$y = \wp'(z)$, $x = \wp(z)$, $\mathbb{C}/\Lambda \to \mathbb{P}^2$; $z \mapsto [\wp(z) : \wp'(z) : 1]$ $\lim_{z\to 0} [\wp(z) : \wp'(z) : 1] =$

$= \lim_{z \to 0} [\wp(z)/\wp'(z) : 1 : 1/\wp'(z)] = [0:1:0] \in \mathbb{P}^2$.

$\mathbb{P}^2$ is covered by charts. In the chart $\{[X:Y:Z] \mid Z \neq 0\}$, local coord. $x = X/Z$, $y = Y/Z$, the equation of the curve is $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$. We should prove that e_1, e_2, e_3 are distinct (to have a smooth curve).

$$\left(\frac{Y}{Z}\right)^2 = 4\left(\frac{X}{Z} - e_1\right)\left(\frac{X}{Z} - e_2\right)\left(\frac{X}{Z} - e_3\right),$$

$$ZY^2 = 4(X - e_1 Z)(X - e_2 Z)(X - e_3 Z).$$

We check it is smooth chart by chart.

$\mathbb{C}/\Lambda \to \mathbb{P}^2$; $z \mapsto [\underset{x}{\underset{\shortparallel}{\wp(z)}} : \underset{y}{\underset{\shortparallel}{\wp'(z)}} : 1]$ $\qquad x = \wp(z) \Rightarrow dx \underset{\text{in the chart } \{Z\neq 0\}}{\underset{\downarrow}{=}} \wp'(z)\,dz \Rightarrow \frac{dx}{y} = dz$.

dz on $\mathbb{C}/\Lambda$ is a hol. diff. form (because it invariant by traslations).

Now, we start from $y^2 = \underbrace{4(x-e_1)(x-e_2)(x-e_3)}_{f(x)}$, e_1, e_2, e_3 distinct; then $\frac{dx}{y}$ is a hol. 1-form on $\underbrace{V(y^2 - f(x))}_{\text{projective curve}}$.

Remark: X, Y cpt conn. Riemann surfaces, $f: X \to Y$ hol. $\Rightarrow$
$\Rightarrow f(X) = \{pt\}$ or Y.

$\frac{dx}{y}$ is hol.: in the chart $Z \neq 0$, $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ are hol. functions, so the only problem is at $y = 0$.

$y^2 = f(x) \Rightarrow 2y\,dy = f'(x)dx \Rightarrow \frac{dx}{y} = \frac{2dy}{f'(x)}$. $y = 0$, $y^2 = f(x) \Rightarrow$
$\Rightarrow x = e_1, e_2, e_3 \Rightarrow f'(x) \neq 0$.

In the chart $Y \neq 0$: $u = X/Y$, $v = Z/Y$ local coord. $\Rightarrow$
$\Rightarrow u = x/y$, $v = 1/y \Rightarrow x = u/v \Rightarrow dx = \frac{v\,du - u\,dv}{v^2} \Rightarrow$

$\Rightarrow \frac{dx}{y} = \frac{v\,du - u\,dv}{v} = du - \frac{u}{v}dv$. We need to check at $(u,v) = (0,0)$: $y^2 = 4x^3 - g_2 x - g_3 \Rightarrow v \overset{(*)}{=} 4u^3 - g_2 u v^2 - g_3 v^3 \Rightarrow$
$\Rightarrow dv = 12u^2 du - g_2 v^2 du - 2g_2 uv\,dv - 3g_3 v^2 dv \Rightarrow$
$\Rightarrow (1 + 2g_2 uv + 3g_3 v^2)\,dv = (12u^2 - g_2 v^2)du \Rightarrow$
$\Rightarrow \frac{u}{v}(1 + 2g_2 uv + 3g_3 v^2)\,dv = \frac{u}{v}(12u^2 - g_2 v^2)du$.

We let $(u,v) \to 0$, so at the LHS it remains what we want and at the RHS it remains $\frac{12u^3}{v}du$. $(*) \Rightarrow$

$\Rightarrow 4u^3 = v + g_2 uv^2 + g_3 v^3 \Rightarrow \frac{12u^3}{v}du = (3 + 3g_2 uv + 3g_3 v^2)du$.

$y^2 = 4(x-e_1)(x-e_2)(x-e_3)$. This is a local 2:1 cover of $\mathbb{C}$: $x \mapsto y = \pm\sqrt{f(x)}$, except at e_1, e_2, e_3. This extends to a 2:1 cover of the proj. curve $\mathbb{P}^2 \supseteq E \to \mathbb{P}^1$, except at $[e_1:0:1], [e_2:0:1], [e_3:0:1], [0:1:0]$. So
$E = (\mathbb{P}^1 \setminus \{4 \text{ pts}\}) \sqcup (\mathbb{P}^1 \setminus \{4\text{ pts}\}) \sqcup \{4 \text{ pts}\} \Rightarrow$
$\Rightarrow \chi(E) = -2 - 2 + 4 = 0$.