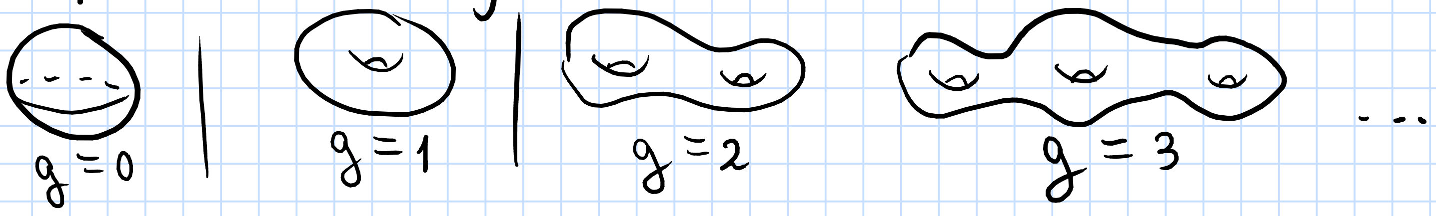
Ex.: M cpt C^∞ boundaryless ori. surface.



$\mathbb{P}^1\mathbb{C}$ $\quad\quad$ $\mathbb{C}$ $\quad\quad$ $\mathbb{D}$

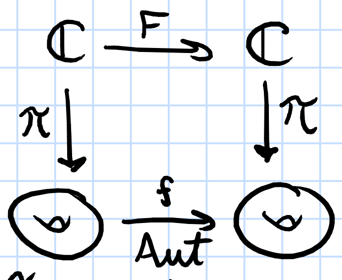
$\tilde{M} \xrightarrow{\pi} M$ universal cover of a comp. man M. $\exists$ comp. structure on $\tilde{M}$ s.t. $\pi : \tilde{M} \to M$. $\forall\, p \in M\ \exists$ neigh $\mathcal{U} \ni p$ s.t. $\pi^{-1}(\mathcal{U}) = \bigsqcup_{\alpha\in A} V_\alpha$, $\pi_{|V_\alpha} : V_\alpha \to \mathcal{U}$ homeo. $\forall \alpha \in A$.

I $(z_1, \dots, z_N)$ is a local coord. system of $\mathcal{U}$, then $(z_1 \circ \pi_{|V_\alpha}, \dots, z_N \circ \pi_{|V_\alpha})$ should be a local coord. system on V_α.

$$\begin{tikzcd} \tilde{M} \arrow[r, "F"] \arrow[d, "\pi"'] & \tilde{M} \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

$f : M \to M \underset{\pi \text{ universal}}{\Rightarrow} F : \tilde{M} \to \tilde{M}$.

If M is a surface, uniformization thm. $\Rightarrow \tilde{M} = \mathbb{P}^1\mathbb{C}, \mathbb{C}, \mathbb{D}$.



To find π, we need to find fixed pts free autos. of $\mathbb{C}$ (deck transformations are fixed pts free autos. of $\tilde{M}$.

Case $\tilde{M} = \mathbb{P}^1$: $Aut(\mathbb{P}^1) =$ Möbius transformation in $PSL_2(\mathbb{C})$; no fixed pts $\Rightarrow z \mapsto \frac{az+b}{cz+d}$, $c=0$, $a=d\neq 0$, but $\infty \mapsto \infty$, so we only have $id \Rightarrow M = \mathbb{P}^1$.

Case $\tilde{M} = \mathbb{C}$: $Aut(\mathbb{C})$ which are fixed pts free are of the form $z \mapsto z + \alpha$.

Case $\tilde{M} = \mathbb{D}$: $Aut(\mathbb{D}) \cong ?$ $\mathbb{D} \cong \mathbb{H}$, $Aut(\mathbb{H}) = PSL_2(\mathbb{R})$.

k (curvature)
0 $\quad \mathbb{C}$ with $\{z \mapsto z + \alpha\}$: the euclidean metric is invariant.
-1 $\quad \mathbb{H}$ with $SL_2(\mathbb{R})$: " hyperbolic " " " .
1 $\quad \mathbb{P}^1$ with $\{id\}$: spherical metric.

So M cpt boundaryless ori. surface $\Rightarrow M$ has a metric of constant curvature s.t. $\pi : M \to \tilde{M}$ is a local isometry.
Each conformal class of metrics contains a representative of curvature k. Take such a metric g, then there exists a set of isothermal coord..

On $\mathbb{D}$: $\frac{1}{1-|z|^2}$; on $\mathbb{C}$: 1; on $\mathbb{P}^1$: $\frac{1}{1+|z|^2}$.

x, y set of coord. as above s.t. $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ is positively ori..
$J(\partial/\partial x) = \partial/\partial y$, $J(\partial/\partial y) = -\partial/\partial x$.
This comes from a comp. structure.
$\dim_{\mathbb{R}} M = 2 \Rightarrow T^{(1,0)}(M)$ has dim. 1. Locally $T^{(1,0)}(M)$ is generated by a section α. $[\underbrace{f\alpha}_{T^{(1,0)}(M)}, \underbrace{g\alpha}] = (f\alpha(g) - \underbrace{g\alpha(f))\alpha}_{T^{(1,0)}(M)}$.

Moduli

Let M be a cpt comp. man. of dim. 1 and genus 1.
Universal cover of M is $\mathbb{C}$: $\int_M k\, ds \underset{\text{Gauss-Bonnet}}{=} 2\pi\chi(M) = 0$, if k is constant it must be 0.
Other way: $\pi_1(M) = \mathbb{Z}\oplus\mathbb{Z}$, so $M \neq \mathbb{P}^1$ because $\pi_1(\mathbb{P}^1) = \{0\}$.
Fixed pts. free autos. of $\mathbb{H}$ which are independent are not abelian, so we can't get $\mathbb{Z}\oplus\mathbb{Z}$.

Ex.: $\mathbb{D}\times\mathbb{D}$ and $B = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 < 1\}$ are not biholo.:
$Aut(\mathbb{D}\times\mathbb{D}) \ncong Aut(B)$.

If $g(M) > 1$, $\pi_1(M)$ is non abelian, so its universal cover must be $\mathbb{D}$.