G/H, H closed subgroup of G (complex Lie groups).

$\mathcal{G} = T_e(G)$; $\gamma \in T_e(G) \rightsquigarrow \underbrace{\tilde{\gamma}(g)}_{\in \mathcal{G}} = (L_g)_* \gamma$.

e: identity of G

$\alpha, \beta \in T_e(G) \rightsquigarrow \tilde{\alpha}, \tilde{\beta} \rightsquigarrow [\tilde{\alpha}, \tilde{\beta}] = [(L_g)_* \alpha, (L_g)_* \beta] = (L_g)_*(\underbrace{\xi}_{= [\alpha, \beta]})$.

Ex.: $G = GL_N(\mathbb{C})$, $\mathcal{G} = gl_N(\mathbb{C})$, $[A, B] = AB - BA$.

$\mathcal{G} \xrightarrow{exp} G$, $\alpha \rightsquigarrow \tilde{\alpha}$ left-invariant vector field,

$exp(\alpha) = \varphi_{\tilde{\alpha}}(e, 1)$.

In the example, $X(t) = exp(tA) = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}$, $X'(t) = X(t)A$.

$\mathcal{G} = \mathcal{C} \oplus \mathcal{H}$ as v.s..

$F: \mathcal{C} \oplus \mathcal{H} \to G$ has bijective diff. at $(0,0) \Rightarrow$

$(c, h) \mapsto exp(c)\,exp(h)$

$\Rightarrow \exists$ neigh. of $(0,0)$ in $\mathcal{C} \oplus \mathcal{H}$, neigh. of e in G s.t.

$g = exp(\underbrace{\gamma}_{\in \mathcal{G}}) = exp(\underbrace{c}_{\in \mathcal{C}})\, exp(\underbrace{h}_{\in \mathcal{H}})$ is a diffeo. $\Rightarrow$

$\Rightarrow G \to G/H$, so $c \mapsto exp(c)H$ is a local bijection from a neigh of 0 in $\mathcal{C}$ to a neigh. of H in G/H.

$g \mapsto gH$, $g = exp(c)\,exp(h)$

M comp. man., $f: M \to \mathbb{C}$ is holo. if f is holo. in each chart.

$p \in M$, $\mathcal{O}_p = \{$germs of holo. functions at $p\}$.

$m_p = \{[f] \in \mathcal{O}_p \mid f(p) = 0\}$. $T_p^{hol}(M)^* := m_p / m_p^2$.

If $(z_1, \dots, z_N)$ are local coord. on a neigh. of p s.t. $z_1(p) = \dots = z_N(p) = 0$, then $T_p^{hol}(M)^* = span\{[z_1], \dots, [z_n]\}$ (Taylor theorem).

$T_p^{hol}(M) = (T_p^{hol}(M)^*)^* = span\left\{ \frac{\partial}{\partial z_1}\Big|_p, \dots, \frac{\partial}{\partial z_N}\Big|_p \right\}$:

$\frac{\partial}{\partial z_j}\Big|_p (gh) = \frac{\partial g}{\partial z_j}\Big|_p h(p) + g(p) \frac{\partial h}{\partial z_j}\Big|_p = 0$ if $g(p) = h(p) = 0$,

so $\frac{\partial}{\partial z_j}\Big|_p \equiv 0$ on m_p^2, and $\frac{\partial}{\partial z_j}\Big|_p (z_k) = \delta_{jk}$. In particular, $[z_j] \longleftrightarrow dz_j$.

Back to $G(r, N)$. $\{f_1(z), \dots, f_r(z)\}$ frame for a family of r-dim subspaces of $\mathbb{C}^N$, $\mathcal{U}: \underbrace{(\mathbb{C}, 0)}_{\text{a neigh.}} \to G(r, N)$.

$\{f_1(0), \dots, f_r(0)\}$ is a frame for $\mathcal{U} = \mathcal{U}(0)$. We want $T_{\mathcal{U}(0)} G(r, N)$.

$\{f_1'(0), \dots, f_r'(0)\}$ is not the answer (it depends on the frame).

$\{\tilde{f}_1(z), \dots, \tilde{f}_r(z)\}$ another choice of frame $\Rightarrow \exists$ a (holo.) functions s.t. $\tilde{f}_j(z) = \sum_k a_{jk}(z) f_k(z) \Rightarrow$

$\Rightarrow \tilde{f}_j'(0) = \sum_k (a_{jk}'(0) f_k(0) + a_{jk}(0) f_k'(0))$.

WLOG $(a_{jk}(0))_{jk} = Id \Rightarrow f_j(0) \mapsto f_j'(0)$ is well def. modulo $\mathcal{U}(0)$.

So $T_{\mathcal{U}(0)} G(r, N) \cong Hom(\mathcal{U}(0), \mathbb{C}^N / \mathcal{U}(0))$.

$f_j(0) \mapsto [f_j'(0)]$

$dim(Hom(\mathcal{U}(0), \mathbb{C}^N / \mathcal{U}(0))) = r(N - r)$.

Complex Structure Tensor

$$T_p^{C^\infty}(M) \otimes_{\mathbb{R}} \mathbb{C} = \underbrace{span_{\mathbb{C}}\left\{ \frac{\partial}{\partial z_1}\Big|_p, \dots, \frac{\partial}{\partial z_N}\Big|_p \right\}}_{T_p^{hol}(M)} \oplus \underbrace{span_{\mathbb{C}}\left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_p, \dots, \frac{\partial}{\partial \bar{z}_N}\Big|_p \right\}}_{\overline{T_p^{hol}(M)}}.$$

$J: T_p^{C^\infty}(M) \otimes_{\mathbb{R}} \mathbb{C} \to T_p^{C^\infty}(M) \otimes_{\mathbb{R}} \mathbb{C}$,

$J|_{T_p^{hol}(M)}$ is multiplication by i,

$J|_{\overline{T_p^{hol}(M)}}$ " " " $-i$.

CR equations become $J(df) = (df)J$.

Def.: an almost complex structure on M is a $J \in End(T(M) \otimes_{\mathbb{R}} \mathbb{C})$ s.t. $J^2 = -1$.

Almost complex structure $\Rightarrow T(M)_{\mathbb{R}} \otimes \mathbb{C} = \underbrace{T^{(1,0)}(M)}_{i} \oplus \underbrace{T^{(0,1)}(M)}_{-i}$.

Theorem (Newlander–Nirenberg): an almost comp. structure J is a comp. structure $\Leftrightarrow [T^{(1,0)}(M), T^{(1,0)}(M)] \subseteq T^{(1,0)}(M)$.