E from previous lesson; we want to show that $E \cong \mathbb{C}/\Lambda$.

$f: E \to \mathbb{C}$, $p \in E$, γ path from p to q, $f(q) = \int_\gamma \frac{dx}{y}$.

$\tilde\gamma$ another pat from p to q, then

$[\tilde\gamma - \gamma] \in \pi_1(E, p)$. A hol. 1-form σ on E is closed: locally $\sigma = g\,dw$, w local hol. coord., g hol.; $d\sigma = \left(\frac{\partial}{\partial w} dw + \frac{\partial}{\partial \bar w} d\bar w\right)\sigma =$
$= \frac{\partial g}{\partial w} \cancel{dw \wedge dw} + \cancel{\frac{\partial g}{\partial \bar w}} d\bar w = 0$.

So $\int_\gamma \frac{dx}{y}$ only depends on the class in homology.

So f is not well def.. $\Lambda = \left\{ \int_\alpha \frac{dx}{y} \,\middle|\, \alpha \in H_1(E, \mathbb{Z}) \right\} \Rightarrow$

$\Rightarrow [f]: E \to \mathbb{C}/\Lambda$. We should check that $\mathrm{rank}_{\mathbb{R}} = 2$.

$[f](q) \mapsto \left[\int_p^q \frac{dx}{y}\right] \in \mathbb{C}/\Lambda$

We also need to check that $[f]$ has inj. diff.: near q,

$f(q) = \int_p^{q_0} \frac{dx}{y} + \int_{q_0}^q \frac{dx}{y} = f(q_0) + \int_{q_0}^q \frac{dx}{y} \Rightarrow$

$\Rightarrow \frac{\partial f}{\partial q} = \frac{\partial}{\partial q} \int_{q_0}^q \frac{dx}{y} = \frac{1}{y(q)} \neq 0$.

Now, WLOG $y^2 = x(x-1)(x-\lambda)$ (Legendre form).

$\frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$. We want $\int_0^1 \frac{dx}{\sqrt{\cdots}}$, $\int_1^\lambda \frac{dx}{\sqrt{\cdots}}$.

↳ we won't do it

In summary: $E = \mathbb{C}/\Lambda$, $\mathrm{rank}_{\mathbb{R}} \Lambda = 2$ embeds as a curve in $\mathbb{P}^2$.

Which $\mathbb{C}^g/\Lambda$ where $\mathrm{rank}_{\mathbb{R}} \Lambda = 2g$ embeds into some $\mathbb{P}^N$ (as a complex subman.).

Let X be a comp. proj. man., $\{\omega_1, \dots, \omega_g\}$ a bases for the hol. 1-forms on X.

Hodge decomposition theorem: $\Lambda = \left\{ \left(\int_\sigma \omega_1, \dots, \int_\sigma \omega_g\right) \,\middle|\, \sigma \in H_1(X, \mathbb{Z}) \right\}$ has real rank $2g$.

$\Lambda \subseteq \mathbb{C}^g$, $\mathrm{Alb}(X) = \mathbb{C}^g/\Lambda$ is an abelian variety with the following universal mapping property: if A is an abelian variety, $f: X \to A$ is a morphism, then f factors through $\mathrm{Alb}(X)$.

$\mathrm{Pic}^0(X) = \{$isomorphism classes of hol. line bundles of X of degree $0\}$.

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^* \to 0 \Rightarrow$$

$$\Rightarrow \underset{(\mathbb{C},+)}{\underset{\parallel}{H^0(X, \mathcal{O})}} \to \underset{(\mathbb{C}^*,\cdot)}{\underset{\parallel}{H^0(X, \mathcal{O}^*)}} \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to \underbrace{H^1(X, \mathcal{O}^*)}_{\text{line bundles}} \to \dots$$

If a line bundle goes to $0 \in H^2(X, \mathbb{Z})$, it is of degree 0.

$$0 \to H^1(X, \mathbb{Z}) \to \underbrace{H^1(X, \mathcal{O})}_{H^0(X, \Omega^1)^*} \to \underbrace{\mathrm{Ker}(H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}))}_{\mathrm{Pic}^0(X)} \to 0.$$

Hodge theory

↳ hol. 1-forms

So $\mathrm{Pic}^0(X) \cong \frac{H^0(X, \Omega^1)^*}{H^1(X, \mathbb{Z})}$, $H^1(X,\mathbb{Z}) \cong H_1(X, \mathbb{Z})$

→ Poincare duality

↳ if $\dim_{\mathbb{C}} X = 1$

Albanesi map: $\{\omega_1, \dots, \omega_g\}$ bases of $H^0(X, \Omega^1)$. Let $p \in X$, γ a path from p to q. $\mathrm{Alb}(q) = \left[\int_\gamma \omega_1, \dots, \int_\gamma \omega_g\right] \in \mathbb{C}^g/\Lambda$.