Remark: Θ is not doubly periodic, but its zero locus is.

A divisor on a comp. man X is a formal sum of codim. 1 subvarieties with $\mathbb{Z}$-coefficients.

$\Theta: \mathbb{C} \to \mathbb{C}$ gives a divisor on $\mathbb{C}/\Lambda$: $Z(\Theta) = \{\text{zero locus of } \Theta\} \subseteq \mathbb{C}$, doubly periodic w r t Λ $\pi: \mathbb{C} \to \mathbb{C}/\Lambda \rightsquigarrow \pi(Z(\Theta))$.

$f: \underset{\llcorner \text{cpt}}{X} \to \mathbb{P}^1$ hol. (f mero. on X), $\operatorname{div}(f) = \sum_{p \in X} \operatorname{ord}_p(f) p$.

What we do with Θ: $\pi\left(\sum_{p \in \mathbb{C}} \operatorname{ord}_p(\Theta) p\right)$ is the Θ-divisor.

Question: when is $\mathbb{C}^g/\Lambda$ proj. alg.?

Answer (Riemann condition): $\exists$ a positive def. hermitian form h on $\mathbb{C}^g$ s.t. $\operatorname{Im} h$ is integral on Λ.

Hermitian metrics

X comp. man. $\Rightarrow X$ has a comp. structure tensor J. A Riemannian metric h on X is hermitian $\overset{\text{def.}}{\Longleftrightarrow} g(J\alpha, J\beta) = g(\alpha, \beta)$.

Basic example: $h = dz \otimes d\bar{z} = (dx + i dy) \otimes (dx - i dy) =$

all on $\mathbb{C}$

$$= (dx \otimes dx + dy \otimes dy) + i(dy \otimes dx - dx \otimes dy).$$

$J(\partial/\partial x) = \partial/\partial x$, $J(\partial/\partial y) = -\partial/\partial x$.

Check the calculations ($J^*(dx) = dx \circ J = -dy$, $J^*(dy) = dy \circ J = dx$).

$\omega(\alpha, \beta) := h(J\alpha, \beta)$. $\omega(\beta, \alpha) = h(J\beta, \alpha) = h(JJ\beta, J\alpha) =$

$$= h(-\beta, J\alpha) = -h(J\alpha, \beta) = -\omega(\alpha, \beta).$$

Def.: h is Kähler if the associated 2-form ω is closed ($d\omega = 0$).

Ex.: $\mathbb{C}^g/\Lambda$. Ex.: $\mathbb{P}^N$.

Def.: a cpt comp. man M is Hodge if it admits a Kähler metric h with Kähler form ω s.t. $[\omega] \in H^2(M, \mathbb{Z})$.

$\mathbb{P}^N$ Kähler $\Rightarrow$ every proj. alg. man. must be Kähler.

$\Theta: \mathbb{C}^g \to \mathbb{C}$, $\Theta(z) = \sum_{m \in \mathbb{Z}^g} \exp(\pi i m^T \Omega m + 2\pi i m^T z)$, where Ω is a $g \times g$ sym. matrix with positive def. imaginary part.

By construction, $\Theta(z + \gamma) = A(\gamma, z)\Theta(z)$, $A(\gamma, z) \neq 0$, $\gamma \in \Lambda \Rightarrow$
$\Rightarrow Z(\Theta)$ is invariant under traslation by $\Lambda \rightsquigarrow Z(\Theta)$ defines a divisor in $\mathbb{C}^g/\Lambda$.

X cpt Riemann surface of genus $g \Rightarrow$ the space of hol. 1-forms on X has dim. g; pick a basis $\omega_1, \dots, \omega_g$, $X \xrightarrow[\text{Abel}]{} \mathbb{C}^g/\Lambda$,
$\Lambda = \{(\int_\gamma \omega_1, \dots, \int_\gamma \omega_g) \mid \gamma \in H_1(X, \mathbb{Z})\}$. Let $p \in X$ be a base pt, we have $X \ni q \mapsto (\int_p^q \omega_1, \dots, \int_p^q \omega_g) \in \mathbb{C}^g/\Lambda$. Hodge-Riemann bilinear relations $\Rightarrow \mathbb{C}^g/\Lambda \to \mathbb{P}^N$ (Λ satisfies Riemann's condition).