

Thm. (Abel):  $\mathbb{C}/\Lambda$ ,  $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ ,  $\pi i \notin \mathbb{R}\Lambda = 2$ .

$\exists f: \mathbb{C}/\Lambda \rightarrow \mathbb{P}^1$  mero. with zeros  $[a_j] \in \mathbb{C}/\Lambda$  of order  $N_j$  and poles  $[b_j]$  of order  $M_j \iff$   
 $\iff \sum N_j \stackrel{(1)}{=} \sum M_j, \sum N_j [a_j] \stackrel{(2)}{=} \sum M_j [b_j]$ .

Proof:  $\Theta(\bar{z}) = \sum_{m \in \mathbb{Z}} \exp(\pi i m^2 \tau) \exp(2\pi i m \bar{z})$ ,  $\tau = \omega_2/\omega_1$ ,  $\text{Im} \tau > 0$ .  $\Theta$  has exactly 1 zero in a fundamental domain for  $\mathbb{C}/\Lambda$ .

$\Theta_\sigma(\bar{z}) := \sum_{m \in \mathbb{Z}} \exp(\pi i (m+1/2)^2 \tau) \exp(2\pi i (m+1/2)(\bar{z} + 1/2))$ .

$\Theta_\sigma(\bar{z}) = \exp(\pi i \tau/4 + \pi i (\bar{z} + 1/2)) \Theta(\bar{z} + (\tau+1)/2) \implies$

$\implies$  zeros of  $\Theta_\sigma$  are translates of zeros of  $\Theta$ ; in particular,

$\Theta_\sigma$  has only one zero in the fund. domain.

$\Theta_\sigma(-\bar{z}) = -\Theta_\sigma(\bar{z}) \implies \Theta_\sigma(\bar{z})$  has a simple zero at 0;

so  $\Theta(\bar{z})$  has a simple zero at  $\frac{\tau+1}{2}$ .

Set  $g(\bar{z}) = \frac{\prod_j (\Theta_\sigma(\bar{z} - a_j))^{N_j}}{\prod_j (\Theta_\sigma(\bar{z} - b_j))^{M_j}}$ . We need  $g$  well def. on  $\mathbb{C}/\Lambda$ .

$\lambda = r\tau + q$ ,  $e(\lambda, \bar{z}) := \exp(\pi i \lambda - \pi i r^2 \tau - 2\pi i r(\bar{z} + \frac{1+\tau}{2}))$ .

$\Theta_\sigma(\bar{z} + \lambda) = e(\lambda, \bar{z}) \Theta_\sigma(\bar{z})$ . So

$g(\bar{z} + \lambda) = g(\bar{z}) \cdot \frac{\prod_j e(\lambda, \bar{z} - a_j)^{N_j}}{\prod_j e(\lambda, \bar{z} - b_j)^{M_j}} \stackrel{(1)}{=} \frac{g(\bar{z}) \exp(2\pi i \sum_j N_j a_j)}{\exp(2\pi i \sum_j M_j b_j)}$ .

Choose  $a_j, b_j$  s.t. this is just  $g(\bar{z})$ .  $\square$

$C$  cpt Riemann surface,  $p \in C$ ,  $AJ_p: C \rightarrow \text{Jac}(C) = \mathbb{C}^g/\Lambda$ .

$\sum_j N_j a_j, \sum_j M_j b_j$  divisors on  $C$  s.t.  $\sum_j N_j = \sum_j M_j$ .

$\sum_j N_j AJ(a_j) = \sum_j M_j AJ(b_j)$ .  $AJ: \text{Div}^0(C) \rightarrow \text{Jac}(C)$ ,

$\ker(AJ) = \{ \sum_j \text{div}(f), f \text{ mero.} \} \rightsquigarrow AJ: \frac{\text{Div}^0(C)}{\text{Div}(\text{mero.})} \hookrightarrow \text{Jac}(C)$ .

It is also surj..