Period Matrix of a Riemann surface (X cpt)

$Q \rightsquigarrow \langle \cdot, \cdot \rangle$ skew sym., pos. def. $\Rightarrow \exists$ basis $\{e_1, \dots, e_g, f_1, \dots, f_g\}$, $g =$ genus of $X > 0$ s.t. $Q(e_a, f_b) = -\delta_{ab}$.

Let $\{\omega_1, \dots, \omega_g\}$ be a basis of $H^{1,0}(X)$.

Write $\omega_j = \sum_{k=1}^{g} \Omega^1_{jk} e_k + \Omega^2_{jk} f_k$. Suppose that $\exists c_1, \dots, c_g \in \mathbb{C}$ s.t. $\sum_{j=1}^{g} c_j \Omega^2_{jk} = 0$, $k = 1, \dots, g \Leftrightarrow (c_1, \dots, c_g) \begin{pmatrix} \Omega^2_{11} & \cdots & \Omega^2_{1g} \\ \vdots & \ddots & \vdots \\ \Omega^2_{g1} & \cdots & \Omega^2_{gg} \end{pmatrix} = (0, \dots, 0)$.

Then $Q(\overbrace{\sum_{j=1}^{g} c_j \omega_j}^{\omega}, \sum_{j=1}^{g} \overline{c}_j \overline{\omega}_j) =$

$= Q\left(\sum_{j,k=1}^{g} c_j \Omega^1_{jk} e_k, \overline{(\dots)}\right) = 0.$

$Q(e_a, e_b) = 0$, $Q(f_a, f_b) = 0$

$Q(\omega, \overline{\omega}) \neq 0$ unless $\omega = 0$ since $\omega \in H^{1,0}(X) \Rightarrow$

$\Rightarrow \sum_{j=1}^{g} c_j \omega_j = 0 \Rightarrow c_1 = \dots = c_g = 0 \Rightarrow (\Omega^2)_{jk} = \Omega^2_{jk}$ is invertible.

$$\left(\frac{\Omega^1}{\Omega^2}\right) = (\omega_1 | \dots | \omega_g).$$

$2g \times g$ matrix

$\hookrightarrow$ in the basis $e_1, \dots, e_g, f_1, \dots, f_g$

$(\Omega^2)^{-1} \left(\frac{\Omega^1}{\Omega^2}\right) = \left(\frac{\tilde{\Omega}^1}{\mathrm{Id}}\right)$. In summary, we can pick the basis $\{\omega_1, \dots, \omega_g\}$ of $H^{1,0}(X)$ s.t. $\omega_j = \left(\sum_{k=1}^{g} \Omega_{jk} e_k\right) + f_j$.

$0 = Q(\omega_a, \omega_b) = \dots$ calculations $\dots = \Omega_{ba} - \Omega_{ab} \Rightarrow \Omega$ is sym..

$Q(H^{1,0}, H^{1,0}) = 0$

$iQ\left(\sum_{k=1}^{g} c_k \omega_k, \sum_{l=1}^{g} \overline{c}_l \overline{\omega}_l\right) = iQ\left(\sum_{k=1}^{g} c_k (f_k + \sum_{j=1}^{g} \Omega_{jk} e_j), (\dots)\right) =$

$= i \sum_{k,l=1}^{g} c_k \overline{c}_l (\overline{\Omega}_{kl} - \Omega_{lk}) = \sum_{k,l=1}^{g} i c_k (\overline{\Omega}_{lk} - \Omega_{lk}) \overline{c}_l =$

$= 2 \sum_{k,l=1}^{g} c_k \operatorname{Im}(\Omega_{lk}) \overline{c}_l$. $0 \neq \omega \in H^{1,0}(X) \Rightarrow i Q(\omega, \overline{\omega}) > 0$, so Ω is a $g \times g$ sym. matrix with pos. def. imaginary part.

Ω is called "the" periodic matrix of X.

$\hookrightarrow$ it depends on the choice of $\{e_1, \dots, e_g, f_1, \dots, f_g\}$, but not " " " " $\{\omega_1, \dots, \omega_g\}$

Ex.: $X = \mathbb{C}/\Lambda$. $\{e, f\}$ symplectic basis of $H^1(X; \mathbb{Z})$:

$Q(e, f) = -1$, $Q(e, e) = Q(f, f) = 0$.

$H^{1,0} = \mathbb{C}(z e + f) \leftrightarrow \mathbb{C}\binom{z}{1}$.

$iQ(z e + f, \overline{z e + f}) = i(\overline{z} - z) = 2 \operatorname{Im}(z) > 0$.

Change of basis: $\mathbb{C} \cdot \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{SL_2(\mathbb{Z})} \binom{z}{1} = \mathbb{C} \cdot \binom{az + b}{cz + d} = \mathbb{C} \cdot \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$:

it's what we already know.