Def.: $0 \to Q_1 \to Q_2 \to G \to 0$ risol. libera,

$\mathrm{Tor}'(A,G) = \mathrm{Ker}(A \otimes Q_1 \to A \otimes Q_2)$.

Oss.: non dipende dalla risoluzione scelta.

Prop.: $\mathrm{Tor}'(A,G) = \mathrm{Tor}(A,G)$.

Dim.: $0 \to P_1 \to P_2 \to A \to 0$ libere

$0 \to Q_1 \to Q_2 \to G \to 0$

Q_1, Q_2 moduli liberi

P_1, P_2 moduli liberi

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \mathrm{Tor}(A,G)=\mathrm{Ker}(\gamma) \arrow[d] & \\
0 \arrow[r] & P_1 \otimes Q_1 \arrow[r] \arrow[d, "\alpha"] & P_1 \otimes Q_2 \arrow[r] \arrow[d, "\beta"] & P_1 \otimes G \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & P_2 \otimes Q_1 \arrow[r] \arrow[d] & P_2 \otimes Q_2 \arrow[r] \arrow[d] & P_2 \otimes G \arrow[r] \arrow[d] & 0 \\
\mathrm{Tor}'(A,G) \arrow[r] & A \otimes Q_1 \arrow[r] \arrow[d] & A \otimes Q_2 \arrow[r] \arrow[d] & A \otimes G \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Guardando le prime due righe come succ. esatta corta di complessi, ho succ. esatta lunga (omologia)

$$\underset{\displaystyle 0}{\mathrm{Ker}\,\alpha} \to \underset{\displaystyle 0}{\mathrm{Ker}\,\beta} \to \underset{\displaystyle \mathrm{Tor}(A,G)}{\mathrm{Ker}\,\gamma} \to \mathrm{coker}\,\alpha \to \mathrm{coker}\,\beta \to \mathrm{coker}\,\gamma \Rightarrow$$

$\Rightarrow \mathrm{Tor}(A,G) = \mathrm{Ker}(\mathrm{coker}\,\alpha \to \mathrm{coker}\,\beta) = \mathrm{Tor}'(A,G)$. $\square$

Def.: $\mathrm{Tor}(A,B) = A * B = B * A$.

Def.: $0 \to K(A) \xrightarrow{i} F(A) \to A \to 0$, applichiamo $\mathrm{Hom}(\cdot, G)$:

$0 \to \mathrm{Hom}(A,G) \to \mathrm{Hom}(F(A),G) \xrightarrow{i^*} \mathrm{Hom}(K(A),G)$,

$\mathrm{Ext}(A,G) := \mathrm{coker}(i^*)$.

Oss.: (come per Tor) la def. di Ext non dipende dalla risol. libera scelta.

Prop. (ex.): A, B $\mathbb{Z}$-moduli,

(1) A libero $\Rightarrow \mathrm{Ext}(A,B) = 0$

(2) $\mathrm{Ext}(\mathbb{Z}/n, B) = B/nB$

(3) $\mathrm{Ext}(\mathbb{Z}/n, B) = 0$ se $B = \mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}$

(4) $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/?$

(5) $\mathrm{Ext}(A_1 \oplus A_2, B) = \mathrm{Ext}(A_1, B) \oplus \mathrm{Ext}(A_2, B)$.

Prop. (teorema dei coefficienti universali per l'omologia): sia $(C_\cdot, \partial)$ un complesso di catene di moduli liberi. Allora abbiamo la succ. esatta

$0 \to H_q(C) \otimes G \xrightarrow{\alpha} H_q(C \otimes G) \xrightarrow{\beta} \mathrm{Tor}(H_{q-1}(C), G) \to 0$

naturale, spezza e $\alpha: [z] \otimes y \mapsto [z \otimes y]$.

Dim.: $0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0$ è esatta, B_{n-1} è libero (perché sottomodulo di libero) $\Rightarrow$

$\Rightarrow 0 \to Z_n \otimes G \to C_n \otimes G \to B_{n-1} \otimes G \to 0$ è esatta e spezza (perché anche quella prima spezzava). Continuo il diagramma:

$$\begin{tikzcd}
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d, "\partial \otimes id = 0"] & C_n \otimes G \arrow[r] \arrow[d, "\partial \otimes id"] & B_{n-1} \otimes G \arrow[r] \arrow[d, "\partial \otimes id = 0"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] \arrow[d] & C_{n-1} \otimes G \arrow[r] \arrow[d] & B_{n-2} \otimes G \arrow[r] \arrow[d] & 0
\end{tikzcd}$$

in omologia ho succ. esatta lunga

$B_n \otimes G \xrightarrow{f_n} Z_n \otimes G \to H_n(C \otimes G) \to B_{n-1} \otimes G \to Z_{n-1} \otimes G$

Ex.: se $i_n : B_n \hookrightarrow Z_n$, $f_n = i_n \otimes id_G$.

$0 \to B_n \xrightarrow{i_n} Z_n \to H_n(C) \to 0$ è risol. libera $\Rightarrow$

$\Rightarrow \mathrm{Ker}(i_n \otimes id_G : B_n \otimes G \to Z_n \otimes G) = \mathrm{Tor}(H_n(C), G)$.

$B_n \otimes G \xrightarrow{i_n \otimes id_G} Z_n \otimes G \to H_n(C) \otimes G \to 0$ esatta $\Rightarrow$

$\Rightarrow \mathrm{coker}(i_n \otimes id_G : B_n \otimes G \to Z_n \otimes G) = H_n(C) \otimes G$

Ho $0 \to H_n(C) \otimes G \to H_n(C \otimes G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0$

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] \arrow[l, bend right, "r"'] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

spezza, dove $r: C_n \to Z_n$ è t.c.

$$\begin{tikzcd}
Z_n \arrow[r, "i"] \arrow[rr, bend right, "id"'] & C_n \arrow[r, "r"] & Z_n
\end{tikzcd}$$

$$\begin{tikzcd}
Z_n(C \otimes G) \subset C_n \otimes G \arrow[r, "r \otimes id_G"] \arrow[rr, bend right, "\varphi"'] & Z_n \otimes G \arrow[r] & H_n(C) \otimes G
\end{tikzcd}$$

φ manda $B_n(C \otimes G)$ in 0 e induce

$\rho: H_n(C \otimes G) \to H_n(C) \otimes G$ t.c. $\rho\alpha = id$. $\square$

Prop. (teorema dei coefficienti universali per la coomologia): sia $(C_\cdot, \partial)$ un complesso di catene di moduli liberi. Allora c'è una successione esatta

$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C, G) \xrightarrow{\alpha} \mathrm{Hom}(H_n(C), G) \to 0$

$\alpha: \varphi \mapsto \alpha(\varphi)$ $(\varphi: C_n \to G)$, $\alpha(\varphi)[c] := \varphi(c)$.

La succ. è naturale (rispetto a mappe di complessi e rispetto a omo. di moduli (G)) e spezza in modo naturale rispetto a G ma non rispetto a C.

Dim.:

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial"] \arrow[l, bend left, "r"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

da cui

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, G) \arrow[r] \arrow[d] & \mathrm{Hom}(C_n, G) \arrow[r] \arrow[d, "\delta"] & \mathrm{Hom}(Z_n, G) \arrow[r] \arrow[d] & 0 \\
 & (n) & (n+1) & (n+1) &
\end{tikzcd}$$

e ottengo succ. esatta lunga in omologia

$0 \to \mathrm{coker}\, d^{n-1} \to H^n(C, G) \to \mathrm{Ker}\, d^n \to 0$

dove $d^n: \mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G)$ indotta da $B_n \subset Z_n$ (ex.).

$0 \to B_n \to Z_n \to H_n(C) \to 0$ è esatta

$0 \to \mathrm{Hom}(H_n(C), G) \to \mathrm{Hom}(Z_n, G) \xrightarrow{d^n} \mathrm{Hom}(B_n, G) \to \mathrm{Ext}(H_n(C), G) \to 0$

$\mathrm{Ker}\, d^n = \mathrm{Hom}(H_n(C), G)$, $\mathrm{coker}\, d^n = \mathrm{Ext}(H_n(C), G)$.

$r: C_n \to Z_n$

$$\begin{tikzcd}
0 \arrow[r] & Z^n(\mathrm{Hom}(C,G)) \arrow[r] & \mathrm{Hom}(C_n, G) \arrow[r, "\delta"] & \mathrm{Hom}(C_{n+1}, G) \\
0 \arrow[r] & \mathrm{Hom}(H_n(C), G) \arrow[r] & \mathrm{Hom}(Z_n, G) \arrow[r, "i^*"] \arrow[u, "r^*"] & \mathrm{Hom}(B_n, G) \arrow[u, "d^*_{n+1}"]
\end{tikzcd}$$

Preso $\varphi \in \mathrm{Ker}\, i^*$, $r^*(\varphi) \in \mathrm{Ker}(\delta)$ e induce $\mathrm{Hom}(H_n(C), G) \to H^n(C, G)$, $\varphi \longmapsto \varphi r$

che dà lo spezzamento. $\square$

Cor.: se $H_{n-1}(C)$ è libero, $\alpha: H^n(C,G) \to \mathrm{Hom}(H_n(C), G)$ è iso..

Cor.: $(C^\cdot, \delta)$ complesso di co-catene di R-moduli liberi,

$0 \to H^q(C) \otimes G \to H^q(C \otimes G) \to \mathrm{Tor}(H^{q+1}(C), G) \to 0$.

Dim.: è una riscrittura dei coefficienti universali in omologia. $\square$

Cor.: se $(C_\cdot, \partial)$ è complesso di catene di R-moduli liberi, G R-modulo fin. gen.. Allora

$0 \to H^q(C) \otimes G \to H^q(C; G) \to \mathrm{Tor}(H^{q+1}(C), G) \to 0$.

Dim.: uso $\mathrm{Hom}(C, R) \otimes G \simeq \mathrm{Hom}(C, G)$ (iso. canonico). $\square$

Data una coppia (X, A) di s.t. ho

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r] & C_n(X) \arrow[r] & C_n(X, A) \arrow[r] \arrow[l, bend right, "s"'] & 0
\end{tikzcd}$$

(succ. esatta di moduli liberi spezza).

$0 \to C_n(A; G) \to C_n(X, G) \to C_n(X, A; G) \to 0$ è esatta $\Rightarrow$

$\Rightarrow$ ho succ. esatta lunga di omologia della coppia a coefficienti in G.

Ho anche

$0 \to C^n(X, A; G) \to C^n(X; G) \to C^n(A; G) \to 0$ esatta

e vale l'analogo di sopra in coomologia.

Da $0 \to C_n^{\mathcal{U}}(X) \to C_n(X) \to C_n(X)/C_n^{\mathcal{U}}(X) \to 0$ (esatta corta di abeliani liberi $\Rightarrow$ spezza), $\hookrightarrow$ omologia banale

applicando $\mathrm{Hom}(\cdot, G)$ o $\otimes G$ ottengo MV.

Analogo per escissione.

Es.: $H^i(S^n; G) = \begin{cases} G & i = 0, n \\ 0 & \text{altrimenti} \end{cases}$

$H^i(\mathbb{CP}^n; G) = \begin{cases} G & i \leq 2n \quad 2 \mid i \\ 0 & \text{altrimenti} \end{cases}$

$H^i(\mathbb{RP}^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \forall\, 0 \leq i \leq n \\ 0 & \text{altrimenti} \end{cases}$

Prodotto cup $\cup$

Sia A anello commutativo, voglio definire una mappa A-bilineare $H^i(X; A) \times H^j(X; A) \xrightarrow{\cup} H^{i+j}(X; A)$ che renda $H^\bullet(X; A) = \bigoplus_{n \geq 0} H^n(X; A)$ un anello graduato con prodotto graded-commutative: $\alpha \cup \beta = (-1)^{|\alpha| \cdot |\beta|} \beta \cup \alpha$.

Dati $(C_\cdot, d_C)$, $(D_\cdot, d_D)$ complessi di catene, definisco

$C \otimes D$ con $(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$ e

$d_n: (C \otimes D)_n \to (C \otimes D)_{n-1}$ dato da

$d_n(\underset{\displaystyle C_p \otimes C_q}{\alpha \otimes \beta}) = d_C(\alpha) \otimes \beta + (-1)^p \alpha \otimes d_D(\beta)$ $\quad (d^2 = 0)$.

Stessa def. per le co-catene.

Prop.: il prodotto di complessi di cui uno contraibile è contraibile.

Dim.: complessi C, D con C contr. $\Rightarrow id_C = d_C k + k d_C$, $k: C_n \to C_{n+1}$. Definisco $\bar{k}: (C \otimes D)_n \to (C \otimes D)_{n+1}$, $\alpha \otimes \beta \longmapsto k(\alpha) \otimes \beta$

Prendo $\alpha \in C_p$,

$d\bar{k}(\alpha \otimes \beta) = d(k(\alpha) \otimes \beta) = d_C k(\alpha) \otimes \beta + (-1)^{p+1} k(\alpha) \otimes d_D(\beta)$

$\bar{k} d(\alpha \otimes \beta) = k d_C(\alpha) \otimes \beta + (-1)^p k(\alpha) \otimes d_D(\beta) \Rightarrow$

$\Rightarrow (d\bar{k} + \bar{k} d)(\alpha \otimes \beta) = id_C(\alpha) \otimes \beta = \alpha \otimes \beta$. $\square$

Cor.: X contraibile $\Rightarrow \tilde{C}_\cdot(X) \otimes \tilde{C}_\cdot(X)$ contraibile.

X o Y contraibile $\Rightarrow \tilde{C}_\cdot(X) \otimes \tilde{C}_\cdot(Y)$ contraibile.