- È ben def. la proiezione ortogonale $\mathcal{H}: H^0(M, \mathcal{A}^{p,q}(M)) \to \mathcal{H}^{p,q}(M)$;
- $\exists\, G: H^0(M, \mathcal{A}^{p,q}(M)) \to H^0(M, \mathcal{A}^{p,q}(M))$ t.c.
  $G(\mathcal{H}^{p,q}(M)) = 0$ e inverte $\Delta_{\bar\partial}$ su $\mathcal{H}^{p,q}(M)^\perp$, $\mathcal{H} + \Delta_{\bar\partial} G = id$,
  inoltre $[G, \bar\partial] = [G, \bar\partial^*] = 0$;
- $\forall\, \psi \in H^0(M, \mathcal{A}^{p,q}(M))$, $\psi = \underset{\text{armonica}}{\underset{\downarrow}{\mathcal{H}(\psi)}} + \underset{\bar\partial\text{-esatto}}{\underset{\downarrow}{\bar\partial\bar\partial^* G(\psi)}} + \underset{\bar\partial^*\text{-esatto}}{\underset{\downarrow}{\bar\partial^*\bar\partial G(\psi)}} \Rightarrow$

$\Rightarrow H^0(M, \mathcal{A}^{p,q}(M)) = \mathcal{H}^{p,q}(M) \oplus^\perp \bar\partial H^0(M, \mathcal{A}^{p,q-1}(M)) \oplus^\perp \bar\partial^* H^0(M, \mathcal{A}^{p,q+1}(M))$:

$(\psi, \bar\partial\eta) = (\underset{0}{\underset{\shortparallel}{\bar\partial^*\psi}}, \eta) = 0$, $(\psi, \bar\partial^*\eta) = (\underset{0}{\underset{\shortparallel}{\bar\partial\psi}}, \eta) = 0$,

$(\bar\partial\eta, \bar\partial^*\psi) = (\underset{0}{\underset{\shortparallel}{\bar\partial^2\eta}}, \psi) = 0$.

Se ψ è $\bar\partial$-chiusa, $\psi = \mathcal{H}(\psi) + \bar\partial\bar\partial^* G(\psi) \Rightarrow \overset{H^{p,q}_{\bar\partial}(M)}{\overset{\Psi}{[\psi]}} = [\underset{\text{armonico}}{\underset{\downarrow}{\mathcal{H}(\psi)}}]$.

Allora $\underset{H^q(M, \Omega^p(M))}{\underset{\cong}{H^{p,q}_{\bar\partial}(M)}} \cong \mathcal{H}^{p,q}(M) \Rightarrow$ tutti di dim. finita.

$* \Delta_{\bar\partial} = \Delta_{\bar\partial} * \Rightarrow *: \mathcal{H}^{p,q}(M) \to \mathcal{H}^{m-p,m-q}(M)$ isomorfismo.

In particolare, $\mathcal{H}^{m,m}(M) = \mathrm{Span}\{\Phi\} = \mathbb{C}$.

Teo. (dualità di Kodaira-Serre): M var. comp. cpt.
- $H^m(M, \Omega^m(M)) \cong \mathbb{C}$;
- $\underset{H^{p,q}_{\bar\partial}(M)}{\underset{\cong}{H^q(M, \Omega^p(M))}} \otimes \underset{H^{m-p,m-q}_{\bar\partial}(M)}{\underset{\cong}{H^{m-q}(M, \Omega^{m-p}(M))}} \xrightarrow{\wedge} \underset{H^{m,m}_{\bar\partial}(M)}{\underset{\cong}{H^m(M, \Omega^m(M))}}$
  è non degenere;
- $H^q(M, \Omega^p(M)) \cong H^{m-q}(M, \Omega^{m-p}(M))$.

La teoria di Hodge si poteva fare anche con ∂ e d.

Def.: una metrica hermitiana su M si dice di Kähler se la $(1,1)$-forma associata è chiusa. In tal caso, M si dice di Kähler.

Es.: $\mathbb{C}^m$ e $\mathbb{P}^m$ sono di Kähler $\Rightarrow$ ogni $M \underset{\text{embedding}}{\hookrightarrow} \mathbb{P}^m$ è di Kähler.
$\dim M = 1 \Rightarrow M$ è di Kähler (ogni metrica hermitiana è di Kähler).

Oss.: la richiesta $d\omega = 0$ è locale, ed è equivalente a:
$\forall\, p \in M\ \exists$ coordinate olo. $z_1, \dots, z_m$ centrate in p t.c.
$ds^2 \overset{\text{loc.}}{=} \sum h_{ij}\, dz_i \otimes d\bar z_j \Rightarrow h_{ij} = \delta_{ij} + (2) \to$ termini nulli in p al 2° ordine, $H = I + (2)$.

Per questioni locali che coinvolgono solo le derivate prime, si può supporre $ds^2 =$ euclidea.

Prop.: M di Kähler. 1) $H^0(M, \Omega^q(M)) \hookrightarrow H^q_{DR}(M)\ \forall\, q$
(le forme olo. sono chiuse e mai esatte se $\neq 0$);
2) $b_{2m}(M) = \dim H^{2m}(M, \mathbb{C}) > 0$, $0 \leq m \leq \dim M$.

Dim.: Se $\eta \in H^0(M, \Omega^q(M))$ è esatta, $\eta = 0$.
$\forall\, z \in M$, in termini di un coframe unitario $\varphi_1, \dots, \varphi_m$ per la metrica di Kähler in un intorno di p $\eta = \sum_{\#I=q} \eta_I \varphi_I \Rightarrow$
$\Rightarrow \eta \wedge \bar\eta = \sum_{I,J} \eta_I \bar\eta_J (\varphi_I \wedge \bar\varphi_J)$; la $(1,1)$-forma della metrica è
$\omega = \frac{i}{2} \sum_{j=1}^m \varphi_j \wedge \bar\varphi_j \Rightarrow \omega^{m-q} = \underset{0}{\underset{\neq}{\text{cost.}}} \cdot \sum_{\#K=m-q} \varphi_K \wedge \bar\varphi_K \Rightarrow$
$\Rightarrow \eta \wedge \bar\eta \wedge \omega^{m-q} = \underset{0}{\underset{\neq}{\text{cost.}}} \cdot \sum_I (\eta_I \bar\eta_I)\, \Phi$; se $\eta \neq 0$,
$\int_M \eta \wedge \bar\eta \wedge \omega^{m-q} \neq 0$. Se $\eta = d\psi$, $d\eta = 0 \Rightarrow d\bar\eta = 0$;
$d\omega = 0 \Rightarrow d\omega^{m-q} = 0$. Allora
$d(\psi \wedge \bar\eta \wedge \omega^{m-q}) = \eta \wedge \bar\eta \wedge \omega^{m-q} \underset{\text{Stokes}}{\Rightarrow} \int_M \eta \wedge \bar\eta \wedge \omega^{m-q} = 0 \Rightarrow \eta = 0$.

$\eta \in H^0(M, \Omega^q(M))$, $\partial\eta \in H^0(M, \Omega^{q+1}(M))$, $d\eta = \partial\eta + \bar\partial\eta = \partial\eta$,
$\partial\eta = 0 \Rightarrow d\eta = 0$. 1) ok.

2) Mostriamo che ω^q è una $2q$-forma chiusa non esatta $\forall\, 1 \leq q \leq m$. Se fosse $\omega^q = d\psi$,
$0 \neq m!\,\mathrm{Vol}(M) = \int_M \omega^m = \int_M \omega^q \wedge \omega^{m-q} = \int_M d(\psi \wedge \omega^{m-q}) \overset{\text{Stokes}}{=} 0$, assurdo. $\square$

M var. comp. cpt con metrica hermitiana ds^2 e $(1,1)$-forma ω.

$$L: H^0(M, \mathcal{A}^{p,q}(M)) \to H^0(M, \mathcal{A}^{p+1,q+1}(M)),\quad \eta \mapsto \eta \wedge \omega$$

$\Lambda = L^* = (-1)^{p+q} * L *$ (verificare).

Se ds^2 è di Kähler ($\Rightarrow d\omega = 0$),
1) $[L, d] = 0$ e, passando agli aggiunti, $[\Lambda, d^*] = 0$;
2) $[L, d^*] = i(\bar\partial - \partial)$ e, " " ", $[\Lambda, d] = i(\bar\partial^* - \partial^*)$,
$[\Lambda, \partial] = i\bar\partial^*$, $[\Lambda, \bar\partial] = -i\partial^*$;
3) $[L, \Lambda] = (p+q-m)\, id$;
4) $[L, \Delta_d] = 0$, $[\Lambda, \Delta_d] = 0$;
5) $\partial\bar\partial^* + \bar\partial^*\partial = \bar\partial\partial^* + \partial^*\bar\partial = 0$;
6) $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$.

1): $d(\eta \wedge \omega) = d\eta \wedge \omega \pm \eta \wedge \underset{0}{\underset{\shortparallel}{d\omega}} = d\eta \wedge \omega$.

2) e 3): in coordinate, supponendo la metrica euclidea.

4): $Ld \overset{1)}{=} dL$, $Ld^* \overset{2)}{=} i(\bar\partial - \partial) + d^*L \Rightarrow L(dd^* + d^*d) =$
$= dLd^* + (i(\bar\partial - \partial) + d^*L)d = d(i(\bar\partial - \partial) + d^*L) + i(\bar\partial - \partial)d + d^*dL =$
$= (dd^* + d^*d)L + i\,d(\bar\partial - \partial) + i(\bar\partial - \partial)d = (dd^* + d^*d)L$.

5): $i\bar\partial^* \overset{2)}{=} \Lambda\partial - \partial\Lambda \Rightarrow i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial =$
$= \partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial = 0$.

6): $\Delta_d = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) =$
$= \partial\partial^* + \partial\bar\partial^* + \bar\partial\partial^* + \bar\partial\bar\partial^* + \partial^*\partial + \partial^*\bar\partial + \bar\partial^*\partial + \bar\partial^*\bar\partial \overset{5)}{=} \Delta_\partial + \Delta_{\bar\partial}$,
basta $\Delta_\partial = \Delta_{\bar\partial}$; $i\bar\partial^* \overset{2)}{=} \Lambda\partial - \partial\Lambda$, $-i\partial^* \overset{2)}{=} \Lambda\bar\partial - \bar\partial\Lambda \Rightarrow$
$\Rightarrow -i\Delta_\partial = \partial(\Lambda\bar\partial - \bar\partial\Lambda) + (\Lambda\bar\partial - \bar\partial\Lambda)\partial = \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial$,
$i\Delta_{\bar\partial} = \bar\partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\bar\partial = \bar\partial\Lambda\partial - \bar\partial\partial\Lambda + \Lambda\partial\bar\partial - \partial\Lambda\bar\partial$.

Cor.: Δ_∂ e $\Delta_{\bar\partial}$ rispettano la decomposizione in tipi $\Rightarrow$
$\Rightarrow [\Delta_d, \pi^{(p,q)}] = 0$.

$H^{p,q}_d(M) = \dfrac{(p,q)\text{-forme globali chiuse}}{\text{" " esatte}}$, $\overset{\bigoplus_{p+q=k} H^{p,q}_d(M)}{\overset{\shortparallel}{H^k_d(M)}} = H^k_{DR}(M)$.

$\mathcal{H}^{p,q}_d(M) = \mathrm{Ker}\,\Delta_d \cap H^0(M, \mathcal{A}^{p,q}(M)) = \mathcal{H}^{p,q}_\partial(M) = \mathcal{H}^{p,q}_{\bar\partial}(M)$.
$\mathcal{H}^k_d(M) = \mathrm{Ker}\,\Delta_d \cap H^0(M, \mathcal{A}^k(M))$,
$\mathcal{H}^k_d(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_d(M)$. Poiché Δ_d è reale, $\overline{\mathcal{H}^{p,q}_d(M)} = \mathcal{H}^{q,p}_d(M)$.

Hodge per Δ_d: $H^{p,q}_d(M) \cong \mathcal{H}^{p,q}_d(M)$.

Teo. (decomposizione di Hodge): M cpt di Kähler,
- $H^r(M, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\bar\partial}(M)$;
- $H^{p,q}_{\bar\partial}(M) \cong \overline{H^{q,p}_{\bar\partial}(M)}$.

L'inclusione $H^0(M, \Omega^p(M)) \hookrightarrow H^{p,0}_d(M) \cong H^{p,0}_{\bar\partial}(M)$ è un isomorfismo.

Cor.: $b_{2p+1}(M)$ è pari. In generale, $b_r(M) = \dim H^r(M, \mathbb{C}) = \sum_{p+q=r} \dim H^{p,q}_{\bar\partial}(M)$.

Cor.: $H^q(\mathbb{P}^m, \Omega^p(\mathbb{P}^m)) = \begin{cases} \mathbb{C} & \text{se } p = q \leq m \\ 0 & \text{altrimenti} \end{cases}$.

$H^{2k+1}(\mathbb{P}^m, \mathbb{C}) = 0\ \forall\, k$, $H^{2k}(\mathbb{P}^m, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{se } k \leq m \\ 0 & \text{se } k > m \end{cases}$.