$\gamma \in H_1(E_\lambda;\mathbb{Z})$, $\pi(\lambda) := \int_\gamma \frac{dx}{y}$.

$\{E_\lambda\}_{\lambda\in\mathbb{P}^1\setminus\{0,1,\infty\}}$ locally C^∞-trivial $\Rightarrow$ deform γ to a topologically constant family of cycles locally.

So $0 = \frac{1}{2}\pi(\lambda) + (4\lambda-2)\frac{\partial\pi}{\partial\lambda}(\lambda) + 2\lambda(\lambda-1)\frac{\partial^2\pi}{\partial\lambda^2}(\lambda)$.

↳ diff. equation with regular singular points

$$\frac{\partial^2\pi}{\partial\lambda^2}(\lambda) + \frac{1}{\lambda}\underbrace{\left(\frac{2\lambda-1}{\lambda-1}\right)}_{p(\lambda)}\frac{\partial\pi}{\partial\lambda}(\lambda) + \frac{1}{\lambda^2}\cdot\underbrace{\frac{\lambda}{4(\lambda-1)}}_{q(\lambda)}\pi(\lambda) = 0,$$

$p(\lambda)$, $q(\lambda)$ are hol. at $\lambda=0$.

Let $\pi(\lambda) = \lambda^r\cdot\sum_{j=0}^{+\infty} a_j\lambda^j$, $a_0\neq 0 \Rightarrow$ ... calculations ... $\Rightarrow$

$\Rightarrow r(r-1) + r\,p(0) + q(0) = 0 \underset{p(0)=1,\ q(0)=0}{\Longrightarrow} r^2 = 0 \Rightarrow r = 0 \Rightarrow$

$\Rightarrow \pi(\lambda) = \sum_{j=0}^{+\infty} a_j\lambda^j$, $(j+1/2)^2 a_j - (j+1)^2 a_{j+1} = 0$, $a_0 = 1 \Rightarrow$

$\Rightarrow a_j = \binom{-1/2}{j}^2 \in \mathbb{Q}$.

$y^2 = x(x-1)(x-\lambda)$, if $\lambda\in\mathbb{Z}$ we get a curve defined by an equation with integers coefficients. How many pts does E_N mod p ($p>2$) have? Answer: $\pi(N)$ mod p, $\pi(\lambda)$ hol. solution.

Flat bundles and representations of $\pi_1(S, s_0)$

$\rho: \pi_1(S,s_0) \longrightarrow GL(V)$ representation. Assume S ha a universal cover $\pi: (\tilde S, \tilde s_0) \longrightarrow (S, s_0)$.

$\tilde V := \tilde S \times V$, $\underbrace{\pi_1(S,s_0)}_{\text{deck transformations of } \tilde S}$ acts on $\tilde V$ by $g\cdot(\tilde s, v) = (g\cdot\tilde s, \rho(g)v)$.

$\mathcal{V} := \tilde V/\pi_1(S, s_0)$.

Ex.: $V = \mathrm{span}\{e, f\}$, $S = \Delta^*$, γ = counterclockwise loop,

$\rho(\gamma) = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ relative to $\{e, f\}$.

$\mathcal{V}$ = vector bundle over Δ^* with flat frame $\{e + \frac{1}{2\pi i}\log s\cdot f, f\}$. ↳ depends on local coord.

$\tilde S = \mathbb{H}$, $\pi: \tilde S \longrightarrow S$, $z \longmapsto e^{2\pi i z}$.

The flat frame on $\mathbb{H}\times V$ is $\{e + zf, f\}$.

$\gamma: z \longmapsto z+1$, $\rho(\gamma)e = e + f$, $\rho(\gamma)f = f \Rightarrow$

$\Rightarrow \gamma\cdot(z, e + zf) = (z+1, e + (z+1)f)$.

So $\{e + zf, f\}$ is a flat frame for $\mathbb{H}\times V$.

$N = \log\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$, $D = \frac{d}{dz}\otimes dz - N\otimes dz$.

$D_{\frac{\partial}{\partial z}}(e + zf) = 0$, $D_{\frac{\partial}{\partial z}} f = 0$.