$U \subseteq \mathbb{C}^N$ domain, $\mathcal{O}_U = \{f: U \to \mathbb{C} \text{ holo.}\}$ is a ring.

Is $\mathcal{O}_U$ an integral domain$\overset{?}{}$ $N=1$: $f \cdot g \equiv 0$ on $U \Rightarrow f \equiv 0$ or $g \equiv 0$.

Suppose $f \not\equiv 0 \Rightarrow \exists\, p \in U$ s.t. $f(p) \neq 0 \Rightarrow f(q) \neq 0$ for q near $p \Rightarrow$
$\Rightarrow g(p) = 0$ for q near $p \Rightarrow g \equiv 0$ on U because U is conn. and g is analytic. It is true also for $N > 1$.

$f: U \to \mathbb{C}$, $U \subseteq \mathbb{C}$ domain, f holo. non-const., $V \subseteq U$ open, $f(V)$ is open.

Goal: inverse function theorem, implicit mapping theorem.

Def.: let $U \subseteq \mathbb{C}^N$ be a domain and $f: U \to \mathbb{C}^m$ be given by
$f(z_1, \dots, z_N) = (f_1(z_1, \dots, z_N), \dots, f_m(z_1, \dots, z_N))$; then
f is holo. $\Leftrightarrow$ each f_j is holo..

Note: f holo. $\Rightarrow f$ is C^∞ as a map $\mathbb{R}^{2N} \supseteq U \to \mathbb{R}^{2m}$.

Def.: $J_f = \left(\frac{\partial f_\lambda}{\partial z_\nu}\right)$ jacobian matrix. If $N = m$,
$\det(J_f)$ is the jacobian of f.

$z_\nu = x_\nu + i y_\nu$, $\nu = 1, \dots, N$, $f_\lambda = u_\lambda + i v_\lambda$, $\lambda = 1, \dots, m$.

$N = m$: $\frac{\partial(u, v)}{\partial(x, y)} = \left| \det \left(\frac{\partial(u_1, v_1, \dots, u_m, v_m)}{\partial(x_1, y_1, \dots, x_N, y_N)} \right) \right|$.

Prop.: f holo., $N = m \Rightarrow \frac{\partial(u, v)}{\partial(x, y)} = \left| \det \left(\frac{\partial f_\lambda}{\partial z_\nu} \right) \right|^2$.

Proof (sketch for $N=2$): $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u_1/\partial x_1 & \partial v_1/\partial x_1 & \partial u_2/\partial x_1 & \partial v_2/\partial x_1 \\ & \text{etc.} & & \end{vmatrix} =$

$\overset{\text{CR equations}}{=} \begin{vmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ -b_{11} & a_{11} & -b_{12} & a_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ -b_{21} & a_{21} & -b_{22} & a_{22} \end{vmatrix}$. Multiply column #2 by i and add to column #1;
" " #4 " i " "
" " #3.

Multiply row #1 by i and subtract from row #2;
" " #3 " i " " " #4.

We get $\frac{\partial(u, v)}{\partial(x, y)} = \left(\begin{array}{cc|cc} g_{11} & g_{12} & * & * \\ g_{21} & g_{22} & * & * \\ \hline 0 & 0 & \bar{g}_{11} & \bar{g}_{12} \\ 0 & 0 & \bar{g}_{21} & \bar{g}_{22} \end{array} \right)$. $\square$

Thm. (inverse mapping theorem): let $f: U \to \mathbb{C}^N$ be holo. on a domain $U \subseteq \mathbb{C}^N$. If $a \in U$ and $\det(J_f(a)) \neq 0$ then $\exists$ an open neigh. W of a s.t. $f|_W$ is bij. onto its image, $f(W)$ is open and $f^{-1}|_{f(W)}$ is holo..

Proof: $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$ on a neigh. of $a \Rightarrow \exists$ a neigh. W of a s.t. $f|_W$ is a bijection onto its image, $f(W)$ is open, $f^{-1}|_{f(W)}$ is C^∞.

Let $\varphi(w) = f^{-1}(w)$, so $z_\mu = \varphi_\mu(f(z))$, $\mu = 1, \dots, N \Rightarrow$

$$\Rightarrow 0 = \frac{\partial z_\mu}{\partial \bar{z}_\nu} = \sum_{\lambda=1}^N \frac{\partial \varphi_\mu}{\partial w_\lambda} \underbrace{\frac{\partial f_\lambda}{\partial \bar{z}_\nu}}_{\overset{\shortparallel}{0}} + \frac{\partial \varphi_\mu}{\partial \bar{w}_\lambda} \frac{\partial \bar{f}_\lambda}{\partial \bar{z}_\nu} \Rightarrow$$

$$\Rightarrow 0 = \sum_{\lambda=1}^N \frac{\partial \varphi_\mu}{\partial \bar{w}_\lambda} \underbrace{\frac{\partial \bar{f}_\lambda}{\partial \bar{z}_\nu}}_{\text{rank } N \text{ matrix}}, \quad \mu, \nu = 1, \dots, N \Rightarrow$$

$$\Rightarrow \frac{\partial \varphi_\mu}{\partial \bar{w}_\lambda} = 0 \text{ for } \lambda = 1, \dots, N. \quad \square$$

Cor. (implicit function theorem): let f_λ for $\lambda = 1, \dots, m$ be holo. functions on a domain $U \subseteq \mathbb{C}^N$. Assume $\left(\frac{\partial f_\lambda}{\partial z_\nu}\right)$ has rank r at each point of U. Suppose $f_1(a) = \dots = f_m(a) = 0$.

WLOG $\left(\frac{\partial f_\lambda}{\partial z_\nu}\right)_{\lambda, \nu = 1, \dots, r}$ is

Then $f_\lambda(z_1, \dots, z_N) = 0$, $\lambda = 1, \dots, N$ has a unique holo. sol.
$z_\lambda = \varphi_\lambda(z_{r+1}, \dots, z_N)$, $a_\lambda = \varphi_\lambda(z_{r+1}, \dots, z_N)$, $\lambda = 1, \dots, r$.

Proof: exc.. $\square$

Complex Manifolds

X = paracpt Hausdorff space.

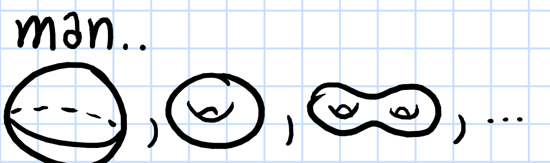
Def.: a local complex coordinate on X is a top. homeo. $z: U \to \mathbb{C}^N$ on a domain $U \subseteq X$.
$z(p) = (z^1(p), \dots, z^N(p))$.

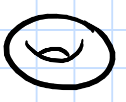
Def.: a system of local complex coord. at X is a collection $\{z_j\}_{j \in I}$ of local comp. coord. $z_j: U_j \to \mathbb{C}^N$ s.t.
(1) $X = \bigcup_{j \in I} U_j$;
(2) if $U_j \cap U_k \neq \emptyset$, $z_j \circ z_k^{-1}: z_k(U_j \cap U_k) \to z_j(U_j \cap U_k)$ is biholo..

Def.: two systems $\{z_j\}_{j \in I}$, $\{w_\lambda\}_{\lambda \in \Lambda}$ are equivalent if $z_j \to w_\lambda$ is biholo. where defined.

Def.: a complex structure on X is an equiv. class of local comp. coord. systems on X. A complex manifold X is a paracpt Hausdorff space with a choice of comp. structure.

Note: a comp. man. is ori. as a C^∞ man..

Real dim.=2, comp. dim.=1. Cpt case: , … .

But there are uncountably many complex structures on, for example, .