Question: when is $\mathbb{C}^g/\Lambda$, $rk_{\mathbb{R}}\Lambda = 2g$ a proj. alg. man.?

$g=1$ ok.

Ex.: $Pic^0(X)$, $Alb(X)$, X proj. smooth man..

$H^1(X)=0 \Rightarrow Alb(X) = Pic^0(X) = 0$.

Ex.: in $Pic^0(\mathbb{P}^N)=0$.

Riemann's condition: $\mathbb{C}^g/\Lambda$, $rk_{\mathbb{R}}\Lambda = 2g$ is a proj. alg. man. $\Leftrightarrow$
$\Leftrightarrow \exists$ a positive def. hermitian form h on $\mathbb{C}^g$ s.t. $\mathrm{Im}\, h_{|\Lambda\times\Lambda}$ is integral.

$g=1$: $\Lambda = \mathbb{Z}\oplus\tau\mathbb{Z}$, $\mathrm{Im}\,\tau>0$, $\Theta: \mathbb{C}\to\mathbb{C}$, $z\mapsto \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n z)$

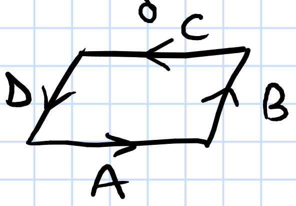
$\Theta(z+1)=\Theta(z)$.

Calculation: $\Theta(z+N\tau) = \exp(-\pi i N^2\tau - 2\pi i N z)\,\Theta(z)$.

$\Theta_{a,b}(z) = \exp(\pi i a^2\tau + 2\pi i a z)\,\Theta(z+a\tau+b)$.

Fix $N\geq 1$ and let V_N = vector space over $\mathbb{C}$ spanned by $\Theta_{a,b}$ where a,b range over $\{0,\frac{1}{N},\dots,\frac{N-1}{N}\}\times\{0,\frac{1}{N},\dots,\frac{N-1}{N}\}$.

Lemma: if $f\in V_N\setminus\{0\}$ then f has N^2 zeros in the parallelogram P with vertices at $0, N, N\tau, N+N\tau$.



Proof: $\#$ zeros $= \frac{1}{2\pi i}\int_\gamma \frac{f'dz}{f}$.

γ ↳ traslation of ∂P not passing through the zeros of f

Sides A and C differ by $N\tau$, B " D " " N. } up to orientation

$\Theta_{a,b}(z+N) = \Theta_{a,b}(z) \Rightarrow \frac{\Theta'_{a,b}(z+N)}{\Theta_{a,b}(z+N)} = \frac{\Theta'_{a,b}(z)}{\Theta_{a,b}(z)} \Rightarrow$

$\Rightarrow \int_{B+D}\frac{f'dz}{f} = 0$.

$\Theta_{a,b}(z+N\tau) = \exp(-N^2\pi i\tau)\exp(-2\pi i N z)\Theta_{a,b}(z) \Rightarrow$

$\Rightarrow \Theta'_{a,b}(z+N\tau) = \exp(-N^2\pi i\tau)(-2\pi i N\exp(-2\pi i N z)\Theta_{a,b}(z) +$
$+ \exp(-2\pi i N z)\Theta'_{a,b}(z)) \Rightarrow$

$\Rightarrow \frac{\Theta'_{a,b}(z+N\tau)}{\Theta_{a,b}(z+N\tau)} = -2\pi i N + \frac{\Theta'_{a,b}(z)}{\Theta_{a,b}(z)} \Rightarrow$

$\Rightarrow \int_{A+C}\frac{f'dz}{f} = N.$ $\square$

Thm.: pick $N>1$, and an ordering (a_j,b_j), $j=1,\dots,N^2$ for the elements of $\{0,\frac{1}{N},\dots,\frac{N-1}{N}\}\times\{0,\frac{1}{N},\dots,\frac{N-1}{N}\}$. Then

$\mathbb{C}/N\cdot\Lambda \xrightarrow{\varphi} \mathbb{P}^{N^2-1}$, $z\mapsto[\Theta_{a_1,b_1}(z):\dots:\Theta_{a_{N^2},b_{N^2}}(z)]$ is an embedding.

Proof (sketch): φ is well def.: $\Theta_{a,b}(z+N) = \Theta_{a,b}(z)$ ok.

$\Theta_{a,b}(z+N\tau) = \exp(-\pi i N^2\tau - 2\pi i N z)\Theta_{a,b}(z)$.

Suppose $\exists\, z_1\neq z_1'$ in $\mathbb{C}/N\cdot\Lambda$ s.t. $\varphi(z_1)=\varphi(z_1') \Leftrightarrow$
$\Leftrightarrow f(z_1) = f(z_1')\ \forall f\in V_N$. Then, by traslations of $\mathbb{Z}\oplus\tau\mathbb{Z}$ we get other pairs of such points, (z_2, z_2').

Since V_N has dimension N^2, we can find a set of distinct pts $\{z_1,\dots,z_{N^2-1}, z_1', z_2'\}$ and $0\neq f\in V_N$ s.t. f vanishes at $z_1,\dots,z_{N^2-1} \Rightarrow$ also at $z_1', z_2' \Rightarrow N^2+1$ zeros, contradiction. $\square$