$X$ cpt Kähler man., $\Delta = dd^* + d^*d$, $\Box = \partial\partial^* + \partial^*\partial$, $\bar{\Box} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, $d = \bar\partial + \partial$.

$\mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, period matrix is $\tau$.

$C: y^2 = 4(x-e_1)(x-e_2)(x-e_3)$, $\alpha, \beta$ basis for $H_1(C,\mathbb{Z})$, $\int_\alpha \frac{dx}{y}, \int_\beta \frac{dx}{y}$.

Thm.(Ehresmann): if $X \xrightarrow{\pi} S$ is a $C^\infty$ map of diff. man. which is a proper $\underset{\text{surj.}}{\vee}$ submersion, then $\pi$ is locally $C^\infty$-trivial.

Proof: let $\xi$ be a nowhere vanishing vector field on an open subset $U$ of $S$. Restrict $X \xrightarrow{\pi} S$ to $U$.

Claim: $\exists$ a vector field $\xi'$ on $X$ or $(\pi^{-1}(U))$ s.t. $\pi_*(\xi') = \xi$.

In a suitable chart, $\pi: (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_m)$. Since $\xi$ is nowhere vanishing, I can use the implicit function thm. to argue that $\xi = \frac{\partial}{\partial x_1} \Rightarrow \pi_*\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_1}$.

This works locally $\Rightarrow \exists\, \xi'$ by partitions of unity.

$X_p \cong X_q$ if $p \sim q$: in local coord., take the vector from $p$ to $q$ as $\xi$; to go from $X_p$ to $X_q$, flow by $\xi'$. $\Box$

Period map for a family of cpt Riemann surfaces

$X \xrightarrow{\pi} S$, $p \in S$ with smooth fiber, $\exists$ neigh. $U$ of $p$ in $S$ s.t. $\pi^{-1}(U) \underset{\text{diff.}}{\cong} U \times \pi^{-1}(p) \Rightarrow$ we have a symplectic basis $\{e_i, f_i\}$ on $U$.

$\mathfrak{D} = \{g \times g$ sym. matrices with pos. def. imaginary part$\}$.

Locally, $\underset{\hookrightarrow \text{period map}}{\varphi}: U \to \mathfrak{D}$ where $U$ is a neigh of $p$ s.t. $\pi^{-1}(U) \cong U \times X_p \Rightarrow$

$\Rightarrow \varphi: S \to \mathfrak{D}/\Gamma$, $\Gamma \subseteq$ symplectic group.

Ex.: $E_\lambda: y^2 = x(x-1)(x-\lambda)$ family of elliptic curves. $\omega = \frac{dx}{y}$ is a holo. 1-form on $E_\lambda$; $\frac{\partial\omega}{\partial\lambda}, \frac{\partial^2\omega}{\partial\lambda^2}$ are also 1-forms on $E_\lambda \underset{\operatorname{rk} H^1(E_\lambda) = 2}{\Longrightarrow}$

$\Rightarrow$ they are linearly dependent (with $\omega$, in $H^1$):

$$d\left(\frac{x^{1/2}(x-1)^{1/2}(x-\lambda)^{1/2}}{(x-\lambda)^2}\right) = -\frac{1}{2}\omega - (4\lambda-2)\frac{\partial\omega}{\partial\lambda} - 2\lambda(\lambda-1)\frac{\partial^2\omega}{\partial\lambda^2}.$$

Leta $\alpha_1, \alpha_2$ be a flat frame for $E_\lambda$ near $\lambda = p$.

$$\pi_j = \int_{\alpha_j} \underbrace{\frac{dx}{y}}_{\omega}. \quad \frac{\partial\pi_j}{\partial\lambda} = \int_{\alpha_j} \frac{\partial\omega}{\partial\lambda}, \quad \frac{\partial^2\pi_j}{\partial\lambda^2} = \int_{\alpha_j} \frac{\partial^2\omega}{\partial\lambda^2}.$$

$$-\frac{1}{2}\pi_j - (4\lambda-2)\frac{\partial\pi_j}{\partial\lambda} - 2\lambda(\lambda-1)\frac{\partial^2\pi_j}{\partial\lambda^2} = \int_{\alpha_j} d(\cdots) = 0.$$