We want to study cpt Riemann surfaces of genus 1.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\Lambda \arrow[r, "f"] & \mathbb{C}/\Lambda \end{tikzcd}$$

Covering transformations: $z \mapsto z+\alpha$.
Case $\pi_1(M) = \langle z \mapsto z+\alpha \rangle$: up to rescale, $z \mapsto z+1 \Rightarrow$
$\Rightarrow \mathbb{C}/\{z \mapsto z+1\} \cong \mathbb{C}^*$.

Case $\langle z \mapsto z+1, z \mapsto z+\alpha, z \mapsto \beta \rangle$: not a man..
Case $\langle z \mapsto z+1, z \mapsto z+\alpha \rangle$: WLOG $M = \mathbb{C}/\Lambda$,
$\Lambda = \Lambda(\tau) = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\operatorname{Im}\tau > 0$.
Question: when are $\mathbb{C}/\Lambda(\tau)$, $\mathbb{C}/\Lambda(\tau')$ biholo.?
$F: \mathbb{C} \to \mathbb{C}$ auto. $\Rightarrow F(z) = az + b$.
We need $F(\Lambda(\tau)) = \Lambda(\tau')$. $F(0) = b \in \Lambda(\tau')$; WLOG $b = 0$.
$F(1) = a = \alpha + \beta\tau'$, $\alpha, \beta \in \mathbb{Z}$, $F(\tau) = a\tau = \gamma + \delta\tau'$, $\gamma, \delta \in \mathbb{Z}$.
So $\underset{\substack{\Uparrow \\ \mathbb{H}}}{\tau} = \frac{a\tau}{a} = \frac{\gamma + \delta\tau'}{\alpha + \beta\tau'} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} . \tau' \in \mathbb{H}$.
↳ acts on

F biholo. $\Rightarrow \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} \in SL_2(\mathbb{Z})$.
Summary: M cpt Riemann surface of genus 1 $\Rightarrow$
$\Rightarrow M = \mathbb{C}/\Lambda(\tau)$, $\tau \in \mathbb{H}$, $\Lambda(\tau) = \mathbb{Z} \oplus \tau\mathbb{Z}$. $\mathbb{C}/\Lambda(\tau) \cong \mathbb{C}/\Lambda(\tau') \Rightarrow$
$\Rightarrow \tau = A.\tau'$, $A \in SL_2(\mathbb{Z})$.
The moduli space of genus 1 of cpt Riemann surfaces is $\mathbb{H}/SL_2(\mathbb{Z})$.

Subvarieties

Def.: a subset S of a comp. man. M is a subvariety if $p \in S \Rightarrow \exists$ local defining equations $\{f_\lambda\}$ of S in some open neigh. $\mathcal{U} \ni p$ in M:
$S \cap \mathcal{U} = \{q \in \mathcal{U} \mid f_\lambda(q) = 0 \ \forall \lambda\}$.
A comp. subvar. S of M is a subman. if $\exists\, r \geq 0$ s.t. $\forall p \in S$ the local def. eqs. $\{f_\lambda\}$ can be selected so that $\operatorname{rank}\left(\frac{\partial f_\lambda}{\partial z_\nu}\right) = r$ for some local coord. $\{z_\nu\}$ in a neigh. of p.
Implicit function thm. $\Rightarrow \underbrace{(z_{r+1}, \dots, z_N)}_{z} \mapsto (g_1(z), \dots, g_r(z), z)$.

Def.: a comp. projective man. is a comp. man which is isomorphic to a closed subman. of $\mathbb{P}^N$.
Ex. (twisted cubic): $\mathbb{P}^1 \to \mathbb{P}^3$, $[x:y] \mapsto [x^3 : x^2y : xy^2 : y^3]$. $f(\mathbb{P}^1)$ is a comp. proj. man. and a subvariety.
Ex. (rational normal curve): $f: \mathbb{P}^1 \longrightarrow \mathbb{P}^N$, $N \geq 3$.
$[x:y] \longmapsto [x^N : x^{N-1}y : \dots : xy^{N-1} : y^N]$

Def.: L, M comp. man., $\pi: L \to M$ hol.. $L \xrightarrow{\pi} M$ is a line bundle if the fibers of π are isomorphic to $\mathbb{C}$ and $p \in M \Rightarrow \exists$ neigh. $\mathcal{U} \ni p$ in M and a hol. local trivialization $\varphi: \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{C}$ s.t. the diagram

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}) \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & \mathcal{U} \times \mathbb{C} \arrow[d] \\ & \mathcal{U} \end{tikzcd}$$

commutes and φ is linear on the fibers.

Ex.: $\mathbb{P}^N = \{\text{line through the origin in } \mathbb{C}^{N+1}\}$, so we have a tautological line bundle $L \xrightarrow{\pi} \mathbb{P}^N$, $\pi^{-1}([\ell]) = \ell$.
Ex.: $L \xrightarrow{\pi} M$ line bundle $\Rightarrow L^* \longrightarrow M$ line bundle.
↳ dual line bundle
$L \xrightarrow{\pi} \mathbb{P}^N$, $\ell \mapsto [\ell]$ tautological line bundle $\Rightarrow L^* \longrightarrow \mathbb{P}^N$ is the hyperplane bundle.
Ex.: $\operatorname{Sym}^N(\mathbb{C}^2)$. $\mathbb{C}^2 = \operatorname{span}\{x, y\}$, $\operatorname{Sym}^N(\mathbb{C}^2) = \operatorname{span}\{x^N, x^{N-1}y, \dots, y^N\}$.

Kodaira embedding theorem

Every cpt Riemann surface is a comp. proj. man..
$L \to M$ hol. line bundle, σ_1, σ_2 non-vanishing hol. sections of $L \Rightarrow$
$\Rightarrow \sigma_1/\sigma_2$ is a well defined hol. function.
$L \to M$ hol. line bundle, $H^0(M, \mathcal{O}(L))$ = space of global hol. sections of M. Hope 1: it is $\neq \{0\}$. Take a basis $\{\sigma_1, \dots, \sigma_m\}$.
Hope 2: $\nexists\, p \in M$ s.t. $\sigma_1(p) = \dots = \sigma_m(p) = 0$. If we have 1 and 2, we get $M \longrightarrow \mathbb{P}^m$, $p \longmapsto [\sigma_1(p) : \dots : \sigma_m(p)]$. Hope 3: embedding.

Next thing we want: $\mathbb{C}/\Lambda \hookrightarrow \mathbb{P}^2$ (explicitly).
Ex.: $E = \mathbb{C}/\Lambda$. The $\mathbb{C}/\Lambda$ is an abelian group $\rightsquigarrow$ we want to see E as such. We have to pick a $0 \in E$.
↳ elliptic curve
$\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\operatorname{Im}\tau > 0$.
$\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \frac{1}{(z - m - n\tau)^2} - \frac{1}{(m + n\tau)^2}$ defines a mero function on $\mathbb{C}$ with poles of order 2 on Λ (and nowhere else) and satisfies $\wp(z + \lambda) = \wp(z) \ \forall z \notin \Lambda$, $\wp(-z) = \wp(z)$.
$\wp(z)$ even $\Rightarrow \wp'(z)$ odd; $\wp(z)$ has polef of order 2 on Λ $\Rightarrow$ $\wp'(z)$ has poles of order 3 on Λ.
$\wp'(1/2) = -\wp'(-1/2) = -\wp'(-1/2 + 1) = -\wp'(1/2) \Rightarrow$
$\Rightarrow \wp'(1/2) = 0$; also $\wp'(\tau/2) = 0$, $\wp'((1+\tau)/2) = 0$.
$e_1 = \wp(1/2)$, $e_2 = \wp(\tau/2)$, $e_3 = \wp((1+\tau)/2)$.
Then a miracle occurs: $\exists\, g_2 \in \mathbb{C}$ s.t.
$\wp'(z)^2 - (4\wp^3(z) - g_2\wp(z))$ has no poles on $\mathbb{C}$ $\Rightarrow$
$\underset{\text{periodic}}{\Rightarrow}$ bounded $\underset{\text{Liouville}}{\Rightarrow}$ constant $\Rightarrow \wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3$.
So $\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$.