Quantum walks on finite graphs

$G=(V,E)$ undirected, connected, no multiple edges, no loops.
$\updownarrow$
A associated adjacency matrix

We know what is a classical random walk on G.
We will work with <u>d-regular</u> graphs, i.e. $\deg(v)=d \ \forall v \in V$.
Ex.: cycles, hypercubes.
Note that A is always sym, $\underset{\substack{\text{transition matrix} \\ \text{with uniform distribution}}}{\underset{\downarrow}{P}}$ is sym for d-reg. graphs.

Quantum version? Main formalism:
- coined quantum walk (well suited for d-reg. graphs);
- Szegedy qw (more general).

Everything so far concerns discrete-time walks.
Continuous-time qw: choose a hamiltonian for the graph (e.g. laplacian or adjacency matrix), model evolution of qw through Schrödinger's eq. $\frac{d}{dt}|\psi(t)\rangle = -iH|\psi(t)\rangle$, H independent of time, $|\psi(t)\rangle = \underbrace{e^{-iHt}}_{\text{unitary evolution operator}}|\psi(0)\rangle$.

Coined qw

$G=(V,E)$ with all the hypothesis, $|V|=N=2^n$.
Hilbert space is $\mathbb{H} = \underset{\substack{\downarrow \\ \text{coin space}}}{\mathbb{H}_C} \otimes \underset{\substack{\downarrow \\ \text{position space}}}{\mathbb{H}_P}$. A state in $\mathbb{H}$ is
$|k,p\rangle = |k\rangle \otimes |p\rangle$, $0 \le p \le N-1$.

For a cycle graph: $\mathbb{H} = \underset{\substack{\downarrow \\ 1 \text{ qubit}}}{\mathbb{H}_C} \otimes \underset{\substack{\downarrow \\ n \text{ qubit}}}{\mathbb{H}_P}$. Coin operator: typical choice is Hadamard.

$C: \mathbb{H} \longrightarrow \mathbb{H}$, $|k,p\rangle \mapsto |Hk,p\rangle$ models a balanced coin.

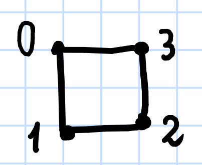
Shift operator: $S: \mathbb{H} \longrightarrow \mathbb{H}$, $|k,p\rangle \mapsto |k, \underbrace{p+(-1)^k}_{\text{mod } N}\rangle$

(some authors: $|k,p\rangle \mapsto |k \overset{\text{mod } 2}{\oplus} 1, p+(-1)^k\rangle$).

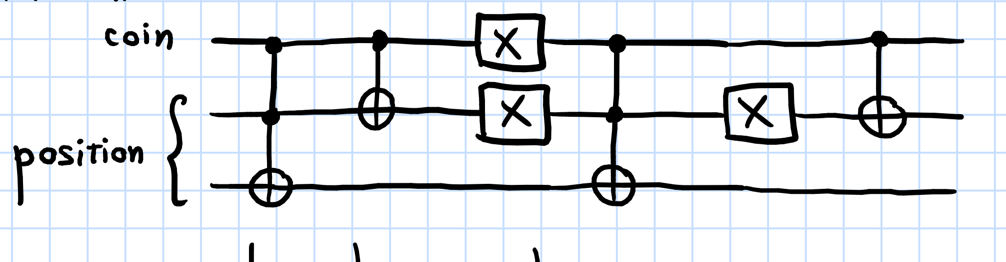
Walk operator is $W=SC$. Each application of W is a qw step ("flip-flop" qw).

Qw algorithm:
1. prepare initial state;
2. repeat T times:
   - apply C;
   - apply S;
3. measure. $\rightarrow$ this is where "randomness" is

Remark: because of unitary evolution, in the quantum case there is no limit ("stationary") distribution. There is instead notion of "limiting" distribution (limit of partial times averages).

Ex.: 4-cycle.


$\mathbb{H} = \underset{\substack{\downarrow \\ 1 \text{ qubit}}}{\mathbb{H}_C} \otimes \underset{\substack{\downarrow \\ 2 \text{ qubit}}}{\mathbb{H}_P}$.

$C = \mathbb{H} \otimes \mathbb{1}$.
S:


Qw on an n-dim. hypercube

Nodes labeled by binary strings of length n. Neighbors differ exactly by one digit.
Ex.: $n=4$, $N=2^4$. Computational basis: $\{|a,v\rangle \mid 0 \le a \le n-1, v \in \{0,1\}^n\}$.
Meaning of coin: if $a=j$, $(j+1)$-th digit changes.
Shift operator: $S|a,v\rangle = |a, v \boxplus e_a\rangle$, $e_a = (0,\dots,0,\underset{\substack{\uparrow \\ (a+1)\text{-th position}}}{1},0,\dots,0)$.
Coin operator is Grover: $G = \frac{2}{n}uu^T - \mathbb{1}$,
$u = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$. Walk operator is $W = SG$.

Remark: this is usually done using Szegedy walks, but here this is equivalent to a coined Grover walk.

We want to use a qw to implement a search algorithm on the hypercube. M = set of marked vertices on hypercube, $m = |M|$. Start qw from suitable state (node or superposition) and perform qw steps $\underset{\substack{\downarrow \\ \text{we can estimate a priori how many steps}}}{\underline{\text{until}}}$ marked vertex is met.

Basis states have two registers: current node; previous node. $\}$ edges

$$|G\rangle = \frac{1}{\sqrt{m}} \sum_{x \in M} |x\rangle |p_x\rangle, \quad |p_x\rangle = \sum_y \sqrt{P_{xy}}\,|y\rangle.$$

$\downarrow$ superposition of "good" states where node in the first register is marked

$|p_x\rangle$: $\downarrow$ uniform superposition of neighbors of x; P_{xy}: $\downarrow$ transition matrix

$$|B\rangle = \frac{1}{\sqrt{N-m}} \sum_{x \notin M} |x\rangle |p_x\rangle.$$

$\varepsilon = \frac{m}{N}$, $\theta = \arcsin\sqrt{\varepsilon}$.

Algorithm:
1) prepare initial state (uniform superposition of all edges):
$|U\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle|p_x\rangle = \sin\theta|G\rangle + \cos\theta|B\rangle$;
2) repeat $O(1/\sqrt{\varepsilon})$ times (compare with Grover):
   2a) apply reflection wrt $|B\rangle$;
   2b) " " " $|U\rangle$; $\}$ rotation in the $|B\rangle$-$|G\rangle$ plane of angle 2θ towards $|G\rangle$

3) Measure.