Complex Manifolds

Def.: $U \subseteq \mathbb{C}$ open, $f: U \to \mathbb{C}$, then f is <u>holomorphic</u> if

1) $\frac{\partial f}{\partial \bar z} = 0$;

$\Updownarrow$

2) $a \in U \Rightarrow f(z) = \sum_{k=0}^{+\infty} c_k (z-a)^k$ (near a).

Def.: a <u>domain</u> is an open conn. $\emptyset \neq U \subseteq \mathbb{C}^N$.

Def.: $f: U \to \mathbb{C}$ is <u>holomorphic</u> if $\forall\, a = (a_1, \dots, a_N) \in U$ f can be represented by a convergent power series $f(z) = \sum c_{k_1 \dots k_N} (z_1 - a_1)^{k_1} \cdot \ldots \cdot (z_N - a_N)^{k_N}$ on some neigh. of a.

Remark: if $p(z) = \sum_K c_K (z - a_1)^{k_1} \cdot \ldots \cdot (z_N - a_N)^{k_N}$ converges at $z = w$ then $p(z)$ converges when $|z_j - a_j| < |w_j - a_j|$ for $j = 1, \dots, N$.

Proof: WLOG $a = 0$. Then $\exists\, c > 0$ s.t. $|c_K w^K| < c \Rightarrow$

$\Rightarrow |c_K z^K| \leq c \left|\frac{z_1}{w_1}\right|^{k_1} \cdot \ldots \cdot \left|\frac{z_N}{w_N}\right|^{k_N} \Rightarrow$

$\Rightarrow \sum |c_K z^K| \leq c \prod_{j=1}^{N} \frac{1}{1 - |z_j / w_j|} < +\infty$. □

Def.: $P(a, r) := \{ z \in \mathbb{C}^N \mid |z_j - a_j| < r_j \text{ for } j = 1, \dots, N \}$.

Def.: a complex valued function of N variables $(z_1, \dots, z_N)$ is <u>continuous</u> if it is cont. as a function of $(x_1, y_1, \dots, x_N, y_N)$ where $z_j = x_j + i y_j$.

Thm. (Osgood): if $f(z) = f(z_1, \dots, z_N)$ is a cont. function from a domain W in $\mathbb{C}^N$ to $\mathbb{C}$ then f is holo. if and only if it is holo. w.r.t. each variable separately.

Proof: ($\Rightarrow$) obvious.

($\Leftarrow$) $a \in W \Rightarrow P(a, r) \subseteq W$. By Cauchy integral theorem

$$f(z_1, \dots, z_N) = \frac{1}{2\pi i} \int_{|w_1 - a_1| = r_1} \frac{f(w_1, z_1, \dots, z_N)}{w_1 - z_1}\, dw_1 = \dots =$$

$$= \left(\frac{1}{2\pi i}\right)^N \int_{|w_1 - a_1| = r_1} \cdots \int_{|w_N - a_N| = r_N} \frac{f(w_1, \dots, w_N)}{(w_1 - z_1) \cdot \ldots \cdot (w_N - z_N)}\, dw_1 \dots dw_N =$$

$$= \left(\frac{1}{2\pi i}\right)^N \int_{|w_1 - a_1| = r_1} \cdots \int_{|w_N - a_N| = r_N} f(w_1, \dots, w_N) \prod_{l=1}^{N} \frac{1}{w_l - a_l} \sum_{j=1}^{+\infty} \left(\frac{z_l - a_l}{w_l - a_l}\right)^j dw_1 \dots dw_N.$$

take $z \in P(a, r)$, so $\left|\frac{z_j - a_j}{w_j - a_j}\right| < 1$

f cont. on a cpt $\Rightarrow$ unif. cont. $\Rightarrow$

$\Rightarrow$ we can exchange $\sum$ e $\int$.

$M = \sup \{ |f(w)| \mid w \in \overline{P(a, r)} \} \Rightarrow c_K \leq \frac{1}{(2\pi)^N} \frac{M}{r_1^{k_1 + 1} \cdot \ldots \cdot r_N^{k_N + 1}}$. □

the coefficients we find

Ex.: $W \subseteq \mathbb{C}$ domain, $a \in W$, then $\frac{1}{z - a}$ has a pole at $a \Rightarrow$

$\Rightarrow f(z) = \frac{1}{z - a}$ in $W \setminus \{a\}$ does not extend to a holo. function $f: W \to \mathbb{C}$.

$W \subseteq \mathbb{C}^2$ domain, $a \in W$, $f: W \setminus \{a\} \to \mathbb{C}$ holo. $\xRightarrow{\text{Hartogs}}$

$\Rightarrow f$ extends to a holo. function $f: W \to \mathbb{C}$.

<u>Cauchy Riemann equations</u>

Thm.: if $W \subseteq \mathbb{C}^N$ is a domain then a cont. function $f: W \to \mathbb{C}$ is holo. if and only if $\frac{\partial f}{\partial \bar z_\nu} = 0$ for $\nu = 1, \dots, N$.

Prop.: let $f(w_1, \dots, w_m)$ and $w_\lambda = g_\lambda(z)$ for $\lambda = 1, \dots, m$ are diff. and the domain of f contains the image of $(g_1, \dots, g_m)$.

Then $\frac{\partial f}{\partial \bar z_\nu} = \sum_{\lambda=1}^{m} \frac{\partial f}{\partial w_\lambda} \frac{\partial w_\lambda}{\partial \bar z_\nu} + \frac{\partial f}{\partial \bar w_\lambda} + \frac{\partial \bar w_\lambda}{\partial \bar z_\lambda}$ and

$\frac{\partial f}{\partial z_\nu} = \sum_{\lambda=1}^{m} \frac{\partial f}{\partial w_\lambda} \frac{\partial w_\lambda}{\partial z_\nu} + \frac{\partial f}{\partial \bar w_\lambda} + \frac{\partial \bar w_\lambda}{\partial z_\lambda}$

If $g_1, \dots, g_m$ and f are holo. then $f \circ g$ is holo..