Ex.: $\mathcal{U} \subseteq \mathbb{C}^N$ domain is a comp. man..

Ex.: M_1, M_2 c.m. $\Rightarrow M_1 \sqcup M_2$ c.m. (if same dim.), $M_1 \times M_2$ c.m..

Thm. (quotient manifold): if a Lie group acts smoothly, freely and properly on a man. M, then M/G is a top. man. with a unique smooth structure s.t. the quotient map $M \to M/G$ is a smooth submersion.

Ex.: $\mathbb{P}^N_{\mathbb{K}} = \{1\text{-dim. subspaces of } \mathbb{K}^{N+1}\}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Ex.: H closed subgroup of $G \Rightarrow H$ acts smoothly, freely and properly on $G \Rightarrow G/H$ is a C^∞-man..

$\mathbb{P}^{N-1}_{\mathbb{R}}$: $G = GL_N(\mathbb{R})$, $x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\mathbb{P}^{N-1}_{\mathbb{R}} \cong G/G^x$. $G^x = \left\{ \left(\begin{array}{c|c} \overset{\in \mathbb{R}\setminus\{0\}}{*} & * \\ \hline 0 & \in GL_{N-1}(\mathbb{R}) \end{array} \right) \right\}$.

G^x is closed in $G_N(\mathbb{R}) \Rightarrow$

$\Rightarrow \mathbb{P}^N_{\mathbb{R}}$ has a unique smooth structure s.t. the quotient map $G \to G/H$, $H = G^x$ is a smooth submersion.

$G = SO_N(\mathbb{R})$ acts transitively on $\mathbb{P}^{N-1}_{\mathbb{R}} \Rightarrow \mathbb{P}^{N-1}_{\mathbb{R}} \cong G/G^x$.

$G \to G/G^x$ is a smooth submersion $\Rightarrow \mathbb{P}^{N-1}_{\mathbb{R}}$ is cpt.

Charts: $\mathbb{R}^{N-1} \xrightarrow{\varphi_j} \mathbb{P}^{N-1}_{\mathbb{R}}$.

$(x_1, \dots, x_{N-1}) \mapsto \operatorname{span}\{(x_1, \dots, x_{j-1}, 1, x_j, \dots, x_{N-1})\}$

$\mathbb{P}^N_{\mathbb{C}}$ is a c.m. using charts and it is cpt (use $U(N+1)$).

Ex.: $G_{\mathbb{K}}(r, N) = \{r\text{-dim. subspaces of } \mathbb{K}^N\}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

$\mathbb{R}$: $G = GL_N(\mathbb{R})$ acts transitively on $G_{\mathbb{R}}(r, N)$.

$G_{\mathbb{R}}(r, N) \cong G/G^x$, $x = \operatorname{span}\{e_1, \dots, e_r\}$, $G^x = \left\{ \left(\begin{array}{c|c} \in GL_r & * \\ \hline 0 & \in GL_{N-r} \end{array} \right) \right\}$

closed in $GL_N(\mathbb{R}) \Rightarrow G \to G/G^x = G_{\mathbb{R}}(r, N)$ is a smooth submersion for a unique smooth structure on $G_{\mathbb{R}}(r, N)$. It is cpt (use $SO(N)$).

Produce charts (in $\mathbb{C}$): $\mathcal{U} \in G_{\mathbb{C}}(r, N) \Rightarrow \mathcal{U}$ has a basis of row vectors $f_1, \dots, f_r$ in $\mathbb{C}^N$, $(f_1, \dots, f_r) \to \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$ $r \times N$ matrix of rank r.

$I = \{$index set for possible $r \times r$ minors of an $r \times N$ matrix$\}$,

$M_j = j^{th}$ minor of M $r \times N$ matrix, $j \in I$.

$\mathcal{U}_j = \{M \mid \det(M_j) \neq 0\}$. After reordering, $\left(\begin{array}{ccc|c} m_{11} & \cdots & m_{1r} & \\ \vdots & \ddots & \vdots & * \\ m_{1r} & \cdots & m_{rr} & \end{array} \right)$ has $\det \neq 0$. Multiply by $\begin{pmatrix} m_{11} & \cdots & m_{1r} \\ \vdots & \ddots & \vdots \\ m_{1r} & \cdots & m_{rr} \end{pmatrix}^{-1} \Rightarrow$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & \underbrace{*}_{\text{coordinates}} \\ 0 & & 1 & \end{array} \right)$. Exc.: check that transition maps are smooth.

$\dim G_{\mathbb{C}}(r, N) = (N - r) r$.

Ex.: $M = \{(f_1, \dots, f_r) \mid f_1, \dots, f_r \in \mathbb{C}^N$ linearly independent$\}$.

$\circlearrowleft$

$GL_N(\mathbb{C})$

Ex.: flag manifold $= \{\{0\} = F^N \subseteq \overset{F^j \text{ v.s.}}{\dots} \subseteq F^0 = \mathbb{C}^N \mid \dim F^j = N - j\}$.