s-sparse if every row of A contains at most s elements $\neq 0$. To solve the problem in this case we can use the conjugate gradient algorithm with $O(Nsk(A)\log(1/\varepsilon))$.

The HHL algorithm, on A positive s-sparse, takes only $O((\log N)s^2k(A)^2/\varepsilon)$ quantum steps.

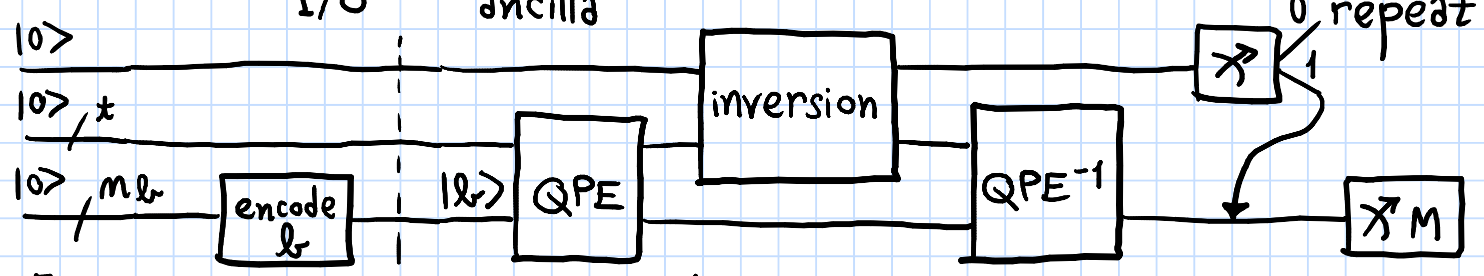
Actually we get a state $\frac{|x\rangle}{\|x\|}$ so we can measure for any observable $M\in\mathbb{C}^{N\times N}$ the quantity $(M)_{\frac{|x\rangle}{\|x\|}} = \frac{\langle x|Mx\rangle}{\|x\|^2}$.

If A is not s-sparse (Wossing, 2018): $O(\sqrt{N}\log N\, k(A)^2)$.

Two pillars of the HHL algorithm:

1) we use QPE on $U=e^{2\pi iA}$, so we need an efficient implementation of controlled U (hamiltonian simulation);
2) there is a "branch selection" step, we measure the ancilla and according to the outcome (1 or 0) we perform further quantum operations on I/O register.

Register $\underbrace{\mathbb{H}^{m_b}}_{I/O}\otimes\underbrace{\mathbb{H}^t\otimes\mathbb{H}}_{ancilla}$ $\to$ we measure this



Inversion gate acts on $\mathbb{H}^t\otimes\mathbb{H}$ on the basis:

$$|l\rangle\otimes|0\rangle \mapsto |l\rangle\otimes\left(f(l)|1\rangle + \sqrt{1-f(l)^2}\,|0\rangle\right),$$

$$|l\rangle\otimes|1\rangle \mapsto |l\rangle\otimes\left(-\sqrt{1-f(l)^2}\,|1\rangle + f(l)|0\rangle\right),$$

where $f(l) = \begin{cases}\frac{\lambda_{min}}{l/2^t} = \frac{c}{l} & \text{if } l\geq c\\ 0 & \text{otherwise}\end{cases}$.

We assume $N=2^{m_b}$ and we have to normalize b ($\|b\|=1$).

Now we do the calculations of the algorithm. To simplify, we do the case where all eigenvalues of A are of the form $\frac{l}{2^t}$.

When we measure the ancilla qubit, $P(1)=\sum_{k=0}^{N-1}|b_k|^2\left(\frac{\lambda_{min}}{\lambda_k}\right)^2$ where λ_k are the eigenvalues of A and b_k are the coefficients of b in the base of eigenstates of A, so $P(1)\geq \cancel{\|b\|^2}(k(A))^{-2}$.

After measuring, I/O register is in state $|x\rangle/\|x\|$.