$0 \to \mathcal{F} \xrightarrow{f} \mathcal{L}_0 \xrightarrow{\varphi_0} \mathcal{L}_1 \xrightarrow{\varphi_1} \dots$ induce

$0 \to \Gamma(\mathcal{F}) \xrightarrow{f} \Gamma(\mathcal{L}_0) \xrightarrow{\varphi_0} \Gamma(\mathcal{L}_1) \xrightarrow{\varphi_1} \dots$ complesso di prefasci di g.a. su X, esatto nei primi tre termini.

$\mathcal{B}^* = \Gamma(X, \mathcal{L}_*)$ complesso di g.a.,

$$H^p(\mathcal{B}^*) = \frac{\operatorname{Ker} \varphi_{p,X}}{\operatorname{Im} \varphi_{p-1,X}} \Rightarrow H^0(\mathcal{B}^*) \cong \Gamma(X, \mathcal{F}).$$

Def.: $\mathcal{F}$ fascio di g.a. su X, $\mathcal{B}^* = \Gamma(X, \operatorname{Can}^*(\mathcal{F}))$,

$$H^p(X,\mathcal{F}) = H^p(\mathcal{B}^*) = \frac{\operatorname{Ker}(\Gamma(X, \operatorname{Can}^p(\mathcal{F})) \to \Gamma(X, \operatorname{Can}^{p+1}(\mathcal{F})))}{\operatorname{Im}(\Gamma(X, \operatorname{Can}^{p-1}(\mathcal{F})) \to \Gamma(X, \operatorname{Can}^p(\mathcal{F})))}$$

coomologia p-esima di X a valori in $\mathcal{F}$.

Oss.: $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$.

Definiamo la coomologia di Čech di X a valori in un prefascio F di g.a. su X.

$\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ ricoprimento aperto di X, $\alpha_0,\dots,\alpha_p \in I$,

$U_{\alpha_0,\dots,\alpha_p} := U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$.

Le p-COCATENE DI ČECH a valori in F subordinate a $\mathcal{U}$:

$$\check{C}^p(\mathcal{U},F) = \prod_{\alpha_0,\dots,\alpha_p\in I} F(U_{\alpha_0,\dots,\alpha_p}).$$

Oss.: $F(\emptyset) = \{0\} \Rightarrow$ bastano gli indici t.c. $U_{\alpha_0,\dots,\alpha_p} \neq \emptyset$.

Una p-cocatena è il dato di

$\sigma_{\alpha_0,\dots,\alpha_p} \in F(U_{\alpha_0,\dots,\alpha_p})$ $\forall \alpha_0,\dots,\alpha_p \in I$ t.c. $U_{\alpha_0,\dots,\alpha_p} \neq \emptyset$.

COBORDO: $d_p : \check{C}^p(\mathcal{U},F) \to \check{C}^{p+1}(\mathcal{U},F)$ omomorfismi t.c. $d_{p+1} \circ d_p = 0$.

$\sigma \in \check{C}^p(\mathcal{U},F)$, $(\sigma_{\alpha_0,\dots,\alpha_p})_{\alpha_0,\dots,\alpha_p \in I}$,

$$\underbrace{(d_p(\sigma))_{\alpha_0,\dots,\alpha_{p+1}}}_{\in F(U_{\alpha_0,\dots,\alpha_{p+1}})} = \sum_{i=0}^{p+1} (-1)^i \rho^{U_{\alpha_0,\dots,\cancel{\alpha_i},\dots,\alpha_{p+1}}}_{U_{\alpha_0,\dots,\alpha_{p+1}}} (\underbrace{\sigma_{\alpha_0,\dots,\cancel{\alpha_i},\dots,\alpha_{p+1}}}_{\in F(U_{\alpha_0,\dots,\cancel{\alpha_i},\dots,\alpha_{p+1}})}).$$

Es.: $\sigma \in \check{C}^0(\mathcal{U},F) \rightsquigarrow \sigma_\alpha \in F(U_\alpha)$,

$$(d\sigma)_{\alpha\beta} = \rho^{U_\beta}_{U_\alpha\cap U_\beta}(\sigma_\beta) - \rho^{U_\alpha}_{U_\alpha\cap U_\beta}(\sigma_\alpha) = \sigma_\beta - \sigma_\alpha.$$

$\eta \in \check{C}^1(\mathcal{U},F) \rightsquigarrow \eta_{\alpha,\beta} \in F(U_\alpha \cap U_\beta)$,

$$(d\eta)_{\alpha\beta\gamma} = \rho^{U_{\beta\gamma}}_{U_{\alpha\beta\gamma}}(\eta_{\beta\gamma}) - \rho^{U_{\alpha\gamma}}_{U_{\alpha\beta\gamma}}(\eta_{\alpha\gamma}) + \rho^{U_{\beta\gamma}}_{U_{\alpha\beta\gamma}}(\eta_{\alpha\beta}) = \eta_{\beta\gamma} - \eta_{\alpha\gamma} + \eta_{\alpha\beta}.$$

$$d^2(\sigma) = (d\sigma)_{\beta\gamma} - (d\sigma)_{\alpha\gamma} + (d\sigma)_{\alpha\beta} = \sigma_\gamma - \sigma_\beta - (\sigma_\gamma - \sigma_\alpha) + \sigma_\beta - \sigma_\alpha = 0.$$

$$\check{C}^p(\mathcal{U},F) \supset \underbrace{Z^p(\mathcal{U},F)}_{p\text{-cocicli di Čech}} = \operatorname{Ker} d_p \supset \operatorname{Im} d_{p-1} = \underbrace{B^p(\mathcal{U},F)}_{p\text{-cobordi di Čech}}$$

$$\check{H}^p(\mathcal{U},F) = Z^p(\mathcal{U},F)/B^p(\mathcal{U},F) = H^p(\check{C}^*(\mathcal{U},F))$$

coomologia di Čech di X a valori in F subordinata a $\mathcal{U}$.

Se $\mathcal{U}' = \{U'_\beta\}_{\beta\in I'}$ ricoprimento aperto di X più fine di $\mathcal{U}$, fissiamo una mappa di raffinamento $\varphi: I' \to I$, $U'_\beta \subset U_{\varphi(\beta)}$.

Costruiamo il morfismo di complessi

$$\rho^*_\varphi : \check{C}^*(\mathcal{U},F) \to \check{C}^*(\mathcal{U}',F),$$

$$\begin{tikzcd} \check{C}^p(\mathcal{U},F) \arrow[r, "\rho^p_\varphi"] \arrow[d, "d"'] \arrow[dr, phantom, "\circlearrowright"] & \check{C}^p(\mathcal{U}',F) \arrow[d, "d'"] \\ \check{C}^{p+1}(\mathcal{U},F) \arrow[r, "\rho^{p+1}_\varphi"] & \check{C}^{p+1}(\mathcal{U}',F) \end{tikzcd}, \quad d'_p \circ \rho^p_\varphi = \rho^{p+1}_\varphi \circ d_p.$$

$\sigma \in \check{C}^p(\mathcal{U},\mathcal{F})$, $\beta_0,\dots,\beta_{p+1} \in I'$,

$$(\rho^p_\varphi(\sigma))_{\beta_0,\dots,\beta_p} = \rho^{U_{\varphi(\beta_0),\dots,\varphi(\beta_p)}}_{U'_{\beta_0,\dots,\beta_p}}(\sigma_{\varphi(\beta_0),\dots,\varphi(\beta_p)}).$$

$\hookrightarrow U'_{\beta_0,\dots,\beta_p} \subset U_{\varphi(\beta_0),\dots,\varphi(\beta_p)}$

$$(d'_p \rho^p_\varphi(\sigma))_{\gamma_0,\dots,\gamma_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho^{U'_{\gamma_0,\dots,\cancel{\gamma_j},\dots,\gamma_{p+1}}}_{U'_{\gamma_0,\dots,\gamma_{p+1}}} (\rho^p_\varphi(\sigma)_{\gamma_0,\dots,\cancel{\gamma_j},\dots,\gamma_{p+1}}) =$$

$$= \sum_{j=0}^{p+1} (-1)^j \rho^{U'_{\gamma_0,\dots,\cancel{\gamma_j},\dots,\gamma_{p+1}}}_{U'_{\gamma_0,\dots,\gamma_{p+1}}} \rho^{U_{\varphi(\gamma_0),\dots,\cancel{\varphi(\gamma_j)},\dots,\varphi(\gamma_{p+1})}}_{U'_{\gamma_0,\dots,\cancel{\gamma_j},\dots,\gamma_{p+1}}} (\sigma_{\varphi(\gamma_0),\dots,\cancel{\varphi(\gamma_j)},\dots,\varphi(\gamma_{p+1})}) =$$

$$= \rho^{U_{\varphi(\gamma_0),\dots,\varphi(\gamma_{p+1})}}_{U'_{\gamma_0,\dots,\gamma_{p+1}}} \left(\sum_{j=1}^{p+1} (-1)^j \rho^{U_{\varphi(\gamma_0),\dots,\cancel{\varphi(\gamma_j)},\dots,\varphi(\gamma_{p+1})}}_{U_{\varphi(\gamma_0),\dots,\varphi(\gamma_{p+1})}} (\sigma_{\varphi(\gamma_0),\dots,\cancel{\varphi(\gamma_j)},\dots,\varphi(\gamma_{p+1})}) \right) =$$

$$= (\rho^{p+1}_\varphi d_p(\sigma))_{\gamma_0,\dots,\gamma_{p+1}}.$$

Allora ρ^*_φ manda p-cobordi in p-cobordi e p-cocicli in p-cocicli $\Rightarrow$

$\Rightarrow$ ρ^*_φ induce $\bar{\rho}^*_\varphi : \check{H}^*(\mathcal{U},F) \to \check{H}^*(\mathcal{U}',F)$ omomorfismo di g.a.

che <u>non</u> dipende da φ, e si dice $\rho^{*\mathcal{U}}_{\mathcal{U}'}$.

Se $\varphi' : I' \to I$ è un'altra mappa di raffinamento,

$$k^p : \check{C}^p(\mathcal{U},F) \to \check{C}^{p-1}(\mathcal{U}',F),$$

$$k^p(\sigma)_{\underbrace{j_0,\dots,j_{p-1}}_{\in I'}} = \sum_{k=0}^{p-1} (-1)^k \rho^{U_{\varphi(j_0),\dots,\varphi(j_k),\varphi'(j_k),\dots,\varphi'(j_{p-1})}}_{U_{j_0,\dots,j_{p-1}}} (\sigma_{\varphi(j_0),\dots,\varphi(j_k),\varphi'(j_k),\dots,\varphi'(j_{p-1})}).$$

$$d'_{p-1} k^p + k^{p+1} d_p = \rho^p_\varphi - \rho^p_{\varphi'} \Rightarrow \bar{\rho}^*_\varphi = \bar{\rho}^*_{\varphi'}.$$

Def.: $\check{H}^p(X,F) = \varinjlim_{\mathcal{U} \text{ ricoprimento aperto di } X} \check{H}^p(\mathcal{U},F)$ è la coomologia di Čech p-esima di X a valori in F.

$s \in \check{H}^p(X,F)$ è il dato di

- $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ ricoprimento aperto di X;
- $s \in \check{H}^p(\mathcal{U},F)$ (che a sua volta è un p-cociclo $\sigma = (\sigma_{\alpha_0,\dots,\alpha_p} \in F(U_{\alpha_0,\dots,\alpha_p}))_{\alpha_0,\dots,\alpha_p\in I}$ t.c. $d\sigma = 0$ e due tali cocicli li consideriamo uguali se differiscono per un cobordo).

$(\mathcal{U}, s)$, $(\mathcal{U}', s')$ li consideriamo uguali se $\exists\, \mathcal{W}$ ricoprimento aperto di X più fine di $\mathcal{U}$ e $\mathcal{U}'$ t.c. $\rho^{\mathcal{U}}_{\mathcal{W}}(s) = \rho^{\mathcal{U}'}_{\mathcal{W}}(s')$.

Def.: $\mathcal{F}$ fascio di g.a. su X, poniamo $\check{H}(X,\mathcal{F}) = \check{H}^p(X,\Gamma(\mathcal{F}))$.

Oss.: F prefascio canonico, allora $\check{H}^0(\mathcal{U},F) = Z^0(\mathcal{U},F) = F(X)$ $\forall \mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ ricoprimento aperto di $X \Rightarrow \check{H}^0(X,F) = F(X)$.

Dato $s \in F(X)$ induce una 0-cocatena $\sigma_\alpha = \rho^X_{U_\alpha}(s) \in F(U_\alpha)$ che è uno 0-cociclo: $(d\sigma)_{\alpha\beta} = \rho^{U_\beta}_{U_\alpha\cap U_\beta} \rho^X_{U_\beta}(s) - \rho^{U_\alpha}_{U_\alpha\cap U_\beta} \rho^X_{U_\alpha}(s) = \rho^X_{U_\alpha\cap U_\beta}(s) - \rho^X_{U_\alpha\cap U_\beta}(s) = 0$.

Viceversa, se $\sigma \in Z^0(\mathcal{U},F) \subset \check{C}^0(\mathcal{U},F)$, $\sigma = (\sigma_\alpha)_{\alpha\in I}$, $\sigma_\alpha \in F(U_\alpha)$.

$0 = (d\sigma)_{\alpha\beta} = \rho^{U_\beta}_{U_\alpha\cap U_\beta}(\sigma_\beta) - \rho^{U_\alpha}_{U_\alpha\cap U_\beta}(\sigma_\alpha)$, cioè ho σ_α dati locali compatibili; F canonico $\underset{(S1)\text{ e }(S2)}{\Rightarrow} \exists!\, s \in X$ t.c. $\rho^X_{U_\alpha}(s) = \sigma_\alpha$ $\forall \alpha \in I$.

Se $\mathcal{F}$ fascio di g.a. su X, $\check{H}^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$ $(= H^0(X,\mathcal{F}))$.

$Y \subset X$ chiuso, $\mathcal{F}$ fascio di g.a. su Y, $\hat{\mathcal{F}}$ l'estensione a 0 $\Rightarrow$

$\Rightarrow \check{H}^p(Y,\mathcal{F}) = \check{H}^p(X,\hat{\mathcal{F}})$.

(ogni ricoprimento aperto di X ne induce uno di Y per intersezione, e viceversa
" " " Y " " " " X aggiungendo $X \setminus Y$)

Teo.: X paracpt, F prefascio di g.a. su X,

$\forall p$ $\check{H}^p(X,F) \cong \check{H}^p(X, \operatorname{Sheaf}(F)) (= \check{H}^p(X, \Gamma(\operatorname{Sheaf}(F))))$.

Dim.: no. $\square$

Teo. (Leray): $\mathcal{U}$ ricoprimento di X aciclico per il prefascio F di g.a. su X

$(\check{H}^q(U_{\alpha_0,\dots,\alpha_p}, F|U_{\alpha_0,\dots,\alpha_p}) = 0$ $\forall q \geq 1$ $\forall \alpha_0,\dots,\alpha_p \in I) \Rightarrow$

$\Rightarrow \check{H}^p(X,F) = \check{H}^p(\mathcal{U},F)$ $\forall p$.

Dim.: no. $\square$